Model Predictive Control of Nonholonomic Mobile Robots without Stabilizing Constraints and Costs*

Karl Worthmann, Mohamed W. Mehrez, Mario Zanon, George K. I. Mann, Raymond G. Gosine, Moritz Diehl

***Abstract*—The problem of steering a nonholonomic mobile robot to a desired position and orientation is considered. In this paper, a model predictive control (MPC) scheme based on tailored non-quadratic stage cost is proposed to fulfil this control task. We rigorously prove asymptotic stability while neither stabilizing constraints nor costs are used. To this end, we first design suitable manoeuvres to construct bounds on the value function. Second, these bounds are exploited to determine a prediction horizon length such that asymptotic stability of the MPC closed-loop is guaranteed. Finally, numerical simulations are conducted to explain the necessity of having non-quadratic running costs.**

***Index Terms*—nonholonomic robots, model predictive control, asymptotic stability, prediction horizon, non-quadratic costs.**

I. Introduction

Unmanned ground vehicles (UGVs) have attracted considerable interest in the recent decades due to their wide range of applicability, cf. [1] or [2] for a thorough review. Nonholonomic differential-drive models, such as unicycle models, are commonly used to describe kinematics of UGVs. Typically, the control objective is to drive the robot between two static poses, which can be identified as set-point regulation (stabilization), cf. [3]. For this problem, Brockett's condition [4] implies that neither the linearized model is stabilizable nor a smooth time-invariant feedback control law exists – a typical characteristic of nonholonomic systems, see also [5]. Nonetheless, various solution strategies like piecewise-continuous feedback control or smooth time-varying control have been reported, see the overview paper [6]. Further control approaches based on differential kinematic control [7], backstepping [8], and vector field orientation feedback [9] have also been proposed. However, these control strategies ignore natural input saturation limits and, thus, require a post processing step in order to scale the calculated control signals to their physical bounds, cf. [9] for details. In addition, determining suitable tuning parameters in order to achieve an acceptable performance remains a challenging task, cf. [10]. In contrast, several successful case studies using model predictive control (MPC) were conducted, see, e.g. [11], [6], [12], [3], [13].

MPC is considered to be one of the most attractive control strategies due to its applicability to constrained nonlinear multiple input multiple output (MIMO) systems. In MPC, a sequence of control inputs minimizing an objective function is computed over a finite prediction horizon; then, the first element of this (optimal) control sequence is applied to the plant. This process is repeated every sampling instant, see, e.g. [14] for further details. Since only finite horizon problems are solved in each MPC step, closed-loop stability may not hold, cf. [15]. Nonetheless, stability can be ensured, e.g. by imposing terminal constraints, cf. [16], [17], or by using bounds on the value function in order to determine a stabilizing prediction horizon length, see, e.g. [18], [19], [20].

For regulation of nonholonomic robots, stabilizing MPC using terminal region constraints and costs has been pursued in [6] while a contraction constraint on the first state in the prediction horizon was used in [3]. Moreover, in [11] a non-quadratic terminal cost was constructed on a terminal region for car-like nonholonomic robots. Here, the desired set point was located at the boundary of the closed terminal region, see also [21] for a robust version. MPC without stabilizing constraints but with terminal costs has been first studied for nonholonomic systems in [22]. For the regulation of differential drive robots, MPC without stabilizing constraints is particularly attractive since computing (possibly time varying) terminal regions for large feasible sets can be an extremely challenging task, cf. [23]. This is especially true if the results shall be generalized to multi robot systems or domains with obstacles.

In this work, a stability analysis of MPC schemes without stabilizing constraints or costs for regulation of nonholonomic mobile robots is performed. Herein, a methodology is proposed, which allows to determine a prediction horizon length such that asymptotic stability of the MPC closed-loop is guaranteed. To this end, a proof of concept for verifying the controllability assumption introduced in [19] is presented. Herein, the running costs are tailored to the design specification of controlling both the position and the orientation. Then, the less conservative technique of [24], [20], [25], [26] is applied in order to rigorously prove asymptotic stability.

While the construction of particular open-loop manoeuvres used to derive the growth condition of [19] heavily relies on the kinematic unicycle model, the pursued approach is outlined such that it can be used as a framework for verifying the above mentioned controllability assumption and, thus, being able to conclude asymptotic stability of the MPC closed-loop also

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for other systems. In particular, the insight provided by our analysis yields guidelines for the design of MPC controllers also for more accurate models of differential drive robots – a topic for future research. An extension of our discrete time results to the continuous time domain based on the presented results can be found in [27].

Finally, we numerically demonstrate that the canonical choice of quadratic running costs is not suited for regulation of nonholonomic mobile robots without (stabilizing) terminal constraints and/or costs. Moreover, the effectiveness of our approach is shown by means of numerical simulations.

This paper is organized as follows: Section II outlines the regulation problem of nonholonomic mobile robots as well as the MPC algorithm. The stability results presented in [20], [26] are revisited in Section III. In Section IV, bounds on the value function are derived by constructing appropriate feasible open-loop trajectories. Based on these bounds, a suitable prediction horizon length can be determined such that the MPC closed-loop is asymptotically stable. Our findings are illustrated by numerical simulations in Section V. Finally, conclusions are drawn in Section VI.

Notation: $\mathbb{R}$ and $\mathbb{N}$ denote real and natural numbers, respectively. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ represents the non-negative integers and $\mathbb{R}_{\geq 0}$ the non-negative real numbers. A continuous function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is zero at zero and strictly monotonically increasing. If it is, in addition, unbounded it is called a class $\mathcal{K}_\infty$-function. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, n) \in \mathcal{K}_\infty$ for all $n \in \mathbb{N}_0$ and $\beta(r, \cdot)$ is strictly monotonically decaying to zero for each $r > 0$.

II. Problem Setup

In this section, a differential drive mobile robot is described by an ordinary differential equation. Then, a corresponding discrete time model is presented and a model predictive control scheme is proposed in order to asymptotically stabilize the robot.

A. Nonholonomic mobile robot

The kinematic model of the mobile robot is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{pmatrix} = \dot{z}(t) = f(z(t), u(t)) = \begin{pmatrix} v(t)\cos(\theta(t)) \\ v(t)\sin(\theta(t)) \\ w(t) \end{pmatrix} \quad (1)$$

with an analytic vector field $f : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$. The first two (spacial) components of the state $z = (x, y, \theta)^T$ (m,m,rad) represent the position in the plane while the angle θ corresponds to the orientation of the robot. The control input is $u = (v, w)^T$ (m/s,rad/s), where v and w are the linear and the angular speeds of the robot, respectively. Assuming piecewise constant control inputs on each interval $[iT, (i+1)T)$, $i \in \mathbb{N}_0$, with sampling period T (seconds), the (exact) discrete time dynamics $f_{e,T} : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ are given by

$$z^+ = f_{e,T}(z, u) = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} \frac{v}{w}\left(\sin(\theta + Tw) - \sin(\theta)\right) \\ \frac{v}{w}\left(\cos(\theta) - \cos(\theta + Tw)\right) \\ Tw \end{pmatrix} \quad (2)$$

for $w \neq 0$. When the robot moves in a straight line (angular speed $w = 0$) the right hand side of (2) becomes

$$z + \lim_{w \to 0} \begin{pmatrix} \frac{v}{w}\left(\sin(\theta + Tw) - \sin(\theta)\right) \\ \frac{v}{w}\left(\cos(\theta) - \cos(\theta + Tw)\right) \\ Tw \end{pmatrix} = z + Tv \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}.$$

The movement is restricted to a rectangle which is modelled by the box constraints

$$\begin{pmatrix} x_{\min} \\ y_{\min} \end{pmatrix} \leq \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \leq \begin{pmatrix} x_{\max} \\ y_{\max} \end{pmatrix} \qquad \forall\, k \in \mathbb{N}_0. \quad (3)$$

The control inputs are limited by

$$\begin{pmatrix} v_{\min} \\ w_{\min} \end{pmatrix} \leq \begin{pmatrix} v(k) \\ w(k) \end{pmatrix} \leq \begin{pmatrix} v_{\max} \\ w_{\max} \end{pmatrix} \qquad \forall\, k \in \mathbb{N}_0 \quad (4)$$

with $v_{\min} < 0 < v_{\max}$ and $w_{\min} < 0 < w_{\max}$. Then, admissibility of a sequence of input signals can be defined as follows.

Definition 1. *Let $Z := [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}] \times \mathbb{R} \subset \mathbb{R}^3$ and $U := [v_{\min}, v_{\max}] \times [w_{\min}, w_{\max}] \subset \mathbb{R}^2$ be given. Then, for a given state $z_0 \in Z$, a sequence of control values $u = (u(0), u(1), \ldots, u(N-1)) \in U^N$ of length $N \in \mathbb{N}$ is called admissible, denoted by $u \in \mathcal{U}^N(z_0)$, if the state trajectory*

$$z_u(\cdot; z_0) = (z_u(0; z_0), z_u(1; z_0), \ldots, z_u(N; z_0))$$

iteratively generated by system dynamics (2) *and $z_u(0; z_0) = z_0$ satisfies $z_u(k; z_0) \in Z$ for all $k \in \{0, 1, \ldots, N\}$. An infinite sequence of control values $u = (u(k))_{k \in \mathbb{N}_0} \subset U$ is said to be admissible for $z_0 \in Z$, denoted by $u \in \mathcal{U}^\infty(z_0)$, if the truncation to its first N elements is contained in $\mathcal{U}^N(z_0)$ for all $N \in \mathbb{N}$.*

B. Model Predictive Control

The goal is to steer the mobile robot to a desired (feasible) state $z^\star \in Z$, which is without loss of generality chosen to be the origin, i.e. $z^\star = 0_{\mathbb{R}^3}$.[1] Indeed, $z^\star$ is a (controlled) equilibrium since $f_{e,T}(z^\star, 0) = z^\star$. More precisely, our goal is to find a static state feedback law $\mu : Z \to U$ such that, for each $z_0 \in Z$, the resulting closed-loop system $z_\mu(\cdot; z_0)$ generated by

$$z_\mu(k+1; z_0) = f_{e,T}(z_\mu(k; z_0), \mu(z_\mu(k; z_0)))$$

and $z_\mu(0; z_0) = z_0$, satisfies the constraints $z_\mu(k; z_0) \in Z$ and $\mu(z_\mu(k; z_0)) \in U$ for all $k \in \mathbb{N}_0$ and is asymptotically stable, i.e. there exists a $\mathcal{KL}$-function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ such that, for each $z_0 \in Z$, the closed-loop trajectory obeys the inequality

$$\|z_\mu(k; z_0)\| \leq \beta(\|z_0\|, k) \qquad \forall\, k \in \mathbb{N}_0.$$

As briefly discussed in the introduction, several control techniques have been developed for this purpose. In this paper we use MPC, which makes use of the system dynamics in order to design a control strategy minimizing a cost function. This cost function sums up given stage costs along predicted

[1] $z^\star$ is supposed to be in the interior of the state constraint set Z.

(feasible) trajectories. We propose to deploy the running (stage) costs $\ell : Z \times U \to \mathbb{R}_{\geq 0}$ defined as

$$\ell(z,u) = q_1 x^4 + q_2 y^2 + q_3 \theta^4 + r_1 v^4 + r_2 w^4 \tag{5}$$

with $q_1, q_2, q_3, r_1, r_2 \in \mathbb{R}_{>0}$. In (5), small deviations in the y-direction are penalized more than deviations with respect to x or θ. The motivation behind this particular choice becomes clear in Subsection IV-B where the different order of y is exploited in order to verify Assumption 1 and, thus, to ensure asymptotic stability. Moreover, in Subsection V-B, we explain why quadratic running costs $\ell(z,u) = z^T Q z + u^T R u$, $Q \in \mathbb{R}^{3\times 3}$ and $R \in \mathbb{R}^{2\times 2}$, are not suited for our example by conducting numerical simulations.

Based on the introduced running costs, a cost function $J_N : Z \times U^N \to \mathbb{R}_{\geq 0}$ and a corresponding (optimal) value function $V_N : Z \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ are defined as

$$J_N(z_0, u) := \sum_{n=0}^{N-1} \ell(z_u(n), u(n)) \quad \text{and}$$
$$V_N(z_0) := \inf_{u \in \mathcal{U}^N(z_0)} J_N(z_0, u)$$

for $N \in \mathbb{N} \cup \{\infty\}$, where $V_N(z_0) = \infty$ if $\mathcal{U}^N(z_0) = \emptyset$ holds. Algorithm 1, which is an MPC scheme without stabilizing constraints or costs, is employed in order to solve this task. For a detailed discussion on MPC we refer to [23], [14].

Algorithm 1 MPC

Initialization: set prediction horizon N and time index $k := 0$.

1: Measure the current state $\hat{z} := z(k)$.
2: Compute $u^\star = (u^\star(0), u^\star(1), \ldots, u^\star(N-1)) \in \mathcal{U}^N(\hat{z})$ satisfying $J_N(\hat{z}, u^\star) = V_N(\hat{z})$.
3: Define the MPC feedback law $\mu_N : Z \to U$ at $\hat{z}$ by $\mu_N(\hat{z}) := u^\star(0)$ and implement $u(k) := \mu_N(\hat{z})$ at the plant. Then, increment the time index k and goto step 1.

Since $0_{\mathbb{R}^2} \in U$ holds, $\mathcal{U}^N(f_{e,T}(\hat{z}, \mu_N(\hat{z}))) \neq \emptyset$ holds, i.e. recursive feasibility of the MPC closed-loop is ensured. Existence of an admissible sequence of control values minimizing $J_N(\hat{z}, \cdot)$ can be infered from compactness of the nonempty domain and continuity of the cost function by applying the Weierstrass theorem, cf. [28] for details. However, since neither stabilizing constraints nor terminal costs are incorporated in our MPC formulation, asymptotic stability is far from being trivial and does, in general, not hold, see, e.g. [15]. In the following, we will show how to ensure asymptotic stability by appropriately choosing the MPC prediction horizon N.

III. Stability of MPC Without Stabilizing Constraints or Costs

In this section, known results from [20], [25] are recalled. Later, these results are exploited in order to rigorously prove asymptotic stability of the exact discrete time model of the mobile robot governed by (2). The following assumption, introduced in [19], is a key ingredient in order to show asymptotic stability of the MPC closed-loop.

Assumption 1. *Let a monotonically increasing and bounded sequence $(\gamma_i)_{i\in\mathbb{N}}$ be given and suppose that, for each $z_0 \in Z$, the estimate*

$$V_i(z_0) \leq \gamma_i \cdot \inf_{u \in \mathcal{U}^1(z_0)} \ell(z_0, u) =: \gamma_i \cdot \ell^\star(z_0) \quad \forall\, i \in \mathbb{N}. \tag{6}$$

holds. Furthermore, let there exist two $\mathcal{K}_\infty$-functions $\underline{\eta}$, $\bar{\eta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$\underline{\eta}(\|z - z^\star\|) \leq \ell^\star(z) \leq \bar{\eta}(\|z - z^\star\|) \qquad \forall\, z \in Z. \tag{7}$$

Based on Assumption 1 and the fact that recursive feasibility trivially holds for our example, as observed in the preceding section, asymptotic stability of the MPC closed-loop can be established, cf. [20, Theorems 4.2 and 5.3] and [26].

Theorem 2. *Let Assumption 1 hold and let the performance index α_N be given by the formula*

$$\alpha_N := 1 - \frac{(\gamma_N - 1)\prod_{k=2}^{N}(\gamma_k - 1)}{\prod_{k=2}^{N}\gamma_k - \prod_{k=2}^{N}(\gamma_k - 1)}. \tag{8}$$

Then, if $\alpha_N > 0$, the relaxed Lyapunov inequality

$$V_N(f_{e,T}(z, \mu_N(z))) \leq V_N(z) - \alpha_N \ell(z, \mu_N(z)) \tag{9}$$

holds for all $z \in Z$ and the MPC closed-loop with prediction horizon N is asymptotically stable.

While Condition (7) holds trivially for the chosen running costs, the derivation of the growth bounds γ_i, $i \in \mathbb{N}_0$, of Condition (6), is, in general, difficult. One option to derive γ_i, is the following proposition.

Proposition 3. *Let a sequence $(c_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}_{\geq 0}$, be given and assume that $\sum_{n=0}^{\infty} c_n < \infty$ holds. In addition, suppose that for each $z_0 \in Z$ an admissible sequence of control values $u_{z_0} = (u_{z_0}(n))_{n\in\mathbb{N}_0} \in \mathcal{U}^\infty(z_0)$ exists such that the inequality*

$$\ell(z_{u_{z_0}}(n; z_0), u_{z_0}(n)) \leq c_n \cdot \ell^\star(z_0) \qquad \forall\, n \in \mathbb{N}_0 \tag{10}$$

holds. Then, the growth bounds γ_i, $i \in \mathbb{N}_0$, of Condition (6) are given by $\gamma_i = \sum_{n=0}^{i-1} c_n$, $i \in \mathbb{N}_0$.

Proof: Let $z_0 \in Z$ and $u_{z_0} \in \mathcal{U}^\infty(z_0)$ be given such that Inequality (10) holds. Then, the definition of the value function V_i yields

$$V_i(z_0) \leq \sum_{n=0}^{i-1} \ell(z_{u_{z_0}}(n; z_0), u_{z_0}(n)) \leq \sum_{n=0}^{i-1} c_n \ell^\star(z_0) = \gamma_i \ell^\star(z_0).$$

While monotonicity of the sequence $(\gamma_i)_{i\in\mathbb{N}}$ results from $c_n \geq 0$, $n \in \mathbb{N}_0$, boundedness follows from the assumed summability of the sequence $(c_n)_{n\in\mathbb{N}_0}$. □

In order to illustrate these results, a simple example taken from [29] is presented for which Condition (10) is deduced.

Example 4. *The system dynamics are given by $x^+ = x + u$ with state and control constraints $X = [-1,1]^2$ and $U = [-\bar{u}, \bar{u}]^2$ for some $\bar{u} > 0$, respectively. The desired equilibrium $x^\star$ is supposed to be contained in X. The running costs are $\ell(x,u) = \|x - x^\star\|^2 + \lambda\|u\|^2$ with weighting factor $\lambda \geq 0$.*

Let $c := \max_{x \in X} \|x - x^\star\|$, i.e. the maximal distance of a feasible point from the desired state $x^\star$. We define inductively a control $u_{x_0} \in \mathcal{U}^N(x_0)$ for some design parameter $\rho \in (0,1)$

$$u(k) = \kappa(x^\star - x_{u_{x_0}}(k;x_0)) \quad \text{with } \kappa = \min\{\bar{u}/c, \rho\}.$$

The choice of κ implies $u(k) \in U$ for $x_{u_{x_0}}(k;x_0) \in X$. Since $x_{u_{x_0}}(k+1;x_0) = x_{u_{x_0}}(k;x_0) + \kappa(x^\star - x_{u_{x_0}}(k;x_0))$ holds, we obtain

$$\|x_{u_{x_0}}(k+1;x_0) - x^\star\| = (1-\kappa)\|x_{u_{x_0}}(k;x_0) - x^\star\|$$

and due to convexity of X and $\kappa \in (0,1)$, feasibility of the state trajectory $(x_{u_{x_0}}(k;x_0))_{k\in\mathbb{N}_0}$ is ensured. Then, Condition (10) *can be deduced by*

$$\begin{aligned}\ell(x_{u_{x_0}}(k), u_{x_0}(k)) &= \|x_{u_{x_0}}(k) - x^\star\|^2 + \lambda\|u_{x_0}(k)\|^2 \\ &= (1+\lambda\kappa^2)\|x_{u_{x_0}}(k) - x^\star\|^2 \\ &= (1+\lambda\kappa^2)(1-\kappa)^{2k}\underbrace{\|x_{u_{x_0}}(0) - x^\star\|^2}_{=\ell^\star(x_0)}\end{aligned}$$

with $x_{u_{x_0}}(k) = x_{u_{x_0}}(k;x_0)$, i.e. Condition (10) *with $c_n = C\sigma^n$ where the parameters $C = 1 + \lambda\kappa^2$ and $\sigma = (1-\kappa)^2$ are used. Hence, an exponential decay is shown which implies the summability of the sequence $(c_n)_{n\in\mathbb{N}_0}$.*

Based on the sequence $(c_n)_{n\in\mathbb{N}_0}$ computed in Example 4, Formula (8) yields $\alpha_2 = 1 - (C + \sigma C - 1)^2$. Hence, $\alpha_2 > 0$ is equivalent to showing $C(1+\sigma) = (1+\lambda\kappa^2)(1+(1-\kappa)^2) < 2$. Supposing $\lambda \in (0,1)$, the left hand side of this inequality is strictly smaller than $(1+\kappa^2)(1+(1-\kappa)^2)$ and, thus the inequality

$$\kappa(1-\kappa) + \kappa(1-\kappa)^2 \geq -\kappa^3(1-\kappa)$$

implies $\alpha_2 > 0$. Hence, Theorem 2 can be used to conclude asymptotic stability for prediction horizon $N = 2$. For, e.g. $\lambda = 0.1$ and $\rho = 0.5$, the performance index α_2 is approximately 0.9209.

Remark 5. *A direct verification of Assumption 1 yields, in general, less conservative bounds on the required prediction horizon in order to ensure that $\alpha_N \in (0,1]$ is satisfied. However, Proposition 3 is instructive for the construction in the subsequent section.*

IV. Stability Analysis of the Unicycle Mobile Robot

In this section, a bounded sequence $(\gamma_i)_{i\in\mathbb{N}_{\geq 2}}$ is constructed such that Assumption 1 holds. For this purpose, first an open set $\mathcal{N}_1 = \mathcal{N}_1(s)$ of initial conditions depending on a parameter $s \in [0,\infty)$ is defined by

$$\left\{ z = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in \mathbb{R}^3 : z \in Z \text{ and } \ell^\star\left(\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\right) < s \right\}. \tag{11}$$

Based on this definition, the feasible set Z is split up into $\mathcal{N}_1$ and $\mathcal{N}_2 := Z \backslash \mathcal{N}_1$ such that $Z = \mathcal{N}_1 \cup \mathcal{N}_2$ holds. Then, bounded sequences $(\gamma_i^{\mathcal{N}_j})_{i\in\mathbb{N}_{\geq 2}}$, $j \in \{1,2\}$, are derived such that

$$V_i(z_0) \leq \gamma_i^{\mathcal{N}_j} \cdot \ell^\star(z_0) \qquad \forall\ z_0 \in \mathcal{N}_j \tag{12}$$

holds for all $i \in \mathbb{N}$. In conclusion, taking into account that the input sequence $(u(k))_{k\in\mathbb{N}_0}$, $u(k) = 0_{\mathbb{R}^2}$, is admissible on the infinite horizon and implies Inequality (6) with $\gamma_i = i$, Inequality (6) holds for all $z_0 \in Z$ with

$$\gamma_i := \min\{i, \max\{\gamma_i^{\mathcal{N}_1}, \gamma_i^{\mathcal{N}_2}\}\}, \qquad i \in \mathbb{N}_{\geq 2}. \tag{13}$$

The motivation behind partitioning the set Z is that we design two different manoeuvres in order to deduce bounded sequences $(\gamma_i^{\mathcal{N}_j})_{i\in\mathbb{N}_{\geq 2}}$, $j \in \{1,2\}$. While in principle one strategy could be sufficient, one of the proposed manoeuvres works for initial states close to the origin (inside the set $\mathcal{N}_1$) while the other becomes more advantageous outside $\mathcal{N}_1$. In this vein, the vehicle is just turned towards the origin $0 \in \mathbb{R}^2$ and, then, drives in that direction before the angle is set to zero if $z_0 \in \mathcal{N}_2$. However, that move does not allow to derive a bounded γ_i-sequence for initial positions $z_0 = (0, y_0, 0)$ whose distance $\ell^\star(z_0)$ tends to zero. But, boundedness is essential in order to deduce asymptotic stability of the MPC closed-loop via Theorem 2.

Before we present the (technical) details in following Subsections IV-A and IV-B, let us briefly explain the strategy used to construct $(\gamma_i^{\mathcal{N}_j})_{i\in\mathbb{N}_{\geq 2}}$, $j \in \{1,2\}$. First, for initial values $z_0 = (x_0, y_0, 0)^T \in \mathcal{N}_j$, a family of particular control sequences $u_{z_0} := (u(k;z_0))_{k\in\mathbb{N}_0} \in \mathcal{U}^\infty(z_0)$ is proposed such that the robot is steered to the origin in a finite number of steps. These input sequences u_{z_0} yield (suboptimal) running costs $\ell(z_{u_{z_0}}(k;z_0), u(k;z_0))$ such that, by definition of optimality, the following quotients can be estimated uniformly with respect to $z_0 = (x_0, y_0, 0)^T \in \mathcal{N}_j$ by

$$\ell(z_{u_{z_0}}(k;z_0), u(k;z_0)) \cdot \ell^\star(z_0)^{-1} \leq c_k \quad \forall\ k \in \mathbb{N}_0 \tag{14}$$

with coefficients $c_k = c_k^{\mathcal{N}_j}$, $k \in \mathbb{N}_0$, i.e. a coefficient sequence $(c_k)_{k\in\mathbb{N}_0}$ such that Inequality (10) holds. Since also the number of steps needed in order to steer the considered initial states z_0 to the origin exhibits a uniform upper bound, there exists $\bar{k}$ such that $c_k = 0$ holds for all $k \geq \bar{k}$. Then, the coefficients $c_1, c_2, \ldots, c_{\bar{k}-1}$ are rearranged in a descending order denoted by $(\bar{c}_k)_{k\in\mathbb{N}_0}$ with $\bar{c}_0 = c_0$, which still implies Condition (6) with $\gamma_i := \sum_{n=0}^{i-1} \bar{c}_n$. Finally, these γ_i-sequences are used in order to ensure Condition (6) for *all* initial states contained in $\mathcal{N}_j$, i.e. also those with $\theta_0 \neq 0$. Due to symmetries (the robot can go back and forth), it is sufficient to consider initial positions with $(x_0, y_0)^T \geq 0_{\mathbb{R}^2}$.

A. Trajectory Generation for $z_0 \in \mathcal{N}_2$

In this section we first consider initial conditions inside $\mathcal{N}_2$ with $\theta_0 = 0$. Subsequently, we will prove that the derived bounds also hold for the case $\theta_0 \neq 0$.

Initial Conditions $z_0 \in \mathcal{N}_2$ with $\theta_0 = 0$: For initial conditions $z_0 = (x_0, y_0, 0)^T$ in the set $\mathcal{N}_2$, the following simple manoeuvre can be employed:

a) choose an angle $\bar{\theta} \in [-\pi, \pi)$ such that the vehicle points towards (or in the opposite direction to) the origin $(0,0)^T \in \mathbb{R}^2$,
b) drive directly towards the origin,
c) turn the vehicle to the desired angle $\theta^\star = 0$.

The number of steps needed in order to carry out this manoeuvre depends on the constraints and the sampling time T which is supposed to satisfy $i \cdot T = 1$ for some integer $i \in \mathbb{N}$. We define the minimal number of steps required to turn the vehicle by 90 degrees as

$$k_T^\star := \left\lceil \frac{\pi/2}{\min\{-w_{\min}, w_{\max}, \pi/2\} \cdot T} \right\rceil$$

assuming reasonable bounds control constraints. We define also the minimal number of steps required to drive the vehicle from the farthest corner of the box defined by the constraints (3) to the origin as

$$l_T^\star := \left\lceil \frac{\sqrt{\max\{-y_{\min}, y_{\max}\}^2 + \max\{-x_{\min}, x_{\max}\}^2}}{\min\{-v_{\min}, v_{\max}\} \cdot T} \right\rceil,$$

respectively. Additionally, the inequality

$$r_2 \leq \frac{q_3 \cdot T}{2} \tag{15}$$

is assumed to hold in order to avoid technical difficulties resulting from not reflecting the sampling time T in the running costs.

Initial values $z_0 = (x_0, y_0, 0)^T \geq 0$ are considered first. Let the angle $\arctan(y_0/x_0) \in [0, \pi/2)$ be denoted by ϕ. The vehicle stays at the initial position without moving for $k_T^\star$ steps, i.e $(v_i, w_i)^T = (0,0)^T$, $i \in \{0, 1, \ldots, k_T^\star - 1\}$, which yields Inequality (14) with $c_i^{\mathcal{N}_2} = 1$, $i = 0, 1, \ldots, k_T^\star - 1$. This artificially added phase is introduced here in order to facilitate the treatment of initial positions with $\theta_0 \neq 0$.

Next, the vehicle turns $k_T^\star$ steps such that $\theta_u(2k_T^\star; z_0) = \phi$ holds by applying the input $u(k_T^\star + i) = (0, \phi \cdot (k_T^\star T)^{-1})^T \in U$ for all $i \in \{0, 1, \ldots, k_T^\star - 1\}$. This control action yields the running costs $\ell(z_u(k_T^\star + i; z_0), u(k_T^\star + i))$ given by

$$q_1 x_0^4 + q_2 y_0^2 + q_3 \left(\frac{i\phi}{k_T^\star}\right)^4 + r_2 \left(\frac{\phi}{k_T^\star T}\right)^4. \tag{16}$$

Since $\phi \in [0, \pi/2)$, $\ell^\star(z_0) \geq s$, and Assumption (15) hold, Inequality (14) is ensured with the coefficients

$$c_{k_T^\star + i}^{\mathcal{N}_2} := 1 + \frac{q_3 \pi^4}{16 {k_T^\star}^4 \cdot s} \left(i^4 + \frac{1}{2T^3}\right), \tag{17}$$

$i = 0, 1, \ldots, k_T^\star - 1$. Then, the vehicle drives towards the origin in $l_T^\star$ steps with constant backward speed $u(2k_T^\star + i) = (-\|(x_0, y_0)^T\| \cdot (l_T^\star T)^{-1}, 0)^T \in U, i \in \{0, 1, \ldots, l_T^\star - 1\}$. This leads to running costs $\ell(z_u(2k_T^\star + i; z_0), u(2k_T^\star + i))$ given by

$$\left(\frac{l_T^\star - i}{l_T^\star}\right)^2 \left[q_1 \left(\frac{l_T^\star - i}{l_T^\star}\right)^2 x_0^4 + q_2 y_0^2\right] + q_3 \phi^4 + r_1 \left(\frac{\|(x_0, y_0)\|}{l_T^\star T}\right)^4$$

Since $\phi \leq \pi/2$ and the control effort is smaller than $\min\{-v_{\min}, v_{\max}\}$, the respective coefficients for Inequality (14) can be chosen as

$$c_{2k_T^\star + i}^{\mathcal{N}_2} := \left(\frac{l_T^\star - i}{l_T^\star}\right)^2 + \frac{q_3 (\pi/2)^4 + r_1 \min\{-v_{\min}, v_{\max}\}^4}{s} \tag{18}$$

for $i \in \{0, 1, \ldots, l_T^\star - 1\}$. Finally, the vehicle turns $k_T^\star$ steps in order to reach $\theta_u(3k_T^\star + l_T^\star; z_0) = 0$ using the input $u(2k_T^\star + l_T^\star + i) = (0, -\phi \cdot (k_T^\star T)^{-1})^T$, $i \in \{0, 1, \ldots, k_T^\star - 1\}$. Thus, the running costs $\ell(z_u(2k_T^\star + l_T^\star + i; z_0), u(2k_T^\star + l_T^\star + i))$ are

$$\left[q_3 \left(\frac{k_T^\star - i}{k_T^\star}\right)^4 + r_2 \left(\frac{1}{k_T^\star T}\right)^4\right] \phi^4. \tag{19}$$

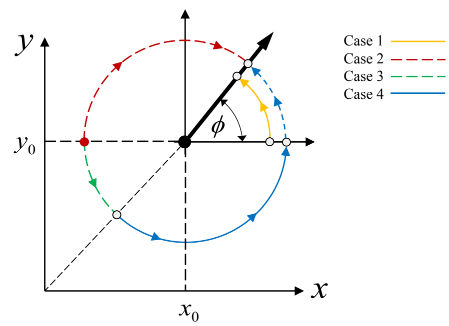


Fig. 1. Classification of the four different cases for $z_0 \in \mathcal{N}_2$ with $\theta_0 \neq 0$.

Then, invoking (15) ensures Inequality (14) with

$$c_{2k_T^\star + l_T^\star + i}^{\mathcal{N}_2} := \frac{q_3 \pi^4}{16 {k_T^\star}^4 \cdot s} \left[(k_T^\star - i)^4 + \frac{1}{2T^3}\right] \tag{20}$$

for $i \in \{0, 1, \ldots, k_T^\star - 1\}$. The calculated coefficients $c_i^{\mathcal{N}_2}$, $i = 1, 2, \ldots, 3k_T^\star + l_T^\star - 1$, are ordered descendingly resulting in a new sequence $(\bar{c}_i^{\mathcal{N}_2})_{i=1}^{3k_T^\star + l_T^\star - 1}$, satisfying $\bar{c}_i^{\mathcal{N}_2} \leq \bar{c}_{i-1}^{\mathcal{N}_2}$ for $i \in \{2, 3, \ldots, 3k_T^\star + l_T^\star - 1\}$. Then, setting $\bar{c}_0^{\mathcal{N}_2} = c_0^{\mathcal{N}_2}$ and $\bar{c}_i^{\mathcal{N}_2} = 0$ for all $i \geq 3k_T^\star + l_T^\star$ yields $(\bar{c}_i^{\mathcal{N}_2})_{i=0}^{\infty}$. Hence, the accumulated bounds $(\gamma_i^{\mathcal{N}_2})_{i \in \mathbb{N}_{\geq 2}}$ of Condition (6) for the first manoeuvre are given by

$$\gamma_i^{\mathcal{N}_2} := \sum_{n=0}^{i-1} \bar{c}_i^{\mathcal{N}_2}, \qquad i \in \mathbb{N}_{\geq 2}. \tag{21}$$

Initial Conditions $z_0 \in \mathcal{N}_2$ with $\theta_0 \neq 0$: In this subsection, we show that Condition (6) holds for arbitrary initial conditions $z_0 \in \mathcal{N}_2$, i.e. $\theta_0 \in [-\pi, 0) \cup (0, \pi)$, using the bounds defined in (21). To this end, we distinguish four intervals in dependence of the initial angular deviation θ_0, see Figure 1. While the basic ingredients are similar to the described manoeuvre for $\theta_0 = 0$, the order of the involved *motions* differs as summarized in Figure 2 in order to facilitate the accountability of the upcoming presentation.

Case 1: let θ_0 be contained in the interval $(0, \phi)$. The robot stays at the initial position without moving for $k_T^\star$ steps; thus, Inequality (14) holds with the coefficients $c_i = 1$, $i = 0, 1, \ldots, k_T^\star - 1$. Then, the control input $u(k_T^\star) = (0, w_{k_T^\star})^T$, $w_{k_T^\star} \in (0, \phi \cdot (k_T^\star T)^{-1}]$ is adjusted such that

$$\exists!\, i^\star \in \{1, \ldots, k_T^\star - 1\} : \theta_0 + T w_{k_T^\star} = \theta_{k_T^\star + i^\star}$$

where $\theta_{k_T^\star + i^\star}$ is one of the achieved angles during the manoeuvre for $\theta_0 = 0$. Then, the robot turns $k_T^\star - i^\star$ steps such that $\theta_u(2k_T^\star - i^\star + 1) = \phi$ is achieved using the input $u(k_T^\star + i) = (0, \phi \cdot (k_T^\star T)^{-1})^T$, $i \in \{1, 2, \ldots, k_T^\star - i^\star\}$. Hence, Inequality (14) is valid with the coefficient $c_{k_T^\star + i + (i^\star - 1)}^{\mathcal{N}_2}$, $i \in \{0, 1, \ldots, k_T^\star - i^\star\}$. The remaining parts of the manoeuvre are performed as for $\theta_0 = 0$. Since we have $\ell^\star(z_0) > s$ but

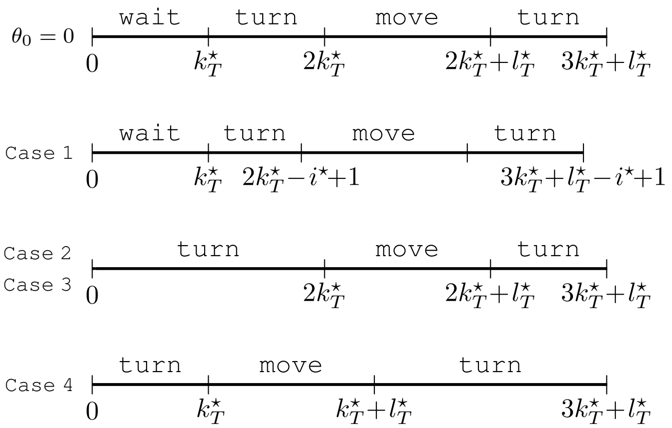


Fig. 2. The manoeuvre for initial conditions $z_0 \in \mathcal{N}_2$ consists of waiting, turning, and moving the differential drive robot. However, the order of these *motions* depends on the initial angular deviation, see Figure 1.

precisely the same running costs, the growth bounds given by (21) can be used to ensure condition (6) for the considered case.

Case 2: let $\theta_0 \in (\phi, \pi]$ hold. The first part of the manoeuvre is performed by turning the robot $2k_T^\star$ steps, such that $\theta_u(2k_T^\star; z_0) = \phi$ is achieved using the input $u(i) = (0, -\Delta\theta \cdot (k_T^\star T)^{-1})^T$, $i = 0, 1, \ldots, 2k_T^\star - 1$, $\Delta\theta = (\theta_0 - \phi)/2$. Hence the running costs $\ell(z_u(i; z_0), u(i))$ are given by

$$q_1 x_0^4 + q_2 y_0^2 + q_3 \left[\theta_0 - i\left(\frac{\Delta\theta}{k_T^\star}\right)\right]^4 + r_2 \left(\frac{\Delta\theta}{k_T^\star T}\right)^4. \quad (22)$$

for $i \in \{0, 1, \ldots, 2k_T^\star - 1\}$. Then, using $\Delta\theta > 0$ and Taylor series expansion theory yields

$$\left[\theta_0 - i\left(\frac{\Delta\theta}{k_T^\star}\right)\right]^4 \leq \theta_0^4 - i\left(\frac{\Delta\theta}{k_T^\star}\right)\theta_0^3$$

and, thus, invoking Assumption (15), i.e. $r_2 \leq \frac{q_3 \cdot T}{2}$, leads to

$$\ell(z_u(i; z_0), u(i)) \leq \ell^\star(z_0) - q_3\left(\frac{\Delta\theta}{k_T^\star}\right)\left[i\theta_0^3 - \frac{1}{2}\left(\frac{\Delta\theta}{k_T^\star T}\right)^3\right] \quad (23)$$

for $i = 0, 1, \ldots, 2k_T^\star - 1$. In conclusion, the right hand side of this inequality is always less than or equal to $\ell^\star(z_0)$ for $i > 0$. Hence, Inequality (14) holds with

$$c_{k_T^\star}^{\mathcal{N}_2}, c_0^{\mathcal{N}_2}, c_1^{\mathcal{N}_2}, \ldots, c_{k_T^\star - 1}^{\mathcal{N}_2}, c_{k_T^\star + 1}^{\mathcal{N}_2}, \ldots, c_{2k_T^\star - 1}^{\mathcal{N}_2}$$

for $i = 0, 1, 2, \ldots, 2k_T^\star - 1$. In particular, the construction of the sequence $(\bar{c}_i^{\mathcal{N}_2})_{i=0}^\infty$ yields $c_{k_T^\star}^{\mathcal{N}_2} + c_0^{\mathcal{N}_2} \leq \bar{c}_0^{\mathcal{N}_2} + \bar{c}_1^{\mathcal{N}_2} = \gamma_2^{\mathcal{N}_2}$. Finally, the remaining parts of the manoeuvre can be dealt with analogously to case 1 showing that Condition (6) is ensured with the accumulated bounds defined by (21).

Case 3: let $\theta_0 \in (-\pi, -\pi + \phi)$ hold. First, the robot is turned $k_T^\star$ steps such that $\theta_{k_T^\star} = -\pi + \phi$ holds using the input $u(i) = (0, \Delta\theta \cdot (k_T^\star T)^{-1})^T$, $i = 0, 1, \ldots, k_T^\star - 1$, with $\Delta\theta = |\theta_0| - \pi + \phi$. The respective running costs are given by (22), which also satisfy Inequality (23) with θ_0 replaced by $|\theta_0|$. Like in Case 2, the inequality $\ell(z_u(i; z_0), u(i)) \leq \ell^\star(z_0)$ holds for $i = 1, 2, \ldots, k_T^\star - 1$. During the second part of the manoeuvre the robot is driven to the origin in $l_T^\star$ steps; thus, Inequality (14) holds with the coefficients defined by (18) — indeed, $q_3(\pi/2)^4$ could have been dropped. Next, the robot is turned $k_T^\star$ steps until $\theta_u(2k_T^\star + l_T^\star; z_0) = -\pi/2$ holds using $u(k_T^\star + l_T^\star + i) = (0, \Delta\theta \cdot (k_T^\star T)^{-1})^T$ with $\Delta\theta = \pi/2 - \phi$ for $i \in \{0, 1, \ldots, k_T^\star - 1\}$. Hence, for $i = 0, 1, \ldots, k_T^\star - 1$, the running costs $\ell(z_u(k_T^\star + l_T^\star + i; z_0), u(k_T^\star + l_T^\star + i))$ are

$$q_3\left(\phi - \pi + \frac{i\Delta\theta}{k_T^\star}\right)^4 + r_2\left(\frac{\Delta\theta}{k_T^\star T}\right)^4.$$

A Taylor expansion of the first term yields

$$\left((\phi - \pi) + \frac{i\Delta\theta}{k_T^\star}\right)^4 \leq (\phi - \pi)^4 - \frac{i\Delta\theta}{k_T^\star}(\pi - \phi)^3.$$

Therefore, using Assumption (15), $\pi - \phi \geq \Delta\theta \cdot (k_T^\star T)^{-1}$, and $|\theta_0| \geq |\phi - \pi|$, one obtains the inequality

$$\ell(z_u(k_T^\star + l_T^\star + i; z_0), u(k_T^\star + l_T^\star + i)) \leq q_3\theta_0^4 \leq \ell^\star(z_0)$$

for $i = 1, 2, \ldots, k_T^\star - 1$. Then, the robot turns another $k_T^\star$ steps such that $\theta_u(3k_T^\star + l_T^\star; z_0) = 0$ holds using the input $u(2k_T^\star + l_T^\star + i) = (0, \pi \cdot (2k_T^\star T)^{-1})^T$, $i \in \{0, 1, \ldots, k_T^\star - 1\}$. The resulting running costs for this part of the manoeuvre are given by (19) with $\phi = \pi/2$ and, thus, also satisfy Inequality (14) with coefficients $c_{2k_T^\star + l_T^\star + i}$, $i = 0, 1, \ldots, k_T^\star - 1$ defined by (20), respectively. We show that Case 3 is less costly than the reference case $\theta_0 = 0$ by the following calculations, in which Assumption (15), i.e. $r_2 \leq \frac{q_3 \cdot T}{2}$, is used:

$$\begin{aligned}
&\ell(z_u(0; z_0), u(0)) + \ell(z_u(k_T^\star + l_T^\star; z_0), u(k_T^\star + l_T^\star)) \\
&= \ell^\star(z_0) + q_3(\pi - \phi)^4 + \frac{r_2 \overbrace{\left[(|\theta_0| - \pi + \phi)^4 + (\frac{\pi}{2} - \phi)^4\right]}^{\leq (|\theta_0| - \frac{\pi}{2})^4 \leq (\frac{\pi}{2})^4}}{(k_T^\star T)^4} \\
&\leq \ell^\star(z_0) + q_3\theta_0^4 + \frac{q_3\pi^4 T}{32(k_T^\star T)^4} \\
&\leq \left(2 + \frac{q_3\pi^4 T}{32(k_T^\star T)^4 \cdot s}\right)\ell^\star(z_0) = \left(c_0^{\mathcal{N}_2} + c_{k_T^\star}^{\mathcal{N}_2}\right) \cdot \ell^\star(z_0)
\end{aligned}$$

In conclusion, the accumulated bounds given by (21) can be used to ensure Condition (6) for the case considered here.

Case 4: let $\theta_0 \in (-\pi + \phi, 0)$ hold. First, for $i = 0, 1, \ldots, k_T^\star - 1$, the robot uses the control inputs $u(i) = (0, \Delta\theta \cdot (k_T^\star T)^{-1})^T$ with $\Delta\theta$ defined as $\max\{0, \phi - \pi/2 - \theta_0\}$ in order to achieve that $\phi - \theta_u(k_T^\star; z_0) \leq \pi/2$ holds. Then, the robot employs $u(k_T^\star + i) = (0, (\phi - \theta_u(k_T^\star; z_0)) \cdot (k_T^\star T)^{-1})^T$ for all $i \in \{0, 1, \ldots, k_T^\star - 1\}$, which yields $\theta_u(2k_T^\star; z_0) = \phi$.

Proceeding analogously to Case 2 leads to Estimate (23) for all $i\{0, 1, \ldots, k_T^\star - 1\}$ while the running costs $\ell(z_u(k_T^\star + i; z_0), u(k_T^\star + i))$ for the next $k_T^\star$ steps are given by

$$q_1 x_0^4 + q_2 y_0^2 + q_3\left(\theta_{k_T^\star} + \frac{i(\phi - \theta_{k_T^\star})}{k_T^\star}\right)^4 + r_2\left(\frac{\phi - \theta_{k_T^\star}}{k_T^\star T}\right)^4$$

with $\theta_{k_T^\star} = \theta_u(k_T^\star; z_0)$ for all $i \in \{0, 1, \ldots, k_T^\star - 1\}$. Invoking Assumption (15), i.e. $r_2 \leq \frac{q_3 \cdot T}{2}$, yields the bound

$$2\ell^\star(z_0) + q_3\left[\underbrace{\theta_{k_T^\star}^4 - \theta_0^4}_{\leq -\Delta\theta \cdot |\theta_0|^3} + \frac{\Delta\theta}{2k_T^\star}\underbrace{\left(\frac{\Delta\theta}{k_T^\star T}\right)^3}_{\leq |\theta_0|^3} + \frac{\phi - \theta_{k_T^\star}}{2k_T^\star}\left(\frac{\phi - \theta_{k_T^\star}}{k_T^\star T}\right)^3\right]$$

for the running costs $\ell(z_u(0;z_0),u(0))+\ell(z_u(k_T^\star;z_0),u(k_T^\star))$ and, thus, allows to derive the estimate

$$\begin{aligned}&\ell(z_u(0;z_0),u(0))+\ell(z_u(k_T^\star;z_0),u(k_T^\star)) \\ \leq\ &\left(2+\frac{q_3\pi^4}{32{k_T^\star}^4T^3\cdot s}\right)\ell^\star(z_0)=\left(c_0^{\mathcal{N}_2}+c_{k_T^\star}^{\mathcal{N}_2}\right)\cdot\ell^\star(z_0).\end{aligned} \tag{24}$$

The running costs $\ell(z_u(i;z_0),u(i))$ can be estimated by $c_i^{\mathcal{N}_2}\ell^\star(z_0)$ for all $i\in\{1,2,\ldots,k_T^\star-1\}\cup\{k_T^\star+1,\ldots,2k_T^\star-1\}$, see Case 2 and the derivation of the coefficients (17) for details while taking $\theta_u(k_T^\star;z_0)+i(\phi-\theta_{k_T^\star})/k_T^\star\leq i\phi/k_T^\star$ into account. Since the remaining parts of the manoeuvre are performed precisely as in Case 1, combining this with Inequality (24) shows that the accumulated bounds given by (21) can be used to ensure Condition (6) for the considered case.

B. Trajectory Generation for $z_0\in\mathcal{N}_1$

We consider initial conditions inside $\mathcal{N}_1$ with $\theta_0=0$ and construct a suitable coefficient sequence $(c_n^{\mathcal{N}_1})_{n\in\mathbb{N}_0}$ satisfying Inequality (14). Here, the particular choice of the stage costs ℓ is heavily exploited in order to successfully steer the robot from a position $(0,y_0,0)^T$ with $y_0\neq 0$ to the origin while simultaneously deriving finitely many bounds $c_n^{\mathcal{N}_1}$, $n\in\mathbb{N}_0$. These bounds are on the one hand uniform in y_0, i.e. Inequality (14) holds independently of y_0 and, thus, also for $y_0\to 0$. On the other hand, the number of coefficients $c_n^{\mathcal{N}_1}$, which are strictly greater than zero, is uniformly bounded. Combining these two properties ensures that the sequence remains summable – an important ingredient to make Proposition 3 applicable in order to ensure Assumption 1. Subsequently, we reorder this sequence in order to get $(\bar{c}_n^{\mathcal{N}_1})_{n\in\mathbb{N}_0}$ and prove that the resulting bounds $\gamma_i^{\mathcal{N}_1}:=\sum_{n=0}^{i-1}\bar{c}_n^{\mathcal{N}_1}$ also yield Inequality (12) for the case $\theta_0\neq 0$.

Initial Conditions $z_0\in\mathcal{N}_1$ with $\theta_0=0$: The following manoeuvre is used in order to derive bounds $\gamma_i^{\mathcal{N}_1}$, $i\in\mathbb{N}_{\geq 2}$, satisfying Inequality (6) for initial condition whose angular deviation is equal to zero:

a) drive towards the y-axis until $(0,y_0,0)^T$ is reached.
b) drive forward while slightly steering in order to reduce the y-component to $y_0/2$; a position $(\bar{x},y_0/2,0)^T$ for some $\bar{x}>0$ is reached.
c) carry out a symmetric manoeuvre while driving backward so that the origin $0_{\mathbb{R}^3}$ is reached.

The number of steps needed in order to perform this manoeuvre depends on the constraints and the sampling time T, which is supposed to satisfy $i\cdot T=1$ for some integer $i\in\mathbb{N}$ as in Subsection IV-A. To this end, we define

$$\begin{aligned}k_T^\star&:=\left\lceil\frac{\pi}{\min\{-w_{\min},w_{\max},\pi\}\cdot T}\right\rceil,\\ l_T^\star&:=\left\lceil\frac{\sqrt[4]{s/q_1}}{\min\{-v_{\min},v_{\max},\sqrt[4]{s/q_1}\}\cdot T}\right\rceil\end{aligned}$$

where the vehicle can turn by 180 degrees in $k_T^\star$ steps and drive to the y-axis in $l_T^\star$ steps, respectively. In addition to Inequality (15), the Condition

$$r_1\leq\frac{q_1\cdot T}{2} \tag{25}$$

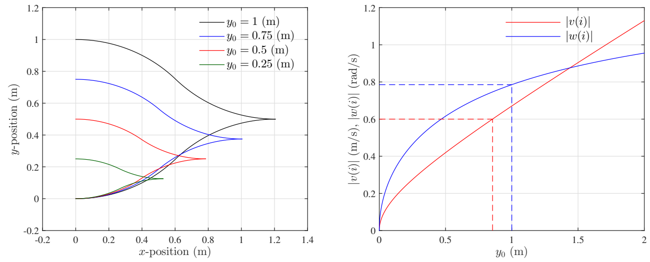


Fig. 3. Trajectories of parts b) and c) of the manoeuvre starting from different initial conditions on the y-axis (left). The respective controls are displayed on the right.

is assumed in order to keep the presentation technically simple.

Initial conditions $z_0=(x_0,y_0,0)^T\geq 0$ are considered first. Firstly, the vehicle does not move for $k_T^\star$ steps. Hence, Inequality (14) holds with $c_i^{\mathcal{N}_1}=1$ for $i\in\{0,1,\ldots,k_T^\star-1\}$. Then, the robot drives towards the y-axis in $l_T^\star$ steps using $u(k_T^\star+i)=(-x_0\cdot(l_T^\star T)^{-1},0)^T\in U$, $i=0,1,\ldots,l_T^\star-1$, which allows to estimate $\ell(z_u(k_T^\star+i;z_0),u(k_T^\star+i))$ first by $q_1x_0^4(1-i/l_T^\star)^4+q_2y_0^2+q_1x_0^4/(2l_T^\star(l_T^\star T)^3)$ using (25) and, then, by

$$\ell^\star(z_0)-q_1\left(\frac{x_0^4}{l_T^\star}\right)\left[i-\frac{1}{2(l_T^\star T)^3}\right].$$

Hence, Inequality (14) holds with $c_0^{\mathcal{N}_1}=1+(2l_T^\star(l_T^\star T)^3)^{-1}$ and $c_{k_T^\star+i}^{\mathcal{N}_1}=1$ for all $i\in\{1,2,\ldots,l_T^\star-1\}$.

The next part of the manoeuvre is performed in four seconds with constant control effort $\|u(\cdot)\|$ such that the angle is decreased to $-\arctan(\sqrt{y_0})$ during the first second and then put back to zero while the y-position of the robot decreases to $y_0/2$. Afterwards, these two moves are carried out backwards in order to reach the origin, cf. Figure 3 on the left. To this end, the controls

$$\begin{aligned}w(i)&=-w(T^{-1}+i)=-w(2T^{-1}+i)=w(3T^{-1}+i),\\ v(i)&=v(T^{-1}+i)=-v(2T^{-1}+i)=-v(3T^{-1}+i),\end{aligned}$$

$i\in\{k_T^\star+l_T^\star,k_T^\star+l_T^\star+1,\ldots,k_T^\star+l_T^\star+T^{-1}-1\}$, with

$$w(i)=-\arctan(\sqrt{y_0})\quad\text{and}\quad v(i)=-\frac{y_0\arctan(\sqrt{y_0})}{\frac{4}{\sqrt{y_0+1}}-4} \tag{26}$$

are employed. Note that this strategy ensures not to move when starting at the origin. The resulting y-positions are given by $y(k_T^\star+l_T^\star+nT^{-1})=(1-n/4)y_0$ for $n\in\{0,1,2,3\}$ while the x-positions are, for $i=k_T^\star+l_T^\star$, given by $x(i)=0$, $x(i+T^{-1})=x(i+3T^{-1})=\sin(w(i))\cdot v(i)/w(i)$, and $x(i+2T^{-1})=2x(i+T^{-1})$. The manoeuvre has to be suitably adapted if either control constraints enforce $v(\cdot)$ or $w(\cdot)$ to be smaller or $x(k_T^\star+l_T^\star+2T^{-1})$ violates the state constraints. However, since this manoeuvre is constructed for small y_0, constraints can be neglected.

Next, we evaluate the running costs and determine coefficients $c_{k_T^\star+l_T^\star+i}^{\mathcal{N}_1}$ such that Inequality (14) holds for all $i\in\{0,1,\ldots,4T^{-1}-1\}$. To this end, the estimates

$$\arctan^2\sqrt{y_0}\leq y_0\quad\text{and}\quad v(i)^4\leq(2+3y_0+y_0^2)^2y_0^2/64$$

for $v(i)$ from (26) are employed where, for the derivation of the latter, the two auxiliary inequalities $(2+y_0)^2 \cdot (y_0+1)^2 \leq (2+3y_0+y_0^2)^2$ and $y_0^2 \leq 2(2+y_0) \cdot (\sqrt{y_0+1}-1)^2$ were exploited. Hence, using $(2+3y_0+y_0^2) \leq (y_0+1.5)^2$ and $q_2 y_0^2 \leq \ell^\star(z_0)$, we obtain that the running costs $\ell(z_u(k_T^\star + l_T^\star); z_0), u(k_T^\star + l_T^\star))$ are bounded by

$$\left(1 + \frac{(\sqrt{s/q_2}+1.5)^4 r_1}{64 q_2} + r_2/q_2\right) \ell^\star(z_0) =: c^{\mathcal{N}_1}_{k_T^\star + l_T^\star} \ell^\star(z_0)$$

and, thus, Inequality (14) holds. Then, since $\sin^2(w(i)) \leq w(i)^2$ holds for $w(i)$ from (26), $\ell^\star(z(k_T^\star + l_T^\star + T^{-1})) \leq q_1 v(i)^4 + (9/16) q_2 y_0^2 + q_3 y_0^2$ this yields Inequality (14) with $c^{\mathcal{N}_1}_{k_T^\star + l_T^\star + T^{-1}}$ given by

$$9/16 + \left(q_3 + r_2 + (q_1 + r_1)(\sqrt{s/q_2}+1.5)^4/64\right) q_2^{-1}.$$

Analogously, the coefficients $c^{\mathcal{N}_1}_{k_T^\star + l_T^\star + 2T^{-1}}$ and $c^{\mathcal{N}_1}_{k_T^\star + l_T^\star + 3T^{-1}}$ defined by $1/4 + \left(r_2 + (16q_1 + r_1)(\sqrt{s/q_2}+1.5)^4/64\right) q_2^{-1}$ and $1/16 + \left(q_3 + r_2 + (q_1 + r_1)(\sqrt{s/q_2}+1.5)^4/64\right) q_2^{-1}$ are derived. For sampling time $T < 1$, further coefficients have to be determined. To this end, the running costs $\ell(\cdot,\cdot)$ at time $k_T^\star + l_T^\star + nT^{-1} + i$, $(n,i) \in \{0,1,2,3\} \times \{1,2,\ldots,T^{-1}-1\}$, are overestimated by plugging in the state

$$\begin{pmatrix} x_u(k_T^\star + l_T^\star + (2.125 - 0.5(n-1.5)^2) \cdot T^{-1}; z_0) \\ (1-0.25n) y_0 \\ \theta_u(k_T^\star + l_T^\star + T^{-1}; z_0) \end{pmatrix}$$

instead of $z_u(k_T^\star + l_T^\star + nT^{-1} + i)$ while leaving the control as it is. This yields, for $i \in \{1,2,\ldots,T^{-1}-1\}$, Inequality (14) with the coefficients $c^{\mathcal{N}_1}_{k_T^\star + l_T^\star + nT^{-1} + i}$ defined by

$$c^{\mathcal{N}_1}_{k_T^\star + l_T^\star + nT^{-1}} + \begin{cases} \left(q_1(\sqrt{s/q_2}+1.5)^4/64 + q_3\right) q_2^{-1}, & n = 0 \\ \left(15 q_1 (\sqrt{s/q_2}+1.5)^4/64\right) q_2^{-1}, & n = 1 \\ q_3 q_2^{-1}, & n = 2 \\ 0, & n = 3 \end{cases}. \tag{27}$$

The coefficients $c_i^{\mathcal{N}_1}$, $i = 1,2,\ldots,k_T^\star + l_T^\star + 4T^{-1} - 1$, are ordered descendingly in order to construct a new sequence $(\bar{c}_i^{\mathcal{N}_1})_{i=1}^{k_T^\star + l_T^\star + 4T^{-1} - 1}$ such that the property $\bar{c}_{i-1}^{\mathcal{N}_1} \geq \bar{c}_i^{\mathcal{N}_1}$ holds for all $i \in \{2,3,\ldots,k_T^\star + l_T^\star + 4T^{-1} - 1\}$. Then, setting $\bar{c}_0^{\mathcal{N}_1} = c_0^{\mathcal{N}_1}$ and $\bar{c}_i^{\mathcal{N}_1} = 0$ for all $i \geq k_T^\star + l_T^\star + 4T^{-1}$ yields $(\bar{c}_i^{\mathcal{N}_1})_{i=0}^\infty$. In conclusion, the accumulated bounds $(\gamma_i^{\mathcal{N}_1})_{i \in \mathbb{N}_{\geq 2}}$ of Condition (6) for the second manoeuvre are given by

$$\gamma_i^{\mathcal{N}_1} := \sum_{n=0}^{i-1} \bar{c}_i^{\mathcal{N}_1}, \qquad i \in \mathbb{N}_{\geq 2}. \tag{28}$$

Initial Conditions $z_0 \in \mathcal{N}_1$ **with** $\theta_0 \neq 0$: Next, we show that Condition (6) with $\gamma_i^{\mathcal{N}_1}$, $i \in \mathbb{N}_{\geq 2}$, holds also for z_0 with $\theta_0 \in [-\pi, 0) \cup (0, \pi)$ and, thus, for all initial conditions $z_0 \in \mathcal{N}_1$. Firstly, the robot turns $k_T^\star$ steps using $u(i) = (0, -\theta_0/k_T^\star T)^T$, $i = 0,1,\ldots,k_T^\star - 1$, such that $\theta_u(k_T^\star; z_0) = 0$ is attained. This yields the running costs $\ell(z_u(i;z_0), u(i))$ given by

$$q_1 x_0^4 + q_2 y_0^2 + q_3 \theta_0^4 \left[1 - \left(\frac{i}{k_T^\star}\right)\right]^4 + r_2 \left(\frac{\theta_0}{k_T^\star T}\right)^4$$

Using $1 - (i/k_T^\star) \in [0,1]$ and Assumption (15) leads to

$$\ell(z_u(i;z_0), u(i)) \leq \ell^\star(z_0) - q_3 \left(\frac{\theta_0^4}{k_T^\star}\right) \left[i - \frac{1}{2(k_T^\star T)^3}\right]$$

for $i \in \{0,1,\ldots,k_T^\star - 1\}$. Hence, the right hand side of this inequality is always less or equal $\ell^\star(z_0)$ for $i > 0$. The remaining parts of the manoeuvre are performed as before. We show that this case is less costly than its counterpart $\theta_0 = 0$ by the following calculations, in which the abbreviation $\Xi := r_1 v(k_T^\star + l_T^\star)^4 + r_2 w(k_T^\star + l_T^\star)^4$ is used:

$$\begin{aligned} &\sum_{i \in \{0, k_T^\star + l_T^\star\}} \ell(z_u(i;z_0), u(i)) \\ &= \ell^\star(z_0) + q_2 y_0^2 + r_2 \left(\frac{\theta_0}{k_T^\star T}\right)^4 + \Xi \\ &\overset{(15)}{\leq} 2 \cdot \ell^\star(z_0) + \Xi \leq \left(c_0^{\mathcal{N}_1} + c^{\mathcal{N}_1}_{k_T^\star + l_T^\star}\right) \cdot \ell^\star(z_0). \end{aligned}$$

In conclusion, the accumulated bounds given by (28) can be used to ensure Condition (6) for initial conditions z_0 with $\theta_0 \neq 0$.

V. Numerical Results

In the preceding section, bounds

$$\gamma_i = \min\{i, \max\{\gamma_i^{\mathcal{N}_1}, \gamma_i^{\mathcal{N}_2}\}\}, \text{ for } i \in \mathbb{N}_{\geq 2}$$

satisfying Assumption 1 were deduced, see (13). Here, the bounds $\gamma_i^{\mathcal{N}_2} = \sum_{n=0}^{i-1} \bar{c}_n^{\mathcal{N}_2}$ were constructed according to the procedure presented in the paragraph before (21) based on the coefficients $c_n^{\mathcal{N}_2}$ displayed in (17), (18), and (20). Similarly, $\gamma_i^{\mathcal{N}_1} = \sum_{n=0}^{i-1} \bar{c}_n^{\mathcal{N}_1}$ are derived using (27). In the following, a prediction horizon N is determined such that the resulting MPC closed-loop is asymptotically stable – based on these bounds γ_i, $i \in \mathbb{N}_{\geq 2}$. To this end, the minimal stabilizing horizon $\hat{N}$ is defined as

$$\min\left\{N \in \mathbb{N}_{\geq 2} : \alpha_N = 1 - \frac{(\gamma_N - 1)\prod_{k=2}^N (\gamma_k - 1)}{\prod_{k=2}^N \gamma_k - \prod_{k=2}^N (\gamma_k - 1)} > 0\right\},$$

a quantity, which depends on the sampling rate T and the weighting coefficients of the running cost $\ell(\cdot,\cdot)$. Then, a comparison with quadratic running costs is presented in Subsection V-B before, in Subsection V-C, numerical simulations are conducted in order to show that MPC without stabilizing constraints or costs steers differential drive robots to a desired equilibrium.

A. Computation of the minimal stabilizing horizon $\hat{N}$

In the considered problem setting of regulating a non-holonomic robot to a desired set point, a particular feature is that the robot may stay at its initial position (including the initial angle) without moving. This property is reflected in Definition (13) of the growth bounds $(\gamma_i)_{i \in \mathbb{N}}$ (using the convention $\gamma_1 := 1$ for simplicity) by $\gamma_i \leq i$, $i \in \mathbb{N}$. The following proposition demonstrates the impact of this observation for asymptotic stability of the MPC closed and is exploited in Algorithm 2.

Proposition 6. *Let $N \in \mathbb{N}_{\geq 3}$ hold and growth bounds $(\gamma_i)_{i=1}^N$ be given by $\gamma_i = i$, $i = 1, 2, \ldots, N-1$, and $\gamma_N = N - 1 + \varepsilon$ with $\varepsilon \in [0, 1)$. Then, the performance index α_N defined by Formula (8) is strictly positive ($\alpha_N > 0$).*

Proof: The following calculation shows the assertion:

$$\begin{aligned}\alpha_N &\overset{(8)}{=} 1 - \frac{(\gamma_N - 1)^2 \prod_{k=1}^{N-2} k}{\gamma_N (N-1) \prod_{k=2}^{N-2} k - (\gamma_N - 1) \prod_{k=2}^{N-2} k} \\ &= 1 - \frac{(\gamma_N - 1)^2}{(N-2)\gamma_N + 1} \\ &= \frac{(1-\varepsilon)(N-2) + (1-\varepsilon)^2}{(N-1+\varepsilon)(N-2)+1} > 0. \quad \square\end{aligned}$$

Let the sets $U = [-0.6, 0.6] \times [-\pi/4, \pi/4]$ and $Z := [-2, 2]^2 \times \mathbb{R}$ be given. Moreover, the weighting parameters $q_1 = 1$, $q_3 = 0.1$, $r_1 = q_1 T/2$, and $r_2 = q_3 T/2$ of the running costs $\ell(\cdot, \cdot)$ are defined depending on the sampling time T. Then, for a given sampling time T and weighting coefficient q_2, the minimal stabilizing horizon $\hat{N}$ can be computed by Algorithm 2.

Algorithm 2 Calculating the minimal stabilizing horizon $\hat{N}$

Given: Control bounds $v_{\min}$, $v_{\max}$, $w_{\min}$, $w_{\max}$, box constraints $x_{\min}$, $x_{\max}$, $y_{\min}$, $y_{\max}$, weighting coefficients q_1, q_2, q_3, r_1, r_2, and sampling time T.
Initialization: Set $N = 1$ and $\alpha = 0$.

1: **while** $\alpha = 0$ **do**
2: Increment N.
3: Minimize $\gamma_N^\star := \max\{\gamma_N^{\mathcal{N}_2}, \gamma_N^{\mathcal{N}_1}\}$ subject to $s \in \mathbb{R}_{\geq 0}$, (21), and (28).
4: Define $\gamma_N := \min\{N, \gamma_N^\star\}$.
5: If $\gamma_N < N$, set $\alpha = 1 - \frac{(\gamma_N - 1)^2}{(N-2)\gamma_N + 1}$, see Proposition 6.
6: **end while**

Output Minimal stabilizing horizon length $\hat{N} = N$ and the performance index $\alpha_{\hat{N}} = \alpha$.

The only optimization is carried out in Step 3 of Algorithm 2. This can be done by a line search. Here, the optimization variable s can be restricted to a compact interval depending on the size of the state and control constraints, and the weighting parameter q_2.

TABLE I
MINIMAL STABILIZING HORIZON $\hat{N}$ IN DEPENDENCE ON THE SAMPLING TIME T AND THE WEIGHTING PARAMETER q_2 FOR $q_1 = 1$, $q_3 = 0.1$, $r_1 = q_1 T/2$, AND $r_2 = q_3 T/2$.

Sampling time T (seconds)	$\hat{N}(\hat{N} \cdot T$(seconds))			
	$q_2 = 2$	$q_2 = 5$	$q_2 = 10$	$q_2 = 100$
1.00	12(12)	10(10)	8(8)	8(8)
0.50	25(12.5)	19(9.5)	16(8)	15(7.5)
0.25	48(12)	37(9.25)	32(8)	29(7.25)
0.10	122(12.2)	93(9.3)	79(7.9)	70(7)

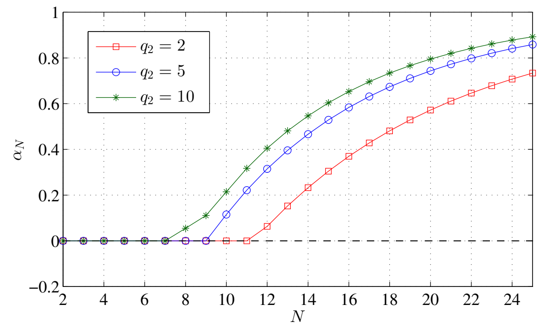


Fig. 4. Dependence of the performance bound α_N on the prediction horizon N for sampling time $T = 1$ and $q_1 = 1$, $q_3 = 0.1$, $r_1 = q_1 T/2$, $r_2 = q_3 T/2$.

The results of Algorithm 2 for sampling time $T = 1$ and weighting coefficient $q_2 \in \{2, 5, 10\}$ are presented in Figure 4. Using a larger weighting coefficient q_2 results in smaller minimal stabilizing horizons $\hat{N}$. Moreover, it can be seen that the suboptimality index α converges to one for prediction horizon N tending to infinity. Furthermore, it is observed that the radius s of the set $\mathcal{N}_1$ increases for larger q_2, i.e. $s = 0.8$ ($q_2 = 2$), $s = 1.4$ ($q_2 = 5$), and $s = 1.7$ ($q_2 = 10$). In contrast to that, the influence of the sampling time T is negligible, cf. Table I.

B. Comparison with quadratic running costs

Here, the proposed MPC scheme without stabilizing constraints or costs, i.e. Algorithm 1, is applied in order to stabilize a unicycle mobile robot to the origin. The constraints and weighting coefficients of the running costs ℓ in this subsection and the subsequent one are the same as in the preceding subsection with $q_2 = 5$ and sampling time $T = 0.25$. In this case, the theoretically calculated minimal stabilizing horizon is given by $\hat{N} = 37$, cf. Table I. All simulations have been run using the *Matlab* routine `fmincon` to solve the optimal control problem in each MPC step. However, for a real time implementation we recommend the ACADO toolkit [30].[2] The MPC performance is investigated through two sets of numerical simulations – on the one hand under the proposed running costs (5); on the other hand using the standard quadratic running costs with weighting matrices $Q = \text{diag}(q_1, q_2, q_3)$ and $R = \text{diag}(r_1, r_2)$.

First, the initial state of the robot is chosen to be $z_0 = (0, 0.1, 0)^T$, i.e. located on the y-axis, close to the origin, and with an orientation angle of zero. Both controllers steer the robot close to the origin, but only the MPC controller with the proposed running costs fulfils the control objective of steering the robot to the origin, cf. Figure 5. This conclusion can be also inferred from the scaled value function $V_N(z_{\mu_N}(n; z_0)) \cdot \ell^\star(z_0)^{-1}$, $n \in \mathbb{N}_0$, evaluated along the closed-loop trajectories as depicted in Figure 6. Since the value function does not

[2] All simulations were also performed with ACADO to investigate the real-time applicability of the proposed MPC scheme. Since all computation times were less than 1 (ms) and, thus, negligible in comparison to the sampling time $T = 0.25$ (s), the implicit assumption that the nonlinear optimization problem of each MPC step can be solved instantaneously seems to be justified.

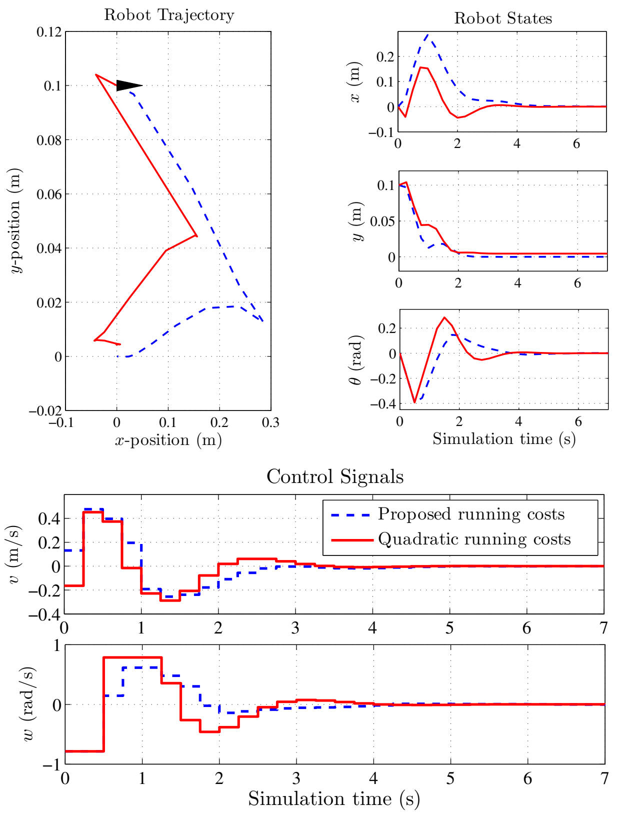


Fig. 5. MPC Closed loop state trajectory and employed controls for sampling time $T = 0.25$ and prediction horizon $N = 37$ under the proposed and quadratic running costs with weighting matrices $Q = \text{diag}(q_1, q_2, q_3)$ and $R = \text{diag}(r_1, r_2)$.

decrease anymore after a few ($n \approx 12$) time steps, MPC with quadratic running costs fails to ensure asymptotic stability for the chosen prediction horizon $N = 37$.

Moreover, since uniform boundedness of $\sup_{z_0 \in Z} V_N(z_0) \cdot \ell^\star(z_0)^{-1}$ with respect to the prediction horizon N is a necessary condition for asymptotic stability of the MPC closed-loop, we further investigate this quantity. To this end, three initial conditions $z_0 = (0, y_0, 0)^T, y_0 \in \{0.1, 0.01, 0.001\}$, are considered, cf. Figure 7. Under the proposed stage costs, the quantity $V_N(z_0) \cdot \ell^\star(z_0)^{-1}$ is bounded for all chosen initial conditions. In contrary to this, for quadratic running costs, the quantity $V_N(z_0) \cdot \ell^\star(z_0)^{-1}$ grows unboundedly for

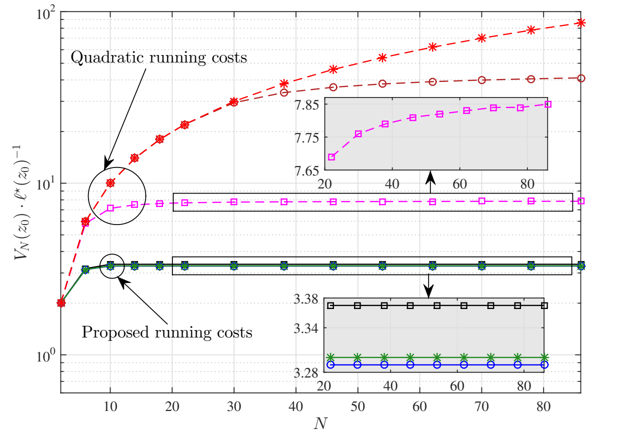


Fig. 7. Evaluation of $V_N(z_0) \cdot \ell^\star(z_0)^{-1}$ for $N = 2, 3, \ldots, 86$ for the proposed and quadratic running costs with initial conditions: $z_0 = (0, 0.1, 0)^T$ (□), (□); $z_0 = (0, 0.01, 0)^T$ (○), (○); and $z_0 = (0, 0.001, 0)^T$ ($*$), ($*$). $T = 0.25$.

decreasing y_0-component, e.g. $y_0 = 10^{-i}$, $i \in \mathbb{N}$, in our numerical simulations. Indeed, this observation was also made for different weighting coefficients and prediction horizons. Even in the setting with stabilizing terminal constraints and costs [11], [21], non-quadratic terminal costs were deployed. We conjecture that Assumption 1 cannot be satisfied for quadratic running costs. In conclusion, using a non-quadratic running cost $\ell(\cdot, \cdot)$ like (5) seems to be necessary in order to ensure asymptotic stability of the MPC closed-loop without stabilizing constraints or costs.

C. Numerical investigation of the required horizon length

In this subsection, the minimal stabilizing horizon $\hat{N}$ is numerically examined for the MPC controller. To this end, the evolution of the value function $V_N(z_{\mu_N}(n; z_0))$, $n \in \mathbb{N}_0$, along the MPC closed-loop, using the proposed running costs (5) for initial conditions $z_0 = (0, 10^{-i}, 0)^T$, $i \in \{0, 1, 2, 3, 4, 5\}$, is considered, cf. Figure 8 (left). If the value function decays strictly, the relaxed Lyapunov inequality (9) holds — a sufficient stability condition, cf. [31]. Hence, we compute the minimal prediction horizon such that this stability condition is satisfied until a numerical tolerance is reached, i.e. $V_N(z_{\mu_N}(n; z_0)) \leq 3 \cdot 10^{-11}$ as shown in Figure 8 (right).

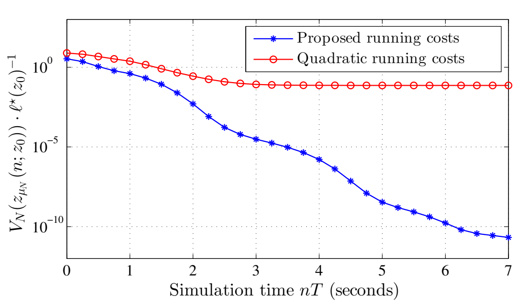
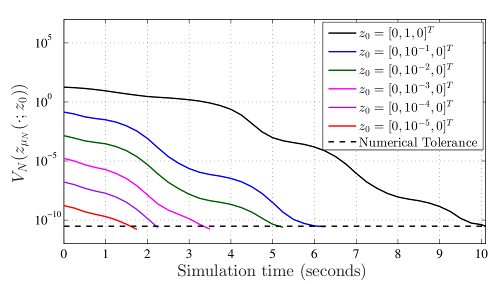


Fig. 6. Evolution of $V_N(z_{\mu_N}(n; z_0)) \cdot \ell^\star(z_0)^{-1}$, $n \in \{0, 1, \ldots, 28\}$, for $z_0 = (0, 0.1, 0)^T$, $T = 0.25$, and $N = 37$.



y_0	$\hat{N}$
10^{-0}	4
10^{-1}	4
10^{-2}	4
10^{-3}	9
10^{-4}	9
10^{-5}	16

Fig. 8. Evolution of $V_N(z_{\mu_N}(\cdot; z_0))$ along the closed-loop trajectories (left) and numerically computed stabilizing prediction horizons $\hat{N}$ (right) for sampling time $T = 0.25$ and different initial conditions.

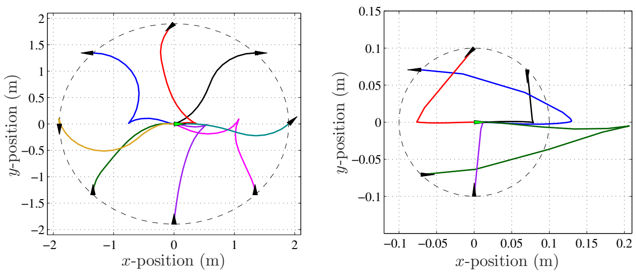


Fig. 9. MPC closed-loop trajectories emanating from initial conditions $(x_0, y_0)^T$ on the circle of radius 1.9 (left; $N = 7$) and 0.1 (right; $N = 15$) using the proposed running costs. The initial state and orientation is indicated by the filled (black) triangles ($T = 0.25$).

So far, we concentrated on very particular initial conditions. Now, the ability of the proposed MPC controller to stabilize a unicycle mobile robot to an equilibrium point is demonstrated. To this end, eight initial positions evenly distributed along a large circle of 1.9 (m) radius, as well as five initial positions distributed along a small circle of 0.1 (m) radius, are selected. The initial orientation angle θ_0 is randomly chosen from the set $\{i \cdot \pi/4 | i \in \{0, 1, 2, 3, 4, 5, 6, 7\}\}$. The prediction horizon N is chosen such that the value function $V_N(z_{\mu_N}(n; z_0))$, $n \in \mathbb{N}_0$, evaluated along the closed-loop reaches a neighbourhood of the origin corresponding to a reference magnitude of 10^{-9} for initial conditions on the large circle depicted and 10^{-11} for initial conditions on the small circle, which is illustrated in Figure 9. It is observed that stabilizing horizons of $N = 7$ and $N = 15$ are required for the initial conditions located on the large and small circles, respectively.

Our numerical simulations show that the required prediction horizon N rapidly grows if the initial condition is located (very) close to the origin. Otherwise, much shorter horizons N are sufficient to steer the robot (very close) to the desired equilibrium. Independently of this observation, the numerically calculated stabilizing prediction horizon is shorter than its theoretically derived bound $\hat{N} = 37$. However, the calculated stabilizing horizon $\hat{N}$ holds for all initial states z_0 in the feasible domain Z. Moreover, both the estimates and the manoeuvres used in order to derive $\gamma_N^{\mathcal{N}_2}$ and $\gamma_N^{\mathcal{N}_1}$ given by (21) and (28), respectively, are not optimal as highlighted in Section IV. Hence, the derived estimate of $\hat{N}$ can be further improved.

VI. Conclusions and Outlook

In this paper, a stabilizing MPC controller is developed for the regulation problem of unicycle nonholonomic mobile robots. Unlike the common stabilizing schemes presented in the literature where terminal constraints and/or costs are adopted, asymptotic stability of the developed controller is guaranteed by the combination of suitably chosen running costs and prediction horizon. Herein, the design of the running costs reflects the task to control both the position and the orientation of the robot and, thus, penalizes the direction orthogonal to the desired orientation more than other directions. Then, open loop trajectories are constructed in order to derive bounds on the value function and to determine the length of the prediction horizon such that asymptotic stability of the MPC closed-loop can be rigorously proven. The presented proof of concept can serve as a blueprint for deducing stability properties of similar applications. Finally, numerical simulations are conducted in order to examine the proposed controller and assess its performance in comparison with a controller based on quadratic running costs.

Compared with the stabilizing MPC controllers presented in the literature for nonholonomic mobile robots, the developed controller stands as a unique one as it relaxes the computational complexities associated with stabilizing constraints and/or costs. Future work will include the extension of the proposed approach to regulation problems for domains with obstacles as well as trajectory tracking and path following.

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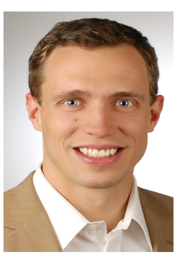

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