An Approach to Transportation Network Analysis Via Transferable Utility Games

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ABSTRACT

Network connectivity is an important aspect of any transportation network, as the role of the network is to provide a society with the ability to easily travel from point to point using various modes. A basic question in network analysis concerns how "important" each node is. An important node might, for example, greatly contribute to short connections between many pairs of nodes, handle a large amount of the traffic, generate relevant information, represent a bridge between two areas, etc. In order to quantify the relative importance of nodes, one possible approach uses the concept of centrality. A limitation of classical centrality measures is the fact that they evaluate nodes based on their individual contributions to the functioning of the network. In the present paper a game theory approach is introduced, based on cooperative games with transferable utility. Given a transportation network, a game is defined taking into account the network topology, the weights associated with the arcs, and the demand based on an origin-destination matrix (weights associated with nodes). The nodes of the network represent the players in such a game. The Shapley value, which measures the relative importance of the players in transferable utility games, is used to identify the nodes that have a major role. For several network topologies, a comparison is made with well-known centrality measures. The results show that the suggested centrality measures outperform the classical ones, and provide an innovative approach for transportation networks analysis.

Keywords: Network analysis, Centrality measures, Transferable utility games, Shapley value, Games on graphs

1 INTRODUCTION

Network connectivity is an important aspect of any transportation network, as the role of the network is to provide a society with the ability to easily travel from point to point using various nodes. Analyzing network connectivity can assist decision makers in identifying weak components, detecting and preventing failures, and improving connectivity in terms of reduced travel time, reduced costs, increased reliability, easy access, etc. Various connectivity measures for general networks have been proposed, including among them the connectivity and strong connectivity of graphs (Ahuja et al., 1993); the cyclomatic number, which measures the number of circuits in a graph; the alpha index, which is the ratio between the number of existing circuits and the maximum number of circuits possible (Black, 2003; Rodrigue et al., 2006). Accessibility measures that can be defined for transportation networks include, for example, the longest length of a shortest path in a network, which is the longest distance traveled among all shortest paths in a network. Another measure is the ratio between the lengths of the network-based shortest path and of the direct line between node pairs. Mishra et al. (2012) provide an extensive overview of connectivity measures. Some studies investigate connectivity measures related to public transportation (Hadas, 2013; Hadas and Ranjitkar, 2012).

A basic question in network analysis asks how "important" each node is. An important node might, for example, greatly contribute to short connections between many pairs of nodes, handle a large amount of the traffic, generate relevant information, represent a bridge between two areas, etc. To quantify the relative importance of nodes, one possible approach uses the concept of centrality. Quite intuitively, nodes that are, in some sense, in the middle of a network play a major role in the functionalities of the network itself. The idea of centrality comes from the literature on social networks (Wasserman and Faust, 1994) and can be intended in different ways: in a pure topological sense, under a quasi-dynamical approach, in a dynamics-based sense, etc. Included among classical measures of centrality are degree centrality, closeness centrality, and betweenness centrality (Wasserman and Faust, 1994, chapter 10).

A limitation of classical centrality measures is that they evaluate nodes based on their individual contributions to the functioning of the network. For instance, the importance of a stop in a transportation network can be computed as the difference between the full network capacity and the capacity when the stop is closed. However, such an approach is inadequate when, for instance, multiple stops can be closed simultaneously. Consequently, the existing centrality measures need to be refined to take into account that the network nodes do not act merely as individual entities, but as members of groups of nodes.

To this end, game theory (González-Díaz et al., 2010; Tijs, 2003) provides a general basis for developing a systematic study of the relationship between rules, actions, choices, and outcomes in situations that can be either competitive or non-competitive. In an informal way, game theory is as old as social theory and was already somewhat implicitly used centuries ago by Thomas Hobbes (1588-1679), John Locke (1632-1704), and Jean-Jacques Rousseau (1712-1778) in investigating the rationale of individuals in drawing up a "social contract." The birth of the "analytical game theory" is usually identified with the publication of the book by von Neumann and Morgenstern (1944), which provides the mathematical foundations of this discipline. Afterwards, its economics-based perspective was expanded towards political science in the late 1960's and evolutionary biology in the early 1970's.

Game theory can be considered a generalization of decision theory with two or more decision makers. A game is a situation involving two or more rational decision makers called players, that make decisions in such a way that each player tries to maximize the payoff obtained as a consequence of the decisions of all players. Games can be cooperative or non-cooperative, each requiring different mathematical tools.

Game theory approaches have been used to develop transportation oriented models. For example, a two-player non-cooperative game was used to model the performance reliability of a network, with one player trying to minimize the travel costs, while the other trying to maximize the costs (Bell, 2000). This model can assist with a cautious approach to network design, based on pessimistic users. A cooperative approach (Szeto, 2011) was developed as well, based on the Stackelberg-Nash and the partial-cooperative Nash formulations to determine travel cost reliability. A non-cooperative, two-player, zero-sum game with perfect information was developed to optimize a utility function when a railway network failure occurs (Laporte et al., 2010). A game theory approach was used to model public transport operators, various integration strategies, and market situations (Roumboutsos and Kapros, 2008). Such a model can assist the policymaker with the identification of the most cost-effective form of intervention.

The idea at the basis of game-theoretic centrality measures is that the nodes are considered players in a cooperative game, where the value of each coalition of nodes is determined by certain graph-theoretic properties. The key advantage of this approach is that nodes are ranked not only according to their individual roles in the network, but also by taking into account how they contribute to the roles of all possible groups of nodes. This is important in various applications in which a group's performance cannot be simply described as the sum of the individual performances of the group members. In the case of transportation networks, suppose a certain budget is available. One possible approach addresses the question of whether the investment of all the money in increasing the capacity and/or service of a transportation component (road section, bridge, transit route, bus stop, etc.) substantially improves the whole network. A preferable approach for the network designer would probably be the consideration of simultaneously improving a possibly small subset of the components. In this case, to evaluate the importance of a component one needs to take into account the potential gain of improving one among several groups of components, and not merely the potential gain of improving the component by itself. This approach can be formalized in terms of cooperative game theory (González-Díaz et al., 2010; Tijs, 2003), in which the nodes are players and their performances are studied in coalition that are subsets of players.

On one hand, by constructing a suitable cooperative game on a network, the game-theoretic solution concepts developed during decades of research can be used. On the other hand, game-theoretic network centrality measures may be computationally very demanding. This is not an issue in the present paper that considers small networks in which the computational effort is not demanding. Moreover, recent studies (Michalak et al., 2013; Szczepański et al., 2012) show that some game-theoretic centrality measures can be efficiently computed in polynomial time with respect to the network dimension. Please refer to Michalak (2016) and the references therein for an introduction to game-theoretic network centrality.

The present study uses methods and tools from cooperative games with transferable utility, also referred to as TU games. It also uses the concept of Shapley value (Shapley, 1953), which represents a criterion according to which each node is attributed a value, so that the larger the value the greater the importance of the node. The Shapley value has mathematical properties well-suited to the present analysis.

More precisely, the present paper defines a TU game that within a transportation network takes into account the network topology, the weights associated with the arcs, and the demand based on an origin-destination matrix (weights associated with nodes). The nodes of the network represent the players in such a TU game and the Shapley values of the nodes are used to identify which nodes play a major role. Of course, the definition of the player changes depending upon whether the analysis focuses on the "physical nodes" or the "physical links." When the transportation nodes (representing, for example, intersections, transit terminals, bus stops, major points of interest, etc.) are analyzed, the network on which the TU game is defined is identical to the physical network. On the other hand, when arcs (e.g., road segments, transit routes, rail lines, etc.) are analyzed, the network can be transformed so that the physical links are modeled as nodes.

The following discussion in the present paper includes (1) a literature review concerning centrality in networks; (2) a proposed TU game model; (3) a demonstration of the model and a comparison with common centrality measures; and (4) conclusions.

2 MEASURING THE CENTRALITY OF NETWORK NODES

2.1 Centrality in networks

Centrality can be considered either an attribute of the network as a whole or an attribute of the network nodes.

According to the first point of view, the aim is to measure how unequal the distribution of centrality is among the network nodes. The broad question asks how centralized the entire network is. For example, in the network presented in Figure 1(A) some nodes are more central than others, no node is outstanding, and the network has a group of central "actors", rather than a single "star." In contrast, with regard to centrality as an attribute of nodes arising from their positions, the question is whether a node plays a central role in the network. The connection between centrality and role is that the more "central" the node in a network, the more likely the node is to be "influential." The motivation is clear since a central node has more opportunities to directly or indirectly influence other nodes. For further clarification consider two extreme cases (Hanneman and Riddle, 2005). The first is called the *star network*. An example is depicted in Figure 1(B). Clearly, node E plays a major role because (i) It is directly connected to all the other nodes; (ii) It is close to more nodes than any other node; and (iii) It lies on all the shortest paths between any pair of other nodes. The second illustrative example is the *circle* *network* in Figure 1(C). Quite intuitively, there seems to be no leader. Indeed, all the nodes have the same degree, are equally close to other nodes, and lie on the same number of shortest paths between any pair of other nodes.

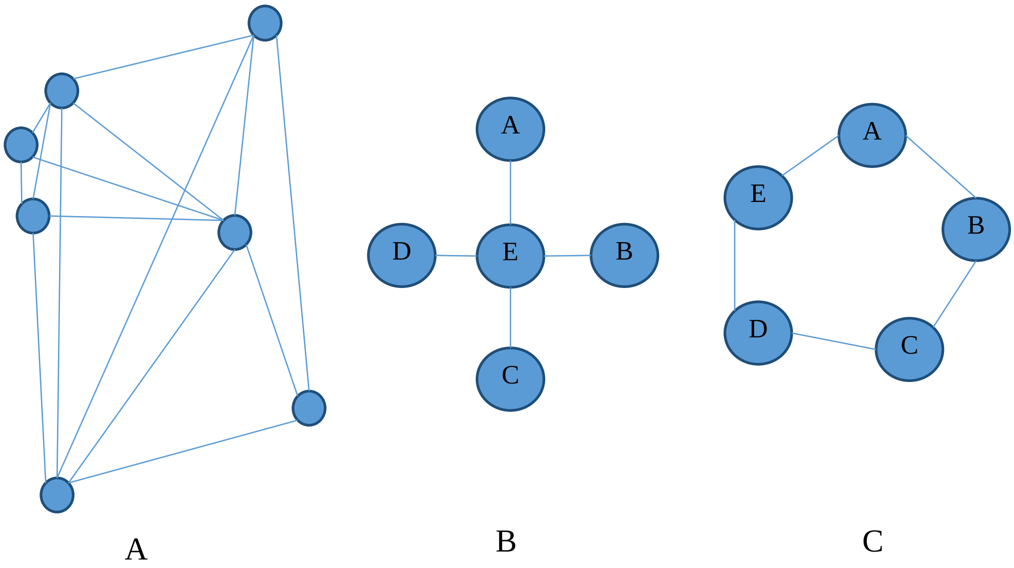

Figure 1 Illustrative networks and node centrality.

The present paper examines centrality as an attribute of nodes. Various centrality measures have been proposed in the literature. Two major families are classical measures and measures based on cooperative game theory.

2.2 Measures of centrality

For simplicity, the following first refers to unweighted and undirected networks. The extensions to the weighted and directed cases are straightforward. Hanneman and Riddle (2005) examine some commonly-used classical measures of centrality that are detailed below and denoted, for each node *i*, by X_i.

The degree-based measure relies on the concept of node degree (i.e., the number of the arcs incident in a node). Nodes that have more direct links than other nodes have more opportunities to interact. Moreover, they are less dependent on other nodes. Formally, the degree-based measure of centrality of node *i* is expressed by $X_i = \sum_j A_{i,j}$, where A is the adjacency matrix of the network, i.e., the matrix that contains 1 in its (i, j)-th entry $A_{i,j}$ if in the network there is an edge between nodes i and j, otherwise 0. Thus, according to this measure the most central node is the one with the highest degree. This is easily computable, but it does not take into account the structure of the entire network.

The closeness-based measure utilizes the concept of shortest path. Nodes that are closer to other nodes have more opportunities to interact. Therefore, the most central node is the one that minimizes the sum of the lengths of the shortest paths between itself and all the other nodes. Hence, node *i*'s closeness-based measure is $X_i = \frac{1}{\sum_j sp_{i,j}}$, where $sp_{i,j}$ is the length of any shortest path between *i* and *j*. In this way, such a node is closer to more nodes than any other one. This measure takes into account the structure of the whole network more than the degree of centrality measure does. However, it is more difficult to compute than the latter and can be applied only to connected components of a network.

The betweenness-based measure is based upon the concept that the most central node is the one with the capability of being an intermediary for the largest number of pairs of other nodes. Thus, node *i*'s betweenness-based measure is defined as $X_i = \sum_{s,t} \frac{nsp(i)_{s,t}}{nsp_{s,t}}$, where, for each node i, $nsp(i)_{s,t}$ denotes the number of shortest paths between any pair (*s,t)* of other nodes, with the additional property that such shortest paths go through i, and $nsp_{s,t}$ denotes the total number of shortest paths between *s* and *t*. Then, the most central node is the one that maximizes X_i. Although this measure can be obtained as a by-product of an algorithm computing closeness centrality, a large proportion of nodes in a network generally do not lie on any shortest path. Therefore they receive the same score of 0.

The eigenvalue-based measure (also known as eigenvector-based measure) assigns relative scores to the nodes based on the idea that for each node a connection to nodes with high scores gives greater contribution to the score of that node than equal connections to low scoring nodes. Then, the eigenvalue-based measure is obtained by solving a suitable algebraic system of linear equations, which relates the centrality of each node (to be determined by solving the system itself) to the ones of the adjacent nodes (which are also to be determined), and is expressed by $X_i = \frac{1}{\lambda_{\max}} \sum_j A_{i,j} X_j$, where $\lambda_{\max}$ is the largest eigenvalue of the adjacency matrix. Hence, the name of this measure derives from the fact that the vector of node scores is related to the largest eigenvalue of the adjacency matrix and the associated eigenvector, which under mild conditions can be selected to have non-negative entries due to an application of Perron-Frobenius theorem. For example, Google's PageRank algorithm implements a variant of the eigenvalue centrality measure. The interested reader is referred to (Erciyes, 2014) for further details on the eigenvalue-based measure.

The above considered measures of centrality can be extended to weighted and directed networks. For instance, the degree may be replaced by the total weight of the arcs entering a node, or by the total weight of the arcs exiting a node. Some measures allow for variations that simultaneously take into account the number of arcs and their weights, using a non-negative parameter to trade off the two contributions (Opsahl et al., 2010).

Cooperative game theory can be utilized both to define new measures of centrality and to improve the classical ones described above. In this framework, the grand coalition corresponds to the set of nodes of the entire graph. The nodes in each subgraph induced by a subset of nodes represent a coalition. For each coalition S, the utility $v(S)$ can be defined on the basis of the network topology and, in the case of a weighted graph, by also taking into account the arc weights. For instance, $v(S)$ may be defined as the cardinality of the set of all the nodes that are in S or are reachable in one step by a node in S. Then one can exploit the solution criteria developed in cooperative game theory to attribute to each node a value that describes its importance.

Among such criteria are the Banzhaf power index and the Shapley value. A notion of centrality based on the Banzhaf power index was first proposed by Grofman and Owen (1982). Recently, Narayanam and Narahari (2011) developed a Shapley value-based approach for the detection of influential nodes in social networks. This paper considers the Shapley value.

Unfortunately, given a cooperative game defined over a network of n nodes, the exact computation of the Shapley value requires considering all possible 2^n coalitions of nodes (unless computational savings are possible, due to the special structure of the utility function). Clearly, this is computationally unmanageable for networks with tens, or even hundreds, of nodes. In some cases the exponential number of computations cannot be avoided, since certain game-theoretic network centralities cannot be computed in polynomial time with respect to the size of the network. In other cases, however, the possibility of polynomial-time computation has been proved (Michalak et al., 2013) such as in the case of the Shapley value-based betweenness centrality (Aadithya et al., 2010; Szczepański et al., 2012; Szczepański et al., 2016).

3 TRANSPORTATION NETWORK TU GAME MODEL

3.1 TU games

A cooperative game with transferable utility, also referred to as a TU game, is a pair (N, v) defined as follows.

$$N := \{1, \ldots, n\} \tag{1}$$

is the set of players, which is called the grand coalition. Each subset $S \subseteq N$ is a (sub)coalition. The real-valued mapping is

$$v : 2^n \to \mathbb{R} \tag{2}$$

such that $v(\varnothing)=0$ is called the utility function (also referred to in the literature as the characteristic function). It assigns to each coalition S its utility $v(S)$ and represents the utility that can be achieved by the players in S, without any contribution from the players in $N \setminus S$. The quantity $v(N)$ is the utility of the grand coalition and, for each player i, $v(\{i\})$ is the utility of the player i without entering any coalition with more than one player. A utility function is superadditive if for every two disjoint subsets of nodes S and U one has $v(S \cup U) \geq v(S)+v(U)$.

In TU games, the utilities can be transferred from one player to another without any loss (e.g., by means of a common "currency" that is valued equally by all the players). TU games can be studied using suitable solution concepts. Each of them is a criterion for dividing the total utility $v(N)$ of the game among individual players. Examples of solution concepts are the core, the nucleolus, and the power indices, such as the Banzhaf power index, and the Shapley value (González-Díaz et al., 2010; Tijs, 2003). Each power index corresponds to a way of allocating the total utility in a "fair way" among all the players. The larger the amount of utility allocated to a player by a power index, the greater the importance of the player in the grand coalition.

Often the power indices are weighted sums (with respect to all possible coalitions) of the marginal utilities associated with the insertion of a player into a coalition that does not contain that player. If player i contributes nothing to the value of the coalition $\{i\}$ and of any extended coalition $S \cup \{i\}$ (i.e., $v(\{i\})=0$ and $v(S \cup \{i\})=v(S)$), then that player should receive 0 in a fair allocation of the utility of the grand coalition. Such a player is called a null player. Instead, if for player i one has $v(S \cup \{i\}) \geq v(S)$ for all coalitions S, and $v(S \cup \{i\}) > v(S)$ for at least one coalition S, then player i should receive more than 0.

The present paper considers the Shapley value, which was introduced axiomatically by (Shapley, 1953) and is defined for every player i as

$$Sh(i) := \sum_{S \subseteq N} \left(\left(v(S) - v(S \setminus \{i\}) \right) \frac{(|S|-1)!(n-|S|)!}{n!} \right), \tag{3}$$

where $|S|$ denotes the cardinality of the set S. $Sh(i)$ can be interpreted in terms of the average marginal contribution that the player i provides to any coalition, considering all possible coalitional scenarios and assuming that all orders are equally likely. Indeed, the factor

$$\frac{(|S|-1)!(n-|S|)!}{n!} \tag{4}$$

comes from taking into account the order of the players entering a coalition. It is worth noting that, in order to compute the Shapley value exactly, in general all possible subsets S of N have to be considered, which makes its evaluation computationally inefficient, unless the utility function has a special structure (e.g., in the simplest case, when it is known a-priori that many subsets S have zero utility). As all permutations need to be considered to obtain the marginal contributions of nodes, use of the Stirling approximation of $n!$ (Cormen et al., 2001) shows that the computational complexity of the Shapley values of the nodes is of order $O\left(\left(\frac{n}{e}\right)^n\right)$ (see Narayanam and Narahari (2011) for details). Thus, more efficient sampling-based approximation approaches have been developed to work in polynomial time (Castro et al., 2009). It is also worth anticipating that the utility functions considered in the paper have a special structure, which makes the computation of the associated Shapley values efficient, at least for some kinds of problems (see Proposition 2 in the paper).

Shapley (1953) proved that the Shapley value is the unique mapping from the space of cooperative TU games to $\mathbb{R}^n$ that satisfies all the following properties: linearity, symmetry, and carrier property, which guarantee that it generally represents a fair way of distributing the utility of the grand coalition among all the players.

3.2 The class of transportation network games

In our approach, nodes can represent intersections, point of interests, transportation centroids, stops, stations, critical transportation components, etc. Arcs can represent physical connections between pairs of nodes, i.e., roads, rails, parts of routes or entire routes, etc. The utility function can be exploited to represent a wide range of transportation attributes, such as connectivity, distance, travel time saving, assigned demand, etc. The transportation network (TN) can also be reconstructed in such a way that the nodes represent the arcs of the original network, i.e., the physical transportation segments (road, rail, etc.).

Assumption 1. The network is connected.

This is not restrictive: if the network is disconnected, the same analysis can be made for each connected subnetwork. Moreover, the approach may provide information about the best way to connect a disconnected network.

Assumption 2. The network is undirected, i.e., an arc represents a bidirectional flow.

This assumption can be relaxed as well, since the model is not necessarily restricted to symmetric matrices.

Assumption 3. Utility transferability holds.

This assumption makes possible the use of the theory of TU games. Moreover, in transportation networks, the more a network is connected, the higher the social gain and the greater the incentive to transfer utility.

The following discussion introduces three specifications of the model: 1) unweighted networks, 2) node-weighted networks, and 3) arc-weighted networks.

3.2.1 Unweighted networks

An unweighted transportation network is a graph $G=(V,E)$ where V is the set of nodes, and E is the set of arcs.

We proceed as follows.

- On the basis of the network topology, a TU game is constructed.
- In order to evaluate the centrality of the nodes in the game, the vector of their Shapley values is computed.

We now define the TU game. The grand coalition is represented by the set V of nodes, with cardinality $n=|V|$. Each coalition S is any subset of nodes of V, which induces a subgraph of G (i.e., the subgraph for which its nodes are the elements of S, and its edges are all the edges in E connecting any two elements of S), which we denote by $G(S)$. Then, we construct the utility as follows. Given a coalition S, first, we compute the matrix $sp(S)$ containing the lengths of all-pair shortest paths in the subgraph induced by S, based on numerous efficient algorithms for this task (Ahuja et al., 1993). We let $x(S)$ be an indication matrix if a (shortest) path exists between node pairs in the subgraph $G(S)$ induced by S:

$$x(S)_{i,j} := \begin{cases} 1 & if \;\; sp(S)_{i,j} < \infty, \\ 0 & \quad otherwise. \end{cases} \tag{5}$$

Based on equation (4), the utility of coalition S is defined as follows:

$$v_1(S) := \sum_{i,j \in S, i \neq j} x(S)_{i,j} \,, \tag{6}$$

meaning that each node contributes a utility to the coalition, proportional to the number of other nodes in the coalition that are connected to it through a shortest path. Note that in equation (4) a path between two nodes i and j contributes twice to the summation, i.e., one has $v_1(S) = 2\sum_{i,j \in S, i<j} x(S)_{i,j}$. We have chosen such a definition in view of the generalization to origin-destination matrices (which is detailed in the next subsection), and also of the possible generalization to directed graphs. By definition, $v_1(\varnothing)=0$. The above-defined mapping $v_1 : 2^n \to \mathbb{R}$ is quite intuitive. In the following, we consider the case of other non-negative utilities.

3.2.2 Node-weighted networks

The above-mentioned formulation assumes that all nodes are a-priori equally important, which is usually not the case with transportation networks. The obvious realization of node weights is the origin-destination demand. In order to integrate demand within the model, we introduce a revised formulation. A transportation network with demand is a 3-tuple (V, E, D) where D is the origin-destination matrix, which reflects the (non-negative) demand between each node pair. By definition, $D_{i,i} := 0$ for every node i.

We alter equation (5) as follows:

$$v_2(S) := \sum_{i,j \in S, i \neq j} \left(D_{i,j} x(S)_{i,j} \right). \tag{7}$$

In other words, if nodes i and j are connected in subgraph $G(S)$, then it is possible to satisfy the demand between the pair, and hence the utility increases accordingly. Otherwise, the pair does not jointly contribute to the utility of the coalition S.

Since $x(S)_{i,j} = x(S)_{j,i}$, one can consider without loss of generality, symmetric origin-destination matrices. Indeed equation (7) can be written as $v_2(S) = \sum_{i,j \in S, i \neq j} \left(\overline{D}_{i,j} x(S)_{i,j} \right)$, where $\overline{D}_{i,j} = \overline{D}_{j,i} := \frac{D_{i,j} + D_{j,i}}{2}$.

3.2.3 Arc-weighted networks

For better modeling of transportation networks, we introduce an enhanced formulation that takes into account the physical properties of the network such as distance, delay, or impedance, computed along the whole path connecting two nodes. Accordingly, we introduce a weight matrix representing the impedance of each arc, so W is a weight matrix such that its (i, j)-th entry is the non-negative weight $w_{i,j}$ if $(i, j) \in E$ and ∞ otherwise. The 3-tuple corresponding to the graph G and the weight matrix W is denoted by (V, E, W). If a destination matrix D is also considered, then the 4-tuple (V, E, D, W) is used.

Since the goal is to evaluate the utility, the following transformation of the weight matrix is required, as the weights correspond to the disutility, not the utility. Let

$$M := \max_{S \subseteq V} \left(\max_{i,j \in S, i \neq j:\, sp(S)_{i,j} < \infty} \left(sp(S)_{i,j} \right) \right) + 1, \tag{8}$$

i.e., M is the largest diameter of any subgraph of the transportation network plus 1.

We now introduce the following utility functions, for unweighted-nodes and weighted-nodes, respectively:

$$v_3(S) := \sum_{i,j \in S, i \neq j:\, sp(S)_{i,j} < \infty} \left(M - sp(S)_{i,j} \right) \tag{9}$$

and

$$v_4(S) := \sum_{i,j \in S, i \neq j:\, sp(S)_{i,j} < \infty} D_{i,j} \left(M - sp(S)_{i,j} \right) \tag{10}$$

Other utility functions can be defined as

$$v_5(S) := \sum_{i,j \in S, i \neq j:\, sp(S)_{i,j} < \infty} \frac{1}{sp(S)_{i,j} + 1} \tag{11}$$

and

$$v_6(S) := \sum_{i,j \in S, i \neq j:\, sp(S)_{i,j} < \infty} \frac{D_{i,j}}{sp(S)_{i,j} + 1} \tag{12}$$

Another possible choice, related to v_5, is

$$v_7(S) := \frac{1}{\sum_{i,j \in S, i \neq j:\, sp(S)_{i,j} < \infty} sp(S)_{i,j} + 1}. \tag{13}$$

However, Section 3.3 shows that despite its similarity with v_5 , the utility v_7 does not share its useful mathematical properties. This is the reason that we do not even present its variation in terms of an origin-destination matrix.

Since the games (N, v_l) above ($l = 1, ..., 7$) are induced by the transportation network, we refer to them as *TN cooperative games*, or *TNc games*. For each l, we denote such a class of games by Υ_l . The *TNc centrality measure* is the measure of centrality that assigns to each node i its Shapley value in the corresponding TNc game Υ_l. The class of TNc games with nodes in N is denoted by Υ^N . Other Shapley-based centrality measures have been proposed in the recent literature (see, for example, Michalak et al. (2013)). The main difference between the proposed TNc centrality measures and the ones in Michalak et al. (2013) is in our more general choice of the utility functions. For instance, one difference is that the measures proposed in the present paper quantify the degree of communication among pairs of nodes belonging to the subgraph induced by a coalition, whereas the measures proposed by (Michalak et al., 2013) consider, instead, the communication among pairs of nodes outside such a subgraph. Another difference is that the utilities (6), (9), and (11) take into account not only the network connectivity properties, but also the demands. Finally, although even the utility functions considered in (Michalak et al., 2013) can be evaluated efficiently, the utility functions considered in this work have a special structure, which makes it possible to use Proposition 2 (which is reported later in the paper) for the efficient computation of the Shapley values of the associated TNc games, at least for special kinds of graphs.

3.3 Mathematical properties of the TNc games

This section investigates properties of the utility functions and associated games defined above, such as superadditivity of the value function, nonconvexity of the game, and the existence of the core. We also provide a way to efficiently compute the Shapley values, at least for certain networks for which the number of connected subgraphs is relatively small.

Superadditivity of the utility function is an incentive to create coalitions. It means that the utility of the union of every two disjoint coalitions in the network is larger than or equal to the sum of the utilities of each of the two coalitions. If the utility function is superadditive, then one can prove that the Shapley value is individually rational, i.e., the Shapley value of each node is larger than or equal to the value of the coalition made only of that node. Hence, each node has the incentive to enter the grand coalition (even if it is not formally obliged to do so). The core of a TU game is the set of allocations of the total utility that in addition to satisfying individual rationality also satisfies coalitional rationality, i.e., for each coalition, its utility is smaller than or equal to the sum of the utilities obtained by its members when they join the grand coalition, and the utility of the grand coalition is distributed according to the allocation. Thus, every two disjoint coalitions have the incentive to form the grand coalition. A TU game is convex if generally, as in Shapley's words, "the incentives for joining a coalition increase as the coalition grows." If a TU game is convex, then its core is nonempty and contains the Shapley value. (For a generic TU game, the core – that is always well-defined – may be empty.) Every convex TU game is also superadditive, but the opposite generally does not hold. For precise definitions and properties of superadditivity, convexity of a TU game, and its core; see (González-Díaz et al., 2010), Tijs (2003), and the Appendix in the present article.

The utility functions defined in Section 3.2 are related to the number and dimensions of the connected components in the subnetwork of the original transportation network ((V,E), (V,E,D), (V,E,W), or (V,E,W,D)) induced by any set of nodes.

The following propositions investigate properties of the utility functions v_l, $l=1,...,6$, which can be useful for making their computation faster (with respect to a direct evaluation using the respective definitions), and also for making the computation of the associated Shapley values faster. The first such proposition shows that the utility of a coalition S can be expressed in terms of the utilities of the coalitions associated with the connected components of the induced subgraph $G(S)$. Like many of the other propositions presented in this paper, the proposition does not extend to the case of the utility function v_7.

Proposition 1. Let the subgraph induced by a subset S of nodes be composed of C_S connected components with sets of nodes $S_1,...,S_k$, respectively. Then

a) $$v_1(S)=\sum_{k=1}^{C_S}|S_k|\left(|S_k|-1\right) \tag{14}$$

b) $$v_2(S)=\sum_{k=1}^{C_S}\sum_{i,j\in S_k,i\neq j}D_{i,j} \tag{15}$$

c) $$v_3(S)=\sum_{k=1}^{C_S}\sum_{i,j\in S_k,i\neq j}\left(M-sp(S_k)_{i,j}\right) \tag{16}$$

d) $$v_4(S)=\sum_{k=1}^{C_S}\sum_{i,j\in S_k,i\neq j}D_{i,j}\left(M-sp(S_k)_{i,j}\right) \tag{17}$$

e) $$v_5(S)=\sum_{k=1}^{C_S}\sum_{i,j\in S_k,i\neq j}\frac{1}{sp(S_k)_{i,j}+1} \tag{18}$$

f) $$v_6(S)=\sum_{k=1}^{C_S}\sum_{i,j\in S_k,i\neq j}\frac{D_{i,j}}{sp(S_k)_{i,j}+1}. \tag{19}$$

The following proposition shows, for $l=1,...6$, how to compute the Shapley value $Sh_l(i)$ in terms of all the connected subgraphs of the original graph. The proposition shows that, as compared to its direct computation through formula (3), $Sh_l(i)$ can be computed much more efficiently when the number of such connected subgraphs is relatively small (particularly, when it is much smaller than the cardinality 2^n of the power set of the set $N=V$ of nodes). This can be the case, e.g., of sparse graphs (i.e., graphs with a small number of edges, as compared to the number of edges $\frac{n(n-1)}{2}$ of a complete graph), graphs with bounded maximum degree, and graphs with a "hierarchical" structure. A specific investigation of graphs with a small number of connected subgraphs and of interest for transportation problems will be the subject of a successive study. In the following, we denote by S_C any subset of nodes of G for which its associated induced subgraph is a connected subgraph of G, and by N_C the set of nodes that are at distance 1 from S_C in the original graph G. Finally, Σ_C is the set containing all such subsets S_C.

Proposition 2. For $l=1,...,6$, the Shapley value $Sh_l(i)$ of the game $\left(N,v_l\right)$ can be expressed as follows:

$$Sh_l(i) = \sum_{S_C \subseteq \Sigma_C, i \in N \setminus (S_C \cup N_C)} \left[v_l(S_C) \sum_{k=|S_C|-1} \left(\frac{|k|!(n-|k|-1)!}{n!} \binom{n-|S_C|-|N_C|-1}{k-|S_C|-1} \right) \right]$$
$$- \sum_{S_C \subseteq \Sigma_C, i \in N_C} \left[v_l(S_C) \sum_{k=|S_C|}^{|N|-|N_C|} \left(\frac{|k|!(n-|k|-1)!}{n!} \binom{n-|S_C|-|N_C|}{k-|S_C|} \right) \right]$$
$$- \sum_{S_C \subseteq \Sigma_C, i \in N \setminus (S_C \cup N_C)} \left[v_l(S_C) \sum_{k=|S_C|}^{|N|-|N_C|} \left(\frac{|k|!(n-|k|-1)!}{n!} \binom{n-|S_C|-|N_C|-1}{k-|S_C|} \right) \right],$$

where

a) $v_1(S_C) = |S_C|(|S_C|-1)$;

b) $v_2(S_C) = \sum_{p,q \in S_C, p \neq q} D_{p,q}$;

c) $v_3(S_C) = \sum_{p,q \in S_C, p \neq q} \left(M - sp(S_C)_{p,q} \right)$;

d) $v_4(S_C) = \sum_{p,q \in S_C, p \neq q} D_{p,q} \left(M - sp(S_C)_{p,q} \right)$;

e) $v_5(S_C) = \sum_{p,q \in S_C, p \neq q} \frac{1}{sp(S_C)_{p,q}+1}$;

f) $v_6(S_C) = \sum_{p,q \in S_C, p \neq q} \frac{D_{p.q}}{sp(S_C)_{p,q}+1}$.

The following propositions relate to basic properties of the utility function v_l, for $l = 1,...,7$. In greater detail, they concern superadditivity, monotonicity, nonconvexity, and existence of the core. As mentioned above, superadditivity is an important property for a TU game since it guarantees that the Shapley value is individually rational.

Proposition 3. For $l = 1,...,6$, the utility function v_l is superadditive, i.e., for every two disjoint subsets of nodes S and U, one has $v_l(S \cup U) \geq v_l(S) + v_l(U)$. Moreover, it is monotonic, i.e., for every $S \subseteq T \subseteq N$, one has $v_l(T) \geq v_l(S)$.

Proposition 4. TNs exist for which the utility function v_7 is neither superadditive nor monotonic.

The following proposition shows that, in general, the TU games defined by the utility functions v_l (for $l = 1,...,7$), may possibly not be convex. Thus, one cannot prove the (possible) existence of the core using convexity of the game.

Proposition 5. For $l = 1,...,7$*, TNs exist for which the associated TNc games* (N, v_l) *are not convex.*

Nevertheless, the next proposition shows the existence of the core for the first 6 classes of TNc games. Since such games are generally not convex, evaluating whether or not the Shapley value belongs to the core would require an additional numerical analysis.

Proposition 6. For $l = 1,...,6$*, the TNc game defined by the utility function* v_l *has a nonempty core.*

4 COMPARING THE TNc MEASURES TO OTHER COMMON CENTRALITY MEASURES

The TNc model was implemented using MATLAB (The MathWorks, 2016) and the TU games toolbox (Holger, 2013). The centrality measures were calculated using MATLAB (The MathWorks, 2016). Several small-size networks were analyzed and the proposed centrality measures were compared to well-known centrality measures: degree, closeness, betweenness, and eigenvalue. Due to the small size of each network, for the case studies reported in the next Sections 4.1 and 4.2 the Shapley value was computed directly starting from its definition (3). We analyzed both unweighted and weighted undirected networks, in which the weights reflect distances or travel times. Furthermore, we illustrated the added value of analyzing weighted networks with origin-destination (OD) demand. For the weighted case, the weighted versions of all the centrality measures were used. Although v_5 and v_6 share similar properties with v_3 and v_4, when compared to the other centrality measures their performance was not as good as that of v_3 and v_4, so that they were omitted from further analysis.

For all the following examples and i = 1,...,6, we denote by TNc(i) the results obtained in correspondence of the TNc game associated with the utility function v_i.

4.1 Unweighted and weighted networks

4.1.1 Example 1

We begin with the analysis of a basic network with 4 nodes and 4 arcs, as illustrated in Figure 2. Despite its simplicity, this analysis provides useful insights into some important features of the proposed approach.

For a better understanding of the suggested approach, let us consider the calculation of the Shapley value $Sh_1(1)$ for the first node in Figure 2 and the utility function v_1. Table 1 provides the utility values for all possible coalitions, based on equations (3) - (6). For example, nodes 1 and 2 are connected, hence coalition {1,2} has a utility value equal to 2, whereas nodes 1 and

3 are not connected, then coalition {1,3} has zero utility. On the other hand, the coalition made of nodes 1, 2, and 3 corresponds to a connected subgraph with 3 nodes, hence its utility value is 6. Furthermore, for the coalition of nodes {1,3,4}, only the nodes 3 and 4 are connected, hence the utility value is 2. Finally, based on equation (3), the Shapley value for node 1 is 2.333, which is the summation on all the possible coalitions *S* of the quantities *Sh(S,1)** defined as follows:

$$Sh\left(S,1\right)^{*}=\left(v\left(S\right)-v\left(S\setminus\{1\}\right)\right)\frac{\left(\left|S\right|-1\right)!\left(n-\left|S\right|\right)!}{n!}.$$

Table 1 Detailed calculation of the Shapley value for node 1 of Example 1 and the utility function v_1.

S	{i}	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
v(S)	0	2	0	0	2	2	2	6	6	2	6	12
v(S\{1})	0	0	0	0	2	2	2	4	0	2	6	6
Eq. (4)	0	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.25
*Sh(S,1)**	0	0.167	0	0	0	0	0	0.167	0.5	0	0	1.5

Table 2 summarizes the results for Example 1, considering this time all the nodes and several centrality measures. As each measure has different units and scale, the relative percentage of each node (i.e., the ratio between its measure and the summation of that measure for all the nodes) is presented as well. All measures except for the betweenness measure provide similar results capturing the centrality of node 2. The betweenness measure assigns zero value to all the nodes with the exception of node 2, while it is evident that the other nodes have some positive centrality within the network (for example, nodes 3 and 4). When the arc weights are included, it is evident that the TNc measure captures better than the other measures the increased centrality of node 3, as it is located on the shortest path to node 4. Both the weighted degree and weighted eigenvalue measures overestimate the centrality of node 3 (they assign higher importance to it than to node 2, which is counterintuitive, as node 2 looks more centrally located), and the closeness measure underestimates that node’s centrality (it assigns lower importance to it than to node 4, which resides on a longer path). The betweenness measure captures the change, but in a rather dichotomic way, as it assigns identical zero importance to nodes 1 and 4, and identical non-zero importance to nodes 2 and 3, while node 2 looks more central within the network.

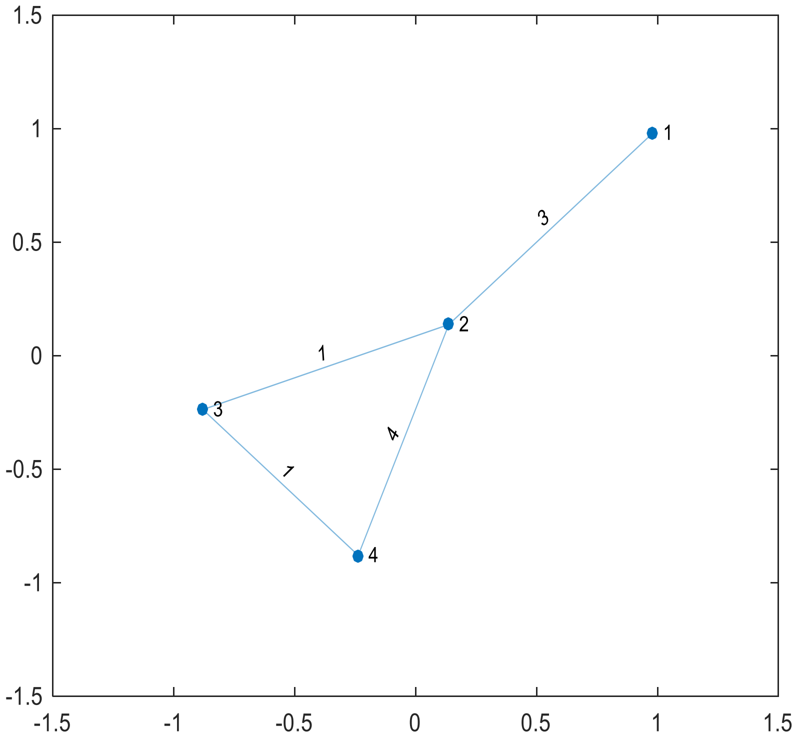

Figure 2 Network topology for Example 1.

Table 2 Centrality measures for Example 1.

	Node	Degree	Closeness	Betweenness	Eigenvalue	TNc(1)
Unweighted	1	1	0.200	0	0.282	2.333
	2	3	0.333	2	0.612	4.333
	3	2	0.250	0	0.523	2.667
	4	2	0.250	0	0.523	2.667
	1	13%	19%	0%	15%	19%
	2	38%	32%	100%	32%	36%
	3	25%	24%	0%	27%	22%
	4	25%	24%	0%	27%	22%
Weighted	Node	Degree	Closeness	Betweenness	Eigenvalue	TNc(3)
	1	0.333	0.198	0	0.113	9.333
	2	1.583	0.281	2	0.532	21.667
	3	2.000	0.133	2	0.666	19.000
	4	1.250	0.242	0	0.511	14.000
	1	6%	23%	0%	6%	15%
	2	31%	33%	50%	29%	34%
	3	39%	16%	50%	37%	30%
	4	24%	28%	0%	28%	22%

4.1.2 Example 2: A grid network

For the following 3×3 grid network, illustrated in Figure 3, three scenarios are analyzed: (i) unweighted network; (ii) arc-weighted network with weights equal to 1 for the arcs connecting node 5 and weights equal to 5 for all other arcs: and (iii) arc-weighted network with weights equal to 5 for the arcs connecting node 5 and weights equal to 1 for all other arcs. The last two scenarios are aimed at analyzing the centrality of node 5 when it resides on some shortest paths (scenario ii) or not (scenario iii).

Once again, similar results summarized in Table 3 are obtained for the unweighted network (except for the betweenness measure). For the weighted networks, there is a tendency for the common measures to attach much greater importance to node 5 in scenario ii, and slightly less importance in scenario iii. Again, the betweenness measure provides much more extreme results for node 5.

Table 3 Centrality measures for the grid network in Example 2.

	Node	Degree	Closeness	Betweenness	Eigenvalue	TNc(1)
i	1	8%	10%	4%	9%	10%
	2	13%	12%	14%	12%	12%
	3	8%	10%	4%	9%	10%
	4	13%	12%	14%	12%	12%
	5	17%	15%	30%	17%	14%
	6	13%	12%	14%	12%	12%
	7	8%	10%	4%	9%	10%
	8	13%	12%	14%	12%	12%
	9	8%	10%	4%	9%	10%
	Node	Degree	Closeness	Betweenness	Eigenvalue	TNc(3)
ii	1	4%	7%	0%	3%	8%
	2	13%	14%	14%	15%	12%
	3	4%	7%	0%	3%	8%
	4	13%	14%	14%	15%	12%
	5	36%	16%	44%	29%	19%
	6	13%	14%	14%	15%	12%
	7	4%	7%	0%	3%	8%
	8	13%	14%	14%	15%	12%
	9	4%	7%	0%	3%	8%
	Node	Degree	Closeness	Betweenness	Eigenvalue	TNc(3)
iii	1	11%	12%	11%	12%	11%
	2	13%	12%	14%	12%	12%
	3	11%	12%	11%	12%	11%
	4	13%	12%	14%	12%	12%
	5	5%	6%	0%	5%	8%
	6	13%	12%	14%	12%	12%
	7	11%	12%	11%	12%	11%
	8	13%	12%	14%	12%	12%
	9	11%	12%	11%	12%	11%

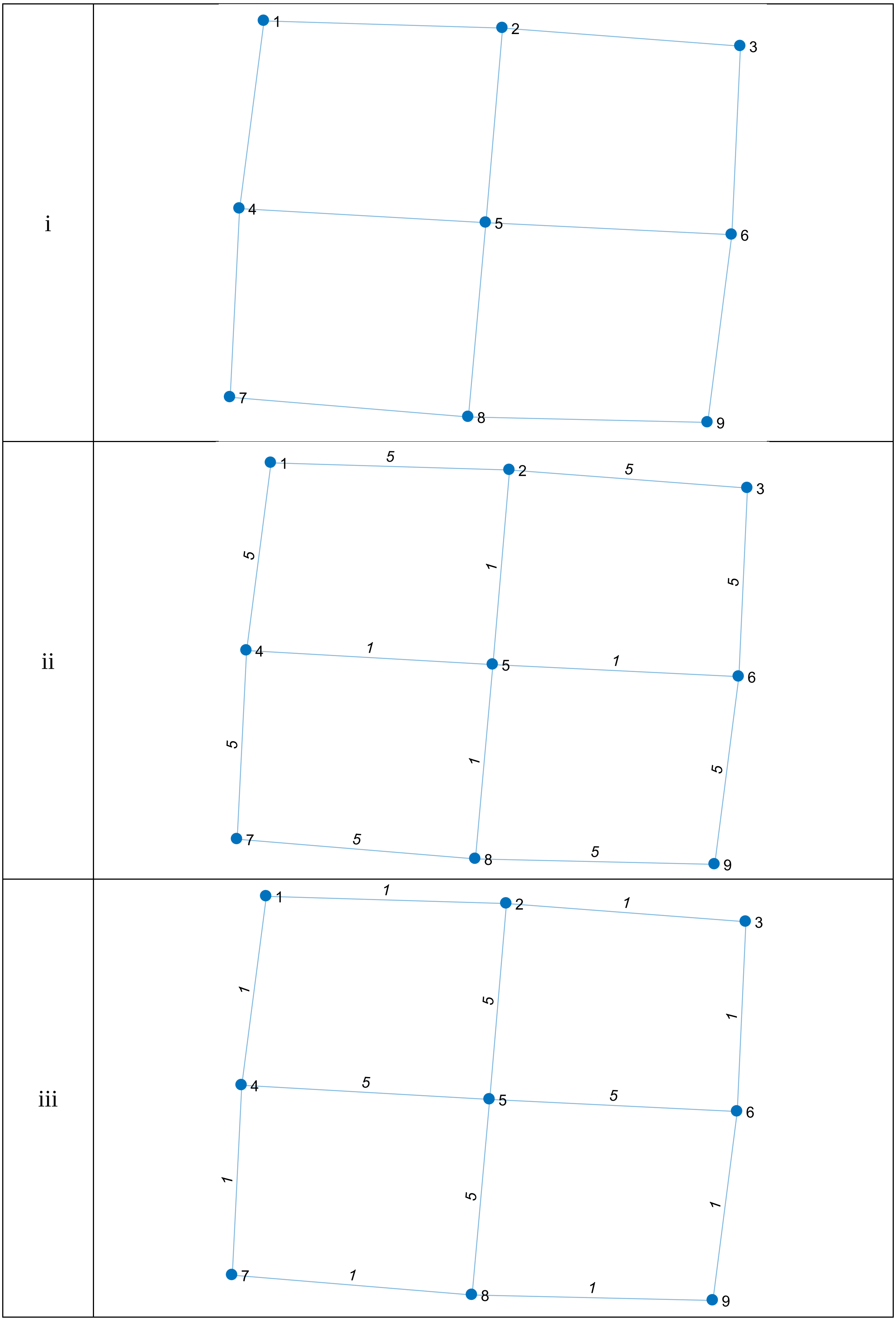

Figure 3 Network topology for the grid network in Example 2.

4.1.3 *Example 3: Weighted hub network*

The following example illustrated in Figure 4 represents a two-tier hub network, with node 13 as the top tier hub, and nodes 4, 8, and 12 as the lower tier hubs. Furthermore, each arc has a different weight representing distance.

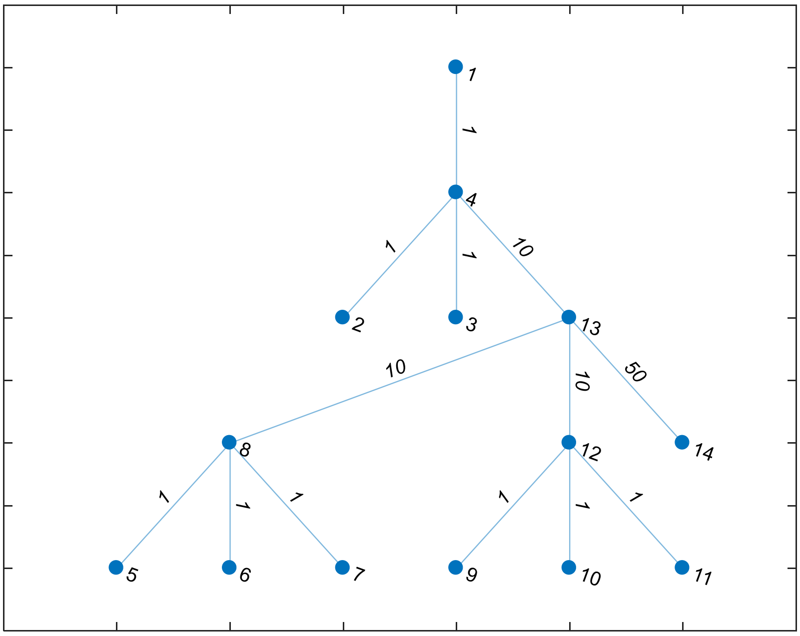

Figure 4 Network topology for the hub network in Example 3.

The results are presented in Figure 5-Figure 9, depicting the graphical representation of the centrality measures of degree, closeness, betweenness, eigenvalue, and TNc(3), respectively. From the five figures, it is clear that all measures except TNc(3) failed to some extent to capture the network effect on the centrality of the different nodes, particularly when considering the highest importance of node 13, followed by the importance of nodes 4, 8, and 12.

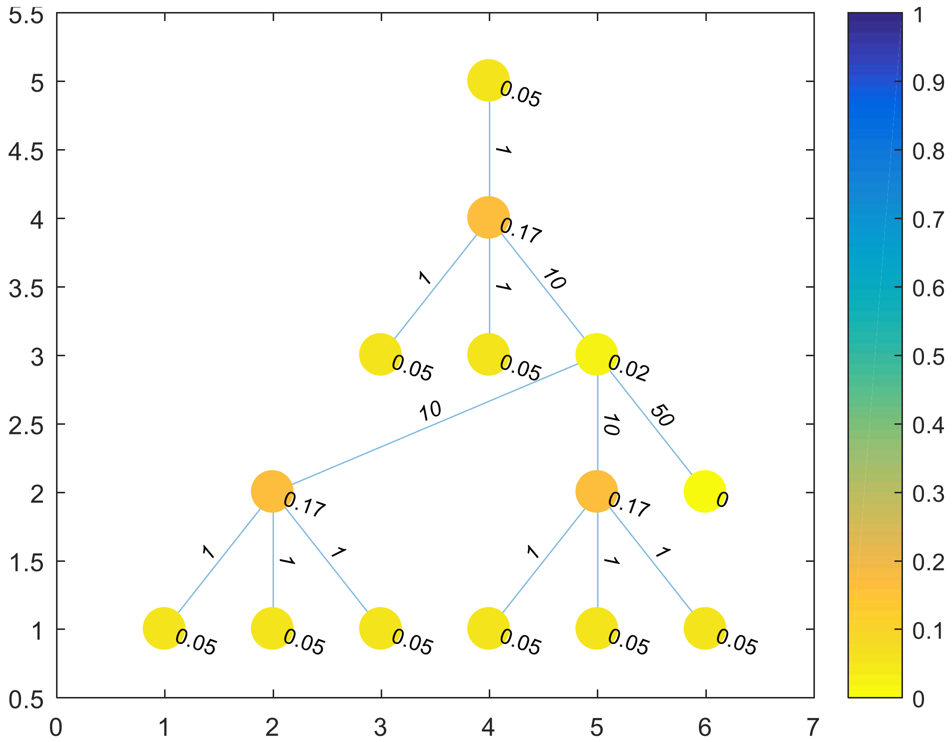

Figure 5 Results for the degree centrality measure in Example 3.

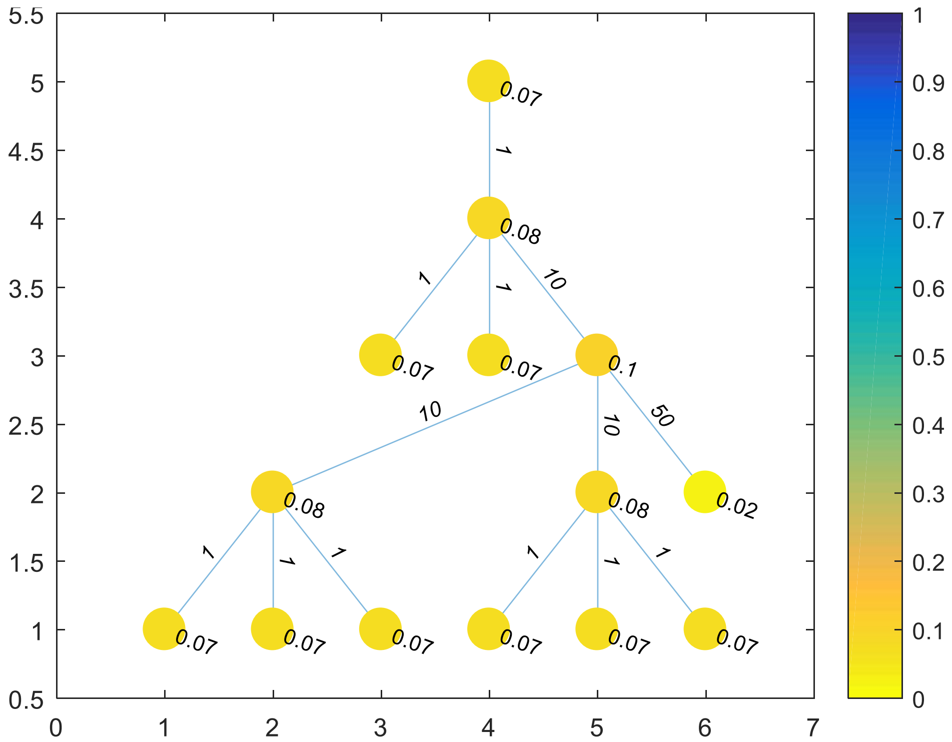

Figure 6 Results for the closeness centrality measure in Example 3.

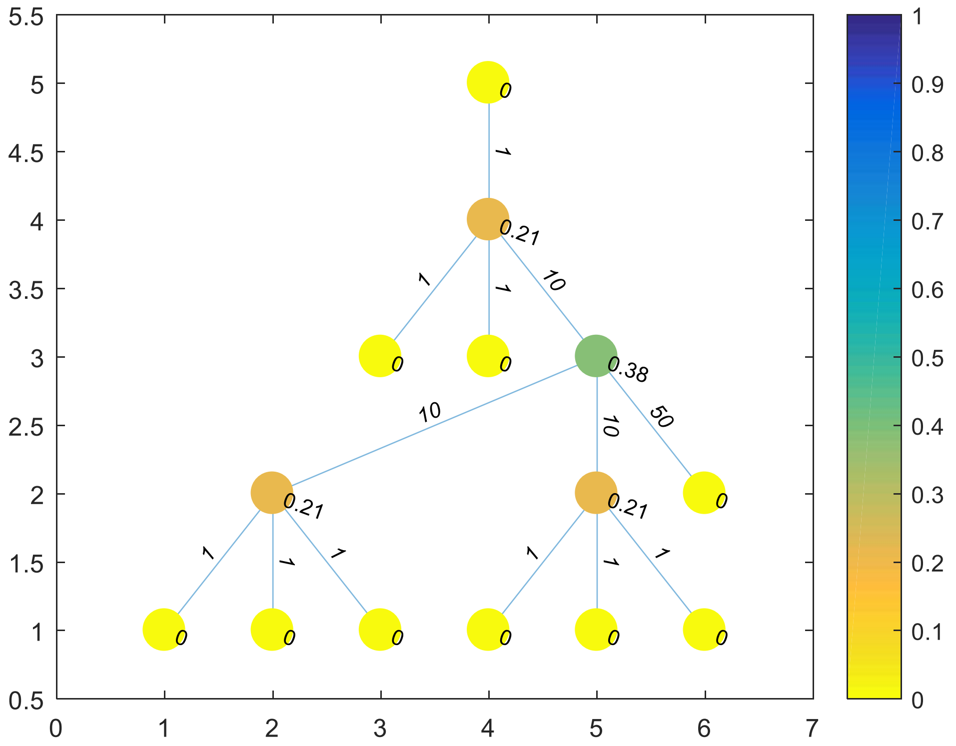

Figure 7 Results for the betweenness centrality measure in Example 3.

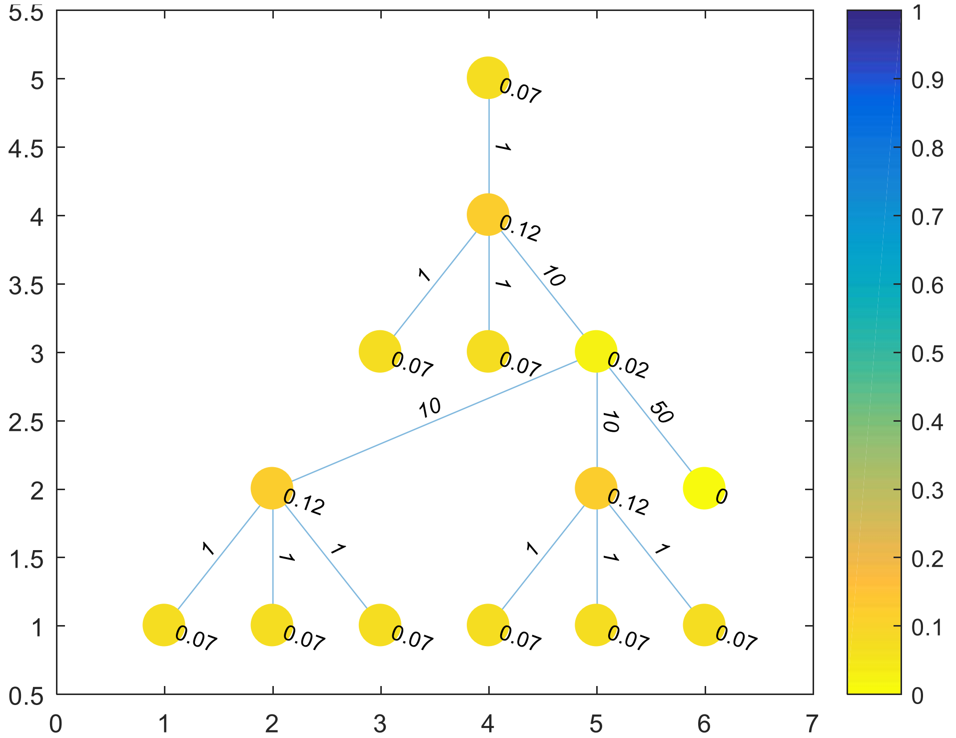

Figure 8 Results for the eigenvalue centrality measure in Example 3.

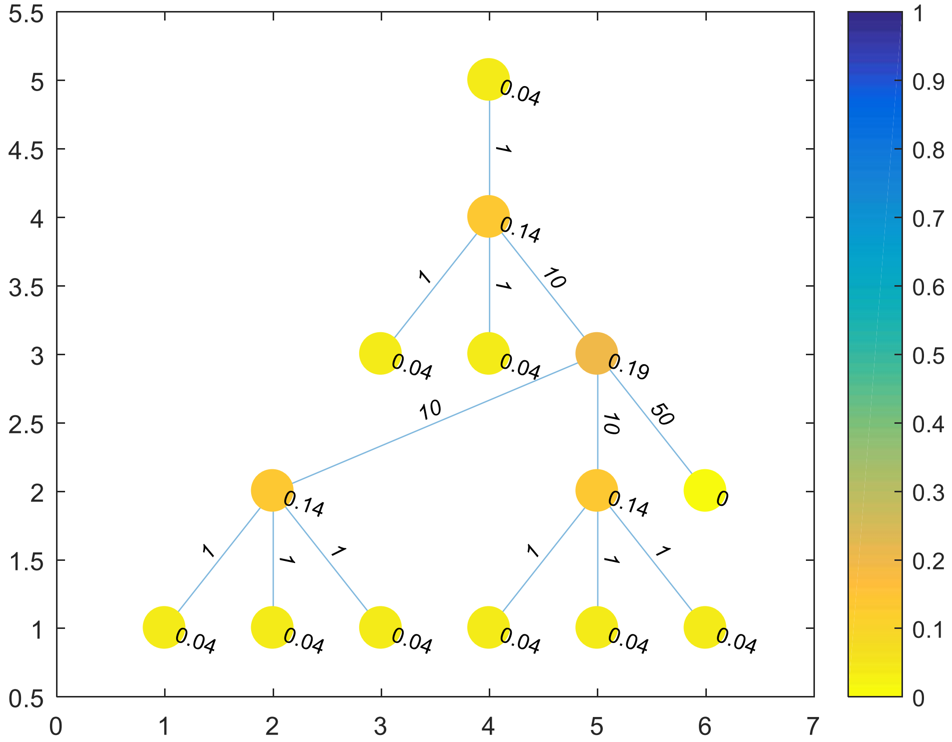

Figure 9 Results for the TNc(3) centrality measure in Example 3.

4.2 Adding demands

Another advantage of the TNc model is its ability to integrate demand, as the utility is calculated directly. For the analysis of networks with demands, Examples 1 and 2 are revisited, with an additional analysis based on the known Sioux Falls network (Bar-Gera, 2001).

4.2.1 Example 4: A 4 nodes network with demands

Table 4 presents the origin-destination demand matrix used for the introduction of the demand, and Table 5 the corresponding results. Based on the demand matrix, the centrality of node 4 increases, since it generates and attracts more demand than node 2.

Table 4 OD demand matrix.

To / From	1	2	3	4	Total
1	0	4	3	6	13
2	1	0	3	4	8
3	2	0	0	3	5
4	1	3	7	0	11
Total	4	7	13	13	

Table 5 Comparison of the centrality measures with and without demands.

	TNc	
Node	Without demand – TNc(3)	With demand – TNc(4)
1	15%	13%
2	34%	27%
3	30%	30%
4	22%	30%

4.2.2 Example 5: A grid network with demands

Consider again the grid network presented in Figure 3, modeling the flow from nodes 1, 2, and 3 to nodes 5, 7, 8, and 9, as presented in Table 6. All other nodes do not generate or attract demand. All flows are equal.

Table 6 OD demand for Example 2.

To / From	1	2	3	4	5	6	7	8	9
1					10		10	10	10
2					10		10	10	10
3					10		10	10	10
4									
5									
6									
7									
8									
9									

Table 7 summarizes the differences in the centrality of the nodes when demand is added to the grid network. The introduction of demand changes the importance of the nodes: (1) Nodes 4 and 6 are transshipment nodes, so that their importance decreases as they are redundant in terms of connectivity. (2) Node 2 increases its importance, as it is both central for the generating nodes as well as a generating node in itself. Furthermore, nodes 7, 8, and 9 are dependent on nodes 1, 2, and 3, as they only attract demand.

Table 7 TNc measures for the grid network, with and without demand.

	Unweighted		*w*=1 for arcs connecting node #5, *w*=5 otherwise		*w*=5 for arcs connecting node #5, *w*=1 otherwise	
Node	No demand – TNc(1)	With demand – TNc(2)	No demand – TNc(3)	With demand – TNc(4)	No demand – TNc(3)	With demand – TNc(4)
1	10%	12%	8%	12%	11%	14%
2	12%	18%	12%	18%	12%	15%
3	10%	12%	8%	12%	11%	14%
4	12%	5%	12%	4%	12%	8%
5	14%	19%	19%	23%	8%	10%
6	12%	5%	12%	4%	12%	8%
7	10%	9%	8%	8%	11%	11%
8	12%	11%	12%	11%	12%	10%
9	10%	9%	8%	8%	11%	11%

4.2.3 Example 6: Sioux Falls network

In order to analyze a larger network with more realistic topology and demand properties, the well-known Sioux Falls network is used (Bar-Gera, 2001). Figure 10 illustrates the network topology while Figure 11 provides a graphical representation of the origin-destination demand matrix, in which the darker cells have a higher demand level for the corresponding OD pair. Finally, Figure 12 summarizes the change in the importance of the nodes when demand is introduced. It is evident that the introduction of demand is reflected logically by the TNc measure, specifically with the higher importance of node 10 that is due to the large amount of demand it generates or attracts. The OD matrix also clearly shows that the higher demand flows within nodes 11-24 are reflected by an increase of importance, and the simultaneous decrease of importance for nodes 1-8.

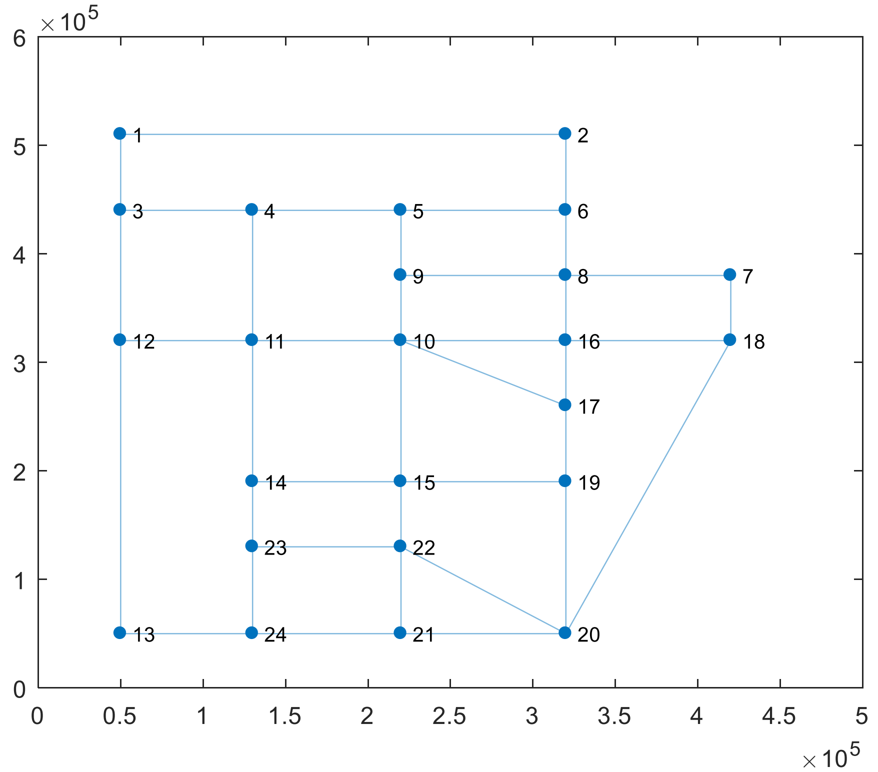

Figure 10 The Sioux Falls network.

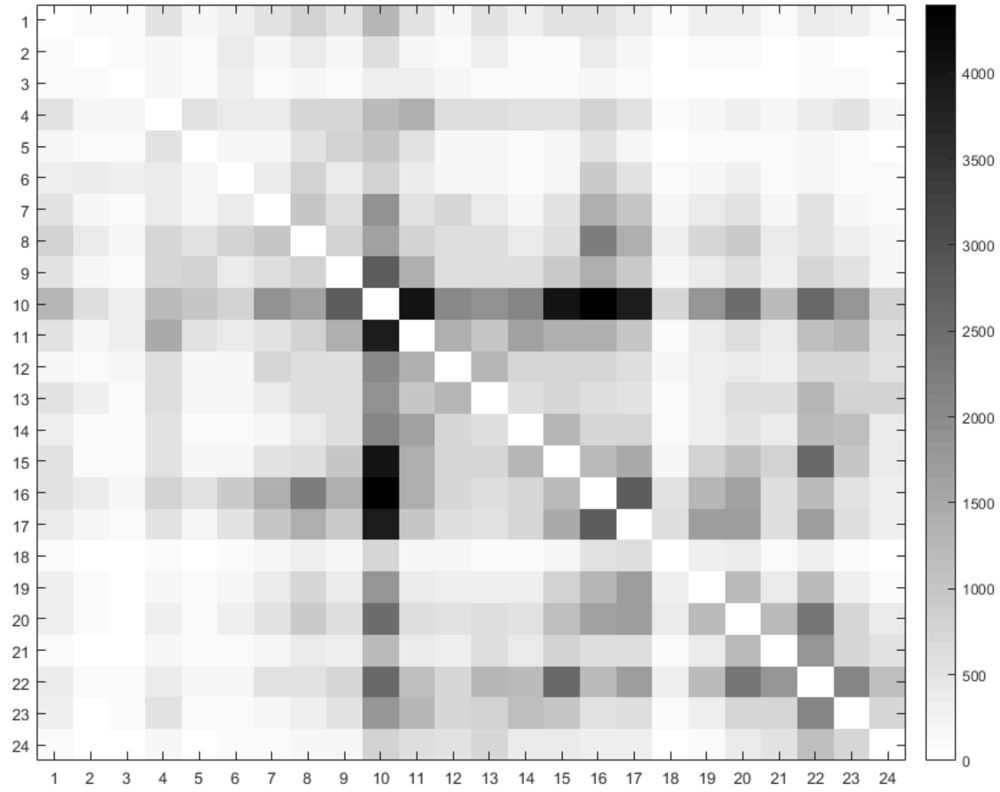

Figure 11 A graphical representation of OD demand matrix of the Sioux Falls network.

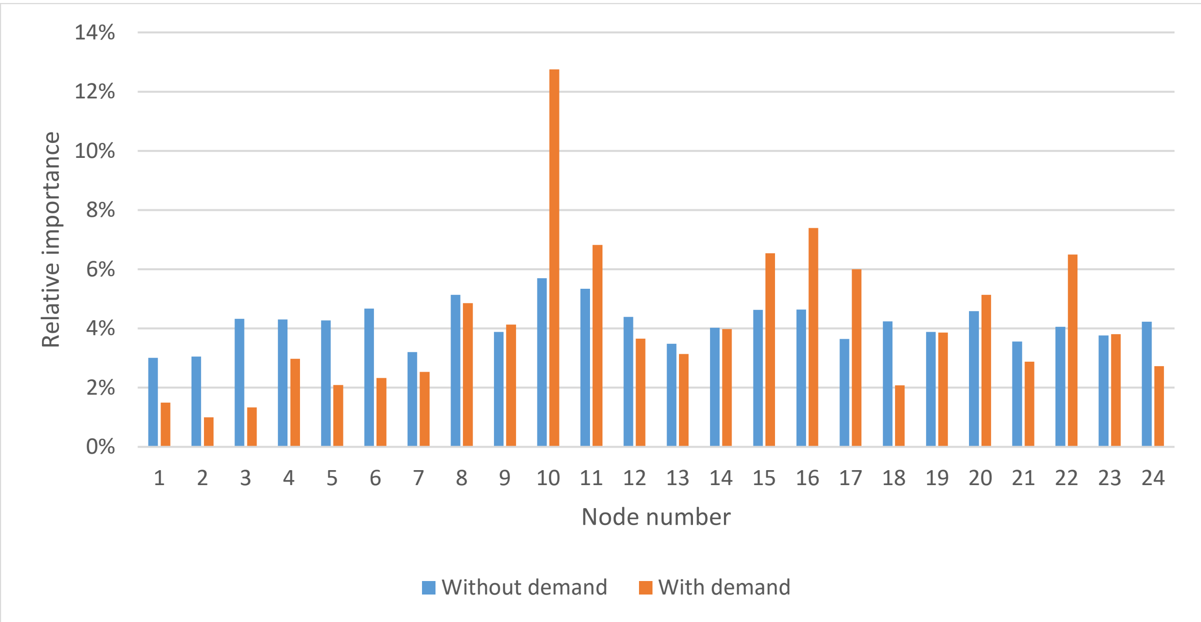

Figure 12 TNc centrality measures with demand (TNc(3)) and without demand (TNc(4)).

4.3 Complexity and algorithmic efficiency

As described in the introduction, the direct calculation of the Shapley value can be time consuming, as it is directly related to the number $2^n - 1$ of non-empty coalitions in the network. As the aim of this paper is to introduce a new approach which is a better suited for the assessment of transportation networks, when compared to common centrality measures, in Section 4.1 and 4.2 the algorithms were not optimized for performance. This is indeed evident from Table 8, which provides a break down of the calculation time for: a) the utility values of all the coalitions, and b) the Shapley values (these times are practically independent from the specific utility function v_i used). Table 2 shows that the first calculation time is much larger than the second one. The main reason for that is a "brute force" approach was used for the calculation of the utility values, since the subgraphs induced by all possible subsets of nodes were considered. In contrast, the computation times for the classical centrality measures were negligible, hence they are not reported in Table 2 (however, the proposed approach provides a better evaluation of centrality, as shown by the numerical results reported in Sections 4.1 and 4.2). Nevertheless, it is possible to reduce the calculation efforts for the proposed measures of centrality if the calculations are based only on the connected subgraphs. Given that, it is also possible to efficiently enumerate all the connected subgraphs of a graph based on a dynamic programming algorithm (Rehrmann, 1998). Furthermore, as the utility value of each connected subgraph is not affected by the ones of the other connected subgraphs, it is possible to utilize also a parallel algorithm for the computations. With regard to the Shapley value calculation, efficient polynomial-time sampling-based approximation approaches could be also used (Castro et al., 2009). It is also worth noting that also Proposition 2 suggests a way to reduce the Shapley value calculation time.

Table 8 Complexity and computation times for the investigated networks.

Problem	Number of nodes	Number of non-empty coalitions	Utility values calculation time	Shapley values calculation time
1 (4-nodes)	4	15	0.29 sec	0.05 sec
2 (grid)	9	511	1.30 sec	0.14 sec
3 (Hub)	14	16,383	35.30 sec	0.50 sec
4 (Sioux Falls)	24	16,777,215	~12 H	133.00 sec

5 CONCLUSIONS

This paper introduces an innovative approach for transportation network analysis, based on TU games. The model analyzes the centrality of nodes, based on the network topology, OD demand matrix (weights of nodes), and distance or delay matrix (weights of arcs). The centrality is determined based on the Shapley value, which calculates the average contribution of each node in relation to all possible coalitions. This model is superior to classical centrality measures due to the use of a utility function that can integrate various network properties of weights and demand. It also provides better assessment of the centrality of nodes. Since these classical centrality measures are not based on congestion, they cannot capture it. Hence, for a fair comparison we have considered utility functions that are not based on congestion. Nevertheless, the proposed approach has the advantage that, in principle, it can be applied to any form of utility function, including those that capture congestion. An extension to this case and a comparison with the results of this paper will be the subject of a future work.

Further related research will include (1) the development of an efficient algorithm for large-scale networks with the integration of other TU games measures, such as the Myerson value (Myerson, 1977), and the use of Monte Carlo methods (i.e., coalition sampling) for the approximate computation of the Shapley value; (2) the incorporation of stochastic properties such as link failure, travel time variability, etc.; (3) the use of more complex utility functions, more suitable, e.g., to model congestion (as an example, at the cost of an additional computational burden, the utility of every coalition - or the one of every connected subgraph - could be defined as the optimal value of an associated traffic assignment problem, such as the Wardrop User Equilibrium assignment problem and its stochastic version: see Patriksson (2015); (4) the modeling of public transport networks, including the realization of routes by travelers, transfers, frequencies, link capacities, etc.; (5) the development of a model with different types of agents such as regular agents (players) and key agents.

As an example related to point (3) above, we conclude this discussion by outlining the following possible formulation of a TNc game based on traffic assignment, whose theoretical and numerical investigation will be the subject of further research. Let *TA(V,E,W',D)* be a traffic assignment algorithm that, given a network topology *(V, E)*, an arc delay matrix *W'*, and a demand matrix *D*, returns the assignment in the form of two three-dimensional arrays, *F* and *T*, defined as follows. The former provides the flow from each origin *i* to each destination *j*, via each path *k* (which belongs to a set *K* of possible paths); while the latter represents the induced travel time from each origin *i* to each destination *j*, along each path *k*. It is worth noting that the traffic assignment algorithm above can be applied not only to the original graph *(V, E)*, but also to each subgraph induced by any subset *S* of its nodes (in this case, we denote by *F(S)* and *T(S)* the corresponding three-dimensional arrays). Following our TNc games' formulation, the TNc game based on traffic assignment (here denoted as TNc(8)) can be formulated in terms of the following utility function:

$$v_8(S) := \sum_{i,j \in S, i \neq j, k \in K:\, T(S)_{i,j,k} < \infty} F(S)_{i,j,k} \left(M - T(S)_{i,j,k} \right), \tag{20}$$

where

$$M := \max_{S \subseteq V} \left(\max_{i,j \in S, i \neq j, k \in K:\, T(S)_{i,j,k} < \infty} \left(T(S)_{i,j,k} \right) \right) + 1. \tag{21}$$

6 APPENDIX

Proof of Proposition 1. Let us first prove this proposition in the case of the utility function v_1. Each ordered pair of nodes (i, j) belonging to a coalition S gives a contribution $x(S)_{i,j} = 1$ to the utility of that coalition if the nodes are in the same connected component of $G(S)$, otherwise $x(S)_{i,j} = 0$. Thus one gets $v_1(S) = \sum_{i,j\in S, i\neq j} x(S)_{i,j} = \sum_{k=1}^{C_S} \sum_{i,j\in S_k, i\neq j} x(S)_{i,j}$. Since for a connected component with $|S_k|$ nodes we have $\sum_{i,j\in S_k, i\neq j} x(S)_{i,j} = |S_k|(|S_k|-1)$, we get $v_1(S) = \sum_{k=1}^{C_S} \sum_{i,j\in S_k, i\neq j} x(S)_{i,j} = \sum_{k=1}^{C_S} |S_k|(|S_k|-1)$. The proof for the other utility functions is analogous to this one (e.g., the term $x(S)_{i,j}$ is replaced by $D_{i,j}x(S)_{i,j}$ for the case of the utility function v_2, which can be different from 0 only if $x(S)_{i,j} = 1$).

QED

Proof of Proposition 2. We first consider the case of the utility function v_1. We begin the proof considering the following expression of the Shapley value:

$$Sh_1(i) := \sum_{S\subseteq N} \left(\left(v_1(S) - v_1(S\setminus\{i\})\right) \frac{(|S|-1)!(n-|S|)!}{n!} \right),$$

which we can rewrite as:

$$Sh_1(i) := \sum_{S\subseteq N} \left(v_1(S\cup\{i\}) \frac{(|S|)!(n-|S|-1)!}{n!} \right) - \sum_{S\subseteq N} \left(v_1(S) \frac{(|S|)!(n-|S|-1)!}{n!} \right). \tag{22}$$

From Proposition 1 a), $v_1(S)$ can be expressed in terms of the sizes of all the connected components of the subgraph of G induced by S, i.e., $v_1(S) = \sum_{k=1}^{C_S} |S_k(S)|(|S_k(S)|-1)$, where we have emphasized the dependence of the S_k on S. Therefore, in order to compute the Shapley value one can calculate for how many subsets S of $N = V$ of given cardinality k, the connected subgraph $G(S_C)$ induced by a generic $S_C \in \Sigma_C$ is a connected component of the subgraph $G(S)$. Clearly $G(S_C)$ is a connected component of $G(S)$ if and only if both the following conditions hold: $S_C \subseteq S$ and $N_C \cap S = \varnothing$. Thus, the number of all subsets S that satisfy these two conditions is $2^{|N|-|S_C|-|N_C|}$. The cardinalities of such subsets take values between $|S_C|$ and $|N|-|N_C|$, and, for $k = |S_C|,...,|N|-|N_C|$, the number of subsets S with cardinality k

that satisfy the two conditions above is $\binom{|N|-|S_C|-|N_C|}{k-|S_C|}$, where we have used the binomial coefficient. Therefore, we can rewrite the second summation $\sum_{S\subseteq N}\left(v_1(S)\frac{(|S|)!(n-|S|-1)!}{n!}\right)$ in (19) as

$$\sum_{S_C\subseteq\Sigma_C}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|}{k-|S_C|}\right)\right].$$

Similarly, we have to find all different subsets of nodes S for which any generic $S_C\in\Sigma_C$ is a connected component of the subgraph $G(S\cup\{i\})$, and their cardinalities. With the same arguments as above, it follows that $G(S_C)$ is a connected component of $G(S\cup\{i\})$ if and only if both the following conditions hold: $S_C\subseteq S\cup\{i\}$ and $N_C\cap(S\cup\{i\})=\varnothing$. In order to compute the number of all such subsets, we have to distinguish among the three following cases:

a) if $i\in S_C$, then the number of such subsets is again $2^{|N|-|S_C|-|N_C|}$;
b) if $i\in N_C$, then the number of such subsets is 0;
c) if $i\in N\setminus(S_C\cup N_C)$, then the number of such subsets is $2^{|N|-|S_C|-|N_C|-1}$.

Correspondingly, the number of such subsets of cardinality k is

a) $\binom{|N|-|S_C|-|N_C|}{k-|S_C|}$ (for $k=|S_C|,...,|N|-|N_C|$);
b) 0;
c) $\binom{|N|-|S_C|-|N_C|-1}{k-|S_C|-1}$ (for $k=|S_C|-1,...,|N|-|N_C|$).

Combining all the results obtained above, we can rewrite the first summation $\sum_{S\subseteq N}\left(v_1(S\cup\{i\})\frac{(|S|)!(n-|S|-1)!}{n!}\right)$ in (22) as

$$\sum_{S_C\subseteq\Sigma_C,i\in S_C}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|}{k-|S_C|}\right)\right]$$
$$+\sum_{S_C\subseteq\Sigma_C,i\in N\setminus(S_C\cup N_C)}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|-1}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|-1}{k-|S_C|-1}\right)\right].$$

We also observe that the following equality holds for the second summation in (22):

$$\sum_{S_C \subseteq \Sigma_C}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|}{k-|S_C|}\right)\right]$$

$$= \sum_{S_C \subseteq \Sigma_C, i\in S_C}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|}{k-|S_C|}\right)\right]$$

$$+ \sum_{S_C \subseteq \Sigma_C, i\in N_C}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|}{k-|S_C|}\right)\right]$$

$$+ \sum_{S_C \subseteq \Sigma_C, i\in N\setminus(S_C\cup N_C)}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|}{k-|S_C|}\right)\right].$$

Finally, combining the last two formulas with (22), and utilizing the formula

$$\binom{n-|S_C|-|N_C|}{k-|S_C|}-\binom{n-|S_C|-|N_C|-1}{k-|S_C|-1}=\binom{n-|S_C|-|N_C|-1}{k-|S_C|},$$

we obtain

$$Sh_1(i) = \sum_{S_C \subseteq \Sigma_C, i\in N\setminus(S_C\cup N_C)}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|-1}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|-1}{k-|S_C|-1}\right)\right]$$

$$- \sum_{S_C \subseteq \Sigma_C, i\in N_C}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|}{k-|S_C|}\right)\right]$$

$$- \sum_{S_C \subseteq \Sigma_C, i\in N\setminus(S_C\cup N_C)}\left[|S_C|(|S_C|-1)\sum_{k=|S_C|}^{|N|-|N_C|}\left(\frac{|k|!(n-|k|-1)!}{n!}\binom{n-|S_C|-|N_C|-1}{k-|S_C|}\right)\right].$$

The proof for the case of each of the other utility functions is obtained by simply replacing the term $|S_C|(|S_C|-1)$ in the proof above (which derives from formula (14)) with the related expression reported in the corresponding formula (15) - (19).

QED

Proof of Proposition 3. Let us first consider the case of the utility function v_1. Let S and U be disjoint subsets of V, and let $T = S \cup U$. Then we obtain

$$v_1(T) = \sum_{i,j\in T, i\neq j} x(T)_{i,j} = \sum_{i,j\in S, i\neq j} x(T)_{i,j} + \sum_{i,j\in U, i\neq j} x(T)_{i,j} + \sum_{i\in S, j\in U} x(T)_{i,j} + \sum_{i\in U, j\in S} x(T)_{i,j}\ .$$

For all i and j, we obtain $x(T)_{i,j} \geq x(S)_{i,j}$ and $x(T)_{i,j} \geq x(U)_{i,j}$ because if a path is present in the subgraph induced by S (or by U), it is also present in the subgraph induced by T. Therefore, from the above we obtain

$$\sum_{i,j\in S,i\neq j} x(T)_{i,j} + \sum_{i,j\in U,i\neq j} x(T)_{i,j} + \sum_{i\in S,\, j\in U} x(T)_{i,j} + \sum_{i\in U,\, j\in S} x(T)_{i,j}$$
$$\geq \sum_{i,j\in S,i\neq j} x(S)_{i,j} + \sum_{i,j\in U,i\neq j} x(U)_{i,j} + \sum_{i\in S,\, j\in U} x(T)_{i,j} + \sum_{i\in U,\, j\in S} x(T)_{i,j}$$
$$\geq \sum_{i,j\in S,i\neq j} x(S)_{i,j} + \sum_{i,j\in U,i\neq j} x(U)_{i,j}$$
$$= v_1(S)+v_1(U),$$

and finally, $v_1(T)\geq v_1(S)+v_1(U)$.

We next prove the superadditivity of v_3. Then with the same notations as before, we obtain

$$\begin{aligned}
v_3(T) &= \sum_{i,j\in T,i\neq j:\, sp(T)_{i,j}<\infty} \left(M - sp(T)_{i,j}\right) \\
&= \sum_{i,j\in S,i\neq j:\, sp(T)_{i,j}<\infty} \left(M - sp(T)_{i,j}\right) + \sum_{i,j\in U,i\neq j:\, sp(T)_{i,j}<\infty} \left(M - sp(T)_{i,j}\right) \\
&\quad + \sum_{i\in S,\, j\in U:\, sp(T)_{i,j}<\infty} \left(M - sp(T)_{i,j}\right) + \sum_{i\in U,\, j\in S:\, sp(T)_{i,j}<\infty} \left(M - sp(T)_{i,j}\right) \\
&\geq \sum_{i,j\in S,i\neq j:\, sp(S)_{i,j}<\infty} \left(M - sp(S)_{i,j}\right) + \sum_{i,j\in U,i\neq j:\, sp(U)_{i,j}<\infty} \left(M - sp(U)_{i,j}\right) \\
&\quad + \sum_{i\in S,\, j\in U:\, sp(T)_{i,j}<\infty} \left(M - sp(T)_{i,j}\right) + \sum_{i\in U,\, j\in S:\, sp(T)_{i,j}<\infty} \left(M - sp(T)_{i,j}\right) \\
&\geq v_3(S)+v_3(U),
\end{aligned}.$$

since for $i,j\in S, i\neq j: sp(S)_{i,j}<\infty$ one has $sp(T)_{i,j}<\infty$ and $sp(T)_{i,j}\leq sp(S)_{i,j}$ and similarly, for $i,j\in U, i\neq j: sp(U)_{i,j}<\infty$ one has $sp(T)_{i,j}<\infty$ and $sp(T)_{i,j}\leq sp(U)_{i,j}$.

Analogously, for the case of v_5, we get

$$\begin{aligned}
v_5(T) &= \sum_{i,j\in T,i\neq j:sp(T)_{i,j}<\infty} \frac{1}{sp(T)_{i,j}+1} \\
&= \sum_{i,j\in S,i\neq j:sp(T)_{i,j}<\infty} \frac{1}{sp(T)_{i,j}+1} + \sum_{i,j\in U,i\neq j:sp(T)_{i,j}<\infty} \frac{1}{sp(T)_{i,j}+1} \\
&\quad + \sum_{i\in S,\, j\in U, sp(T)_{i,j}<\infty} \frac{1}{sp(T)_{i,j}+1} + \sum_{i\in U,\, j\in S:sp(T)_{i,j}<\infty} \frac{1}{sp(T)_{i,j}+1} \\
&\geq \sum_{i,j\in S,i\neq j:sp(S)_{i,j}<\infty} \frac{1}{sp(S)_{i,j}+1} + \sum_{i,j\in U,i\neq j:sp(U)_{i,j}<\infty} \frac{1}{sp(U)_{i,j}+1} \\
&\quad + \sum_{i\in S,\, j\in U, sp(T)_{i,j}<\infty} \frac{1}{sp(T)_{i,j}+1} + \sum_{i\in U,\, j\in S:sp(T)_{i,j}<\infty} \frac{1}{sp(T)_{i,j}+1} \\
&\geq v_5(S)+v_5(U).
\end{aligned}$$

Finally, the superadditivity of v_2, v_4 and v_6, respectively, is proved in a similar way as the respective proof for v_1, v_3 and v_5, utilizing the non-negativity of the elements of the origin-destination matrix D.

Monotonicity can be proved either directly from the definitions of the utility functions, or by observing that it follows from the superadditivity and the fact that each node has 0 utility if taken alone (i.e., for $l=1,...6$ and for each of $S \subset N$ and $i \in N \setminus S$, one has $v_l(S \cup \{i\}) \geq v_l(S) + v_l(\{i\}) = v_l(S)$).

QED

Proof of Proposition 4. A simple counterexample in which superadditivity of the utility function v_7 does not hold is provided by a triangular graph, assuming all the arc weights are equal to 1. Indeed, if $S=\{1,2\}$ and $U=\{3\}$, one obtains

a) $v_7(S) = \dfrac{1}{2\cdot 1+1} = \dfrac{1}{3}$;

b) $v_7(U) = \dfrac{1}{0+1} = 1$;

c) $v_7(S \cup U) = \dfrac{1}{6\cdot 1+1} = \dfrac{1}{7}$.

Thus, $v_7(S \cup U) = \dfrac{1}{7} < v_7(S) + v_7(U) = \dfrac{1}{3} + 1 = \dfrac{4}{3}$, and v_7 is not superadditive. Incidentally, the counterexample also shows that v_7 is not monotonic because $v_7(S \cup U) < v_7(S)$.

QED

Proof of Proposition 5. According to Theorem 17.54 in Maschler et al. (2013), one of the equivalent characterizations of a convex TU game is as follows: a TU game is convex if and only if, for all $S \subseteq T \subseteq N$, and every $i \in N \setminus T$, one has $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. Now, for $v = v_1$, let us consider a TN for which all the following hold:

a) The subgraph of G induced by the subset S is made of $|S|$ disconnected nodes;

b) The node i connects all the nodes of S through edges in G;

c) $T = S \cup \{j\}$, where the node $j \neq i$ also connects all the nodes of S through edges in G;

d) $|S| \geq 3$.

From the construction above, we obtain:

a) $v_1(S \cup \{i\}) = (|S|+1)|S|$ (because $G(S \cup \{i\})$ is connected, and has $|S|+1$ nodes);

b) $v_1(S) = 0$ (because $G(S)$ is completely disconnected);

c) $v_1(T \cup \{i\}) = (|T|+1)|T| = (|S|+2)(|S|+1)$ (because $G(T \cup \{i\})$ is connected, and has $|T|+1 = |S|+2$ nodes);

d) $v_1(T) = |T|(|T|-1) = (|S|+1)|S|$ (because $G(T)$ is connected, and has $|T| = |S|+1$ nodes).

Therefore, we obtain

$$v_1(S\cup\{i\})-v_1(S)=\left(|S|+1\right)|S|-0=\left(|S|+1\right)|S|,$$

and

$$v_1(T\cup\{i\})-v_1(T)=\left(|S|+2\right)\left(|S|+1\right)-\left(|S|+1\right)|S|=2\left(|S|+1\right).$$

In conclusion, by solving the quadratic inequality $2\left(|S|+1\right)<|S|^2+|S|$, we obtain

$$v_1(S\cup\{i\})-v_1(S)>v_1(T\cup\{i\})-v_1(T),$$

since $|S|\geq 3$. Therefore, we conclude that this particular TNc game $\left(N,v_1\right)$ is not convex.

Similarly, for the case of the game $\left(N,v_3\right)$, we consider a TN for which all the following hold:

a) The subgraph of G induced by the subset S is made of $|S|$ disconnected nodes;
b) The node i connects all the nodes of S through edges in G;
c) $T=S\cup\{j\}$, where the node $j\neq i$ also connects all the nodes of S through edges in G;
d) $|S|\geq 3$;
e) $V=S\cup\{i\}\cup\{j\}$.

From the construction above, we obtain:

a) $M=3$;
b) $v_3(S\cup\{i\})=(3-2)\cdot\dfrac{|S|\left(|S|-1\right)}{2}\cdot 2+(3-1)\cdot|S|\cdot 2=|S|\left(|S|-1\right)+4|S|=|S|^2+3|S|$;
c) $v_3(S)=0$ (because $G(S)$ is completely disconnected);
d) $$\begin{aligned}v_3(T\cup\{i\})&=(3-2)\cdot\frac{|S|\left(|S|-1\right)}{2}\cdot 2+(3-1)\cdot|S|\cdot 2+(3-1)\cdot|S|\cdot 2+2\\&=|S|\left(|S|-1\right)+8|S|+2=|S|^2+7|S|+2.\end{aligned}$$
e) $v_3(T)=(3-2)\cdot\dfrac{|S|\left(|S|-1\right)}{2}\cdot 2+(3-1)\cdot|S|\cdot 2=|S|\left(|S|-1\right)+4|S|=|S|^2+3|S|$.

Therefore, we obtain

$$v_3(S\cup\{i\})-v_3(S)=|S|^2+3|S|-0=|S|^2+3|S|,$$

and

$$v_3(T\cup\{i\})-v_3(T)=|S|^2+7|S|+2-\left(|S|^2+3|S|\right)=4|S|+2.$$

In conclusion, by solving the quadratic inequality $4|S|+2<|S|^2+3|S|$, we obtain

$$v_3(S\cup\{i\})-v_3(S)>v_3(T\cup\{i\})-v_3(T)$$

since $|S|\geq 3$. Therefore, we conclude that this particular TNc game (N,v_3) is not convex.

For the case of the utility function v_5, we consider the same graph as above for v_1, with unit weights for the arcs. We obtain the following:

a) $v_5(S\cup\{i\})=\frac{1}{2}\cdot 2|S|+\frac{1}{3}\frac{|S|(|S|-1)}{2}\cdot 2=|S|+\frac{1}{3}|S|(|S|-1)$;

b) $v_5(S)=0$;

c) $v_5(T\cup\{i\})=|S|+\frac{1}{3}|S|(|S|-1)+|S|+1$;

d) $v_5(T)=\frac{1}{2}\cdot 2|S|+\frac{1}{3}\frac{|S|(|S|-1)}{2}\cdot 2=|S|+\frac{1}{3}|S|(|S|-1)$.

Therefore we obtain

$$v_5(S\cup\{i\})-v_5(S)=|S|+\frac{1}{3}|S|(|S|-1)-0=\frac{|S|^2+2|S|}{3},$$

and

$$v_5(T\cup\{i\})-v_5(T)=|S|+\frac{1}{3}|S|(|S|-1)+|S|+1-\left(|S|+\frac{1}{3}|S|(|S|-1)\right)=|S|+1.$$

In conclusion, by solving the quadratic inequality $|S|+1<\frac{|S|^2+2|S|}{3}$, we obtain

$$v_5(S\cup\{i\})-v_5(S)>v_5(T\cup\{i\})-v_5(T)$$

since $|S|\geq 3$. Therefore, we conclude that this particular TNc game (N,v_5) is not convex.

Finally, counterexamples similar to those above can be constructed with regard to the utility functions v_2, v_4, and v_6, starting from the counterexamples reported above for the utility functions v_1, v_3, and v_5, respectively.

For the case of the utility function v_7, one can consider the same counterexample presented in the proof of Proposition 6, since convexity implies superadditivity, which does not hold for that counterexample.

QED

Proof of Proposition 6. The proof is constructive, as it produces an element of the core. As an example, we consider the case of the game defined by the utility function v_1. (Similar proofs hold for all the other 5 successive cases.) We define the vector $y \in \mathbb{R}^n$ for which its elements y_i are defined as follows:

$$y_i := \sum_{j \in V, i \neq j} x(V)_{i,j} \ .$$

Since, for every coalition $S \subseteq N = V$, one has $x(N)_{i,j} \geq x(S)_{i,j}$, we conclude that the following holds:

$$\sum_{i \in S} y_i = \sum_{i \in S} \sum_{j \in V, i \neq j} x(V)_{i,j} \geq \sum_{i \in S} \sum_{j \in S, i \neq j} x(S)_{i,j} = \sum_{i,j \in S, i \neq j} x(S)_{i,j} = v_1(S) \ ,$$

Therefore, the vector y belongs to the core.

QED

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