

Crack bridging using linear elastic spring distribution with power-law variation of stiffness

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ABSTRACT

This article establishes a theoretical foundation and provides numerical evidence for a novel computational approach to simulate crack nucleation and propagation by finite crack advances within the framework of the coupled stress and energy criterion (CC) of finite fracture mechanics (FFM). The key innovation lies in bridging prospective crack paths, predicted by the stress criterion of CC, using a suitable Winkler-type continuous linear-elastic spring distribution. This allows a damage variable to be defined along these prospective finite crack paths, here called fictitious cracks. The effect of the spring stiffness variation on the elastic solution near a fictitious crack is analysed, focusing on how a power-law variation in spring stiffness affects the fictitious-crack tip stress singularity. Power-law variations of spring stiffness that eliminate this fictitious singularity are identified. Such a damageable spring distribution along the fictitious crack enables the seamless application of the energy criterion of CC to modelling virtual crack propagation through the progressive deactivation of springs within the bridged zone. Numerical tests show the effect of the power-law exponent of spring stiffness variation on both the stress state at the fictitious crack and on the energy release rate (ERR) calculated at the virtual-crack tip. Consequently, an optimal power law is proposed for modelling virtual crack propagation that successfully balances the elimination of the fictitious-crack tip singularity against the computational overhead associated with highly stiff spring distributions to achieve accurate ERR values. Finally, a new formulation of the path-independent J -integral is derived for a crack along a spring interface with varying stiffnesses.

1. Introduction

1.1. Minimisation of total energy in fracture mechanics

The (incremental) principle of minimum total energy (PMTE) forms the basis of many modern rate-independent models for a wide range of physical phenomena in solid mechanics. In this context, total energy is defined as the sum of potential and dissipated energy. For a comprehensive treatment of this methodology, starting from Biot's equation and leading to the concept of energetic solutions, see [1]. PMTE is typically applied in time-stepping procedures in incremental form. In the field of fracture mechanics, PMTE was introduced in the seminal work of [2], where the total energy is minimised with respect to crack advancement, that is, the creation of new crack surface. In fracture mechanics, a wide variety of crack representations and models have been developed.

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Nomenclature	
Latin Symbols	
Symbol	Definition
a	Real crack (length)
Δa	Fictitious crack (length)
Δa_v	Virtual crack (length)
a'	Modified crack length, $a' = a + \Delta a$
c	Constant in crack opening asymptotic law
d	Damage variable (0 = active spring, 1 = inactive spring)
E, E'	Young's modulus and reduced Young's modulus
G	ERR
G_I	Mode I ERR
G_{II}	Mode II ERR
G_{Griffith}	Analytical ERR from Griffith's solution
G_{Leblond}	Analytical ERR from Leblond's asymptotic formula
G_c	Fracture energy
k_0	Characteristic spring stiffness
$k(r)$	Spring stiffness with power-law variation
k_n	Normal spring stiffness
k_t	Tangential spring stiffness
k_s	Softening slope in cohesive law
K_I	Mode I SIF
K_{II}	Mode II SIF
K^{load}	SIF due to external loading
K_I^{bridge}	SIF due to bridging tractions
L	Plate length
M	Dimensionless multiplier for spring stiffness
N	Number of nodes (springs) along fictitious crack
r	Distance from the fictitious crack tip
$\mathbf{u}$	Displacement vector
W	Plate half-width
x, y	Cartesian coordinates
Greek Symbols	
α	Exponent in power-law spring stiffness variation
δ	Displacement jump vector on the fictitious crack, $\delta = [\delta_n, \delta_t]^T$
δ_n	Normal (opening) displacement jump
δ_t	Tangential (sliding) displacement jump
δ_{Lenci}	Dimensionless Lenci's parameter
Γ, Γ'	Upper and lower crack surfaces
ξ	Dimensionless parameter, $\xi = k_t/k_n$
κ	Kolosov's constant in plane strain/stress elasticity
λ	Stress singularity exponent
μ	Shear modulus
ν, ν'	Poisson's ratio and reduced Poisson's ratio
σ	Traction vector on the fictitious crack, $\sigma = [\sigma, \tau]^T$
σ	Normal traction on the fictitious crack
σ_c	Tensile strength
σ_{nom}	Nominal applied stress
σ_{yy}	Stress component in y -direction
τ	Shear traction on the fictitious crack
Subscripts and Superscripts	
$(\cdot)_n$	Normal component
$(\cdot)_t$	Tangential component
$(\cdot)_0$	Reference/characteristic value
$(\cdot)^+$	Positive part (Macaulay bracket)
$(\cdot)^-$	Negative part (Macaulay bracket)
$(\cdot)_{\text{Griffith}}$	From Griffith's analytical solution
$(\cdot)_{\text{Leblond}}$	From Amestoy and Leblond's asymptotic solution
$(\cdot)_v$	Virtual crack

$(\cdot)_{\text{bridge}}$	Due to bridging tractions
$(\cdot)_{\text{load}}$	Due to external loading
Acronyms	
Acronym	Full Form
AMA	Alternating Minimisation Algorithm
BVP	Boundary Value Problem
CC	Coupled Criterion
CZM	Cohesive Zone Model
ERR	Energy Release Rate
FEA	Finite Element Analysis
FEM	Finite Element Method
FFM	Finite Fracture Mechanics
LEBIM	Linear-Elastic Brittle Interface Model
LEFM	Linear Elastic Fracture Mechanics
PFM	Phase-Field Model (for fracture)
PMTE	Principle of Minimum Total Energy
PMTE-SC	PMTE subjected to a Strength Criterion
PUM	Partition of Unity Method
SIF	Stress Intensity Factor
VEM	Virtual Element Method
XFEM	eXtended Finite Element Method

From the standpoint of continuum solid mechanics, a key distinction concerns whether the crack surface is represented as a discontinuity in the solid or in a diffused, regularised form.

In recent decades, diffuse representations of crack surfaces have become particularly popular owing to their significant mathematical and computational advantages in variational approaches based on total energy minimisation. The most prominent example is provided by phase-field models for fracture (PFMs) [3–6].

Despite this progress, traditional representations of cracks as discontinuities in the solid domain continue to offer important advantages in theoretical developments and computational fracture models. These approaches are notable for their accuracy, computational efficiency, and reliable agreement between theoretical predictions and experimental observations. Consequently, there has been intensive recent development of discontinuous crack representations in computational mechanics, often in connection with modern finite element techniques, such as partition of unity methods (PUMs) [7], including the extended finite element method (XFEM) [8], unfitted finite element methods [9–11], and the virtual element method (VEM) [12,13].

1.2. Strength and energy criteria for fracture

Throughout the history of solid mechanics, two families of criteria have coexisted for predicting structural failure: local criteria based on strength and global criteria based on energy. Within the framework of finite fracture mechanics (FFM) [14], these two approaches were effectively integrated in the pioneering works of Leguillon [15,16], leading to the coupled criterion (CC). Precursors of this CC formulation can be found in [17–20]. Over the past 25 years, numerous successful applications of CC to challenging crack initiation and propagation problems have demonstrated the robustness of this approach, highlighting both the simplicity of its theoretical formulation and its strong agreement with experimental results. Alternative formulations of CC can be found in [21,22], while comprehensive reviews are given in [23,24]. It is worth recalling the limitations of classical fracture mechanics in predicting crack initiation at stress concentrations or weak stress singularities, due to its reliance on the assumption of pre-existing cracks.

The need to incorporate strength criteria into purely energetic fracture approaches, as introduced in [15,16], has also become evident in PMTE-based methods such as PFMs; see, for example, [6,25–32].

From now on, the focus will be on discontinuous representations of crack surfaces. When PMTE is coupled with a strength criterion, so that the minimisation of the total energy is subjected to fulfilment of a strength criterion, then it is essentially equivalent to CC of FFM [33]. Furthermore, it was shown in [33] that a finite increment in the crack surface as predicted by the PMTE-SC (principle of minimum total energy subjected to a strength criterion) is associated with tunnelling through the total-energy barrier, see [19, Section 12.6], and also [34,35] for clear illustrations of this phenomenon by analytical studies using beam theory.

1.3. Damage variable along a prospective crack path bridged by Winkler springs

The ultimate goal of the present study is to develop a novel framework for modelling the onset and propagation of internal cracks in the bulk material using PMTE-SC within a load-stepping procedure. However, minimising total energy with respect to crack advancement leads to a highly non-convex problem, rendering the development of a general, reliable, and robust computational strategy particularly challenging. To overcome this difficulty, an efficient approach is herein proposed in which a damage variable is introduced along a prospective crack path Δa , permitted by a strength criterion, by bridging Δa with a continuous distribution of

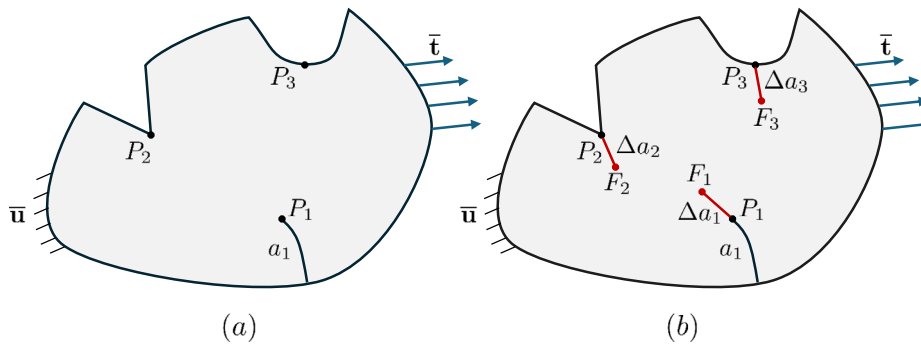


Fig. 1. Schematic of the fracture problem. (a) Physical configuration of a solid containing a crack a_1 , a V-notch, and a rounded notch, with a crack-tip singularity at P_1 , a weak stress singularity at P_2 , and a stress concentration at P_3 . (b) Corresponding numerical model with fictitious cracks Δa_i predicted by a strength criterion, with fictitious crack tips F_i ($i = 1, 2, 3$). Each Δa_i is bridged by a linear-elastic spring distribution with a suitable power-law stiffness variation to eliminate spurious stress singularities at F_i .

sufficiently stiff linear-elastic (Winkler) springs [36,37]. The state of these springs (active or inactive) is governed by the damage variable, in a manner similar to that employed in [38,39] for interface crack problems with prescribed crack paths.

The key advantage of such a quite simple approach is that the functional of total energy, the sum of the potential and dissipated energy, is separately convex with respect to the displacement field in the solid and the damage variable along the bridged prospective crack path Δa in the bulk. Hereinafter Δa is called the fictitious crack¹. The separate convexity of the total energy allows the application of a highly efficient and robust minimisation strategy referred to as alternate minimisation algorithm (AMA), sometimes referred to as fractional-step strategy or staggered scheme, see [1,3,38,39]. In the minimisation of total energy by AMA, the damage variable can activate or deactivate springs along the bridged zone Δa , modelling in this way the so-called virtual crack $\Delta a_v \subset \Delta a$.

Notably the size of Δa permitted by the strength criterion is often of the order of the Irwin length (fracture energy $\times$ elastic modulus/ strength²), which substantially reduces the set of possible crack advances to be considered in the restricted minimisation of the total energy when applying the PMTE-SC, in comparison with unrestricted minimisation in PMTE, as usually considered, e.g., in PFMs.

However, the introduction of a spring distribution along the fictitious crack Δa , although Δa is usually very small, necessarily alters the elastic solution in the neighbourhood of Δa when compared to the elastic solution for the case of perfect bonding along Δa , corresponding to the limit case of infinite stiffness of the spring distribution therein. Therefore, the challenging requirement for such a suitable spring distribution is that the elastic state in a fictitious crack neighbourhood must essentially be the same as in the case of perfect bonding along the bridged zone. Note that due to the Saint-Venant principle the effect of introducing a spring distribution along Δa , replacing the perfect bonding, usually rapidly vanishes with the distance from the fictitious crack Δa .

In the present approach, after numerically solving the problem shown in Fig. 1(a), the strength criterion of CC is applied to predict possible fictitious cracks Δa_i . These fictitious cracks are then equipped with a suitable spring distribution and an associated damage variable, resulting in the problem shown in Fig. 1(b). This problem is then used in the AMA to minimise the total energy and resolve competition between crack propagation at P_1 and crack initiations at P_2 and P_3 .

The specific objective of the present study is therefore to find a suitable continuous spring distribution that bridges the fictitious crack Δa , such that the difference between the elastic solution with this spring distribution along Δa and the case of perfect bonding along Δa is negligible in the bulk near Δa . This ensures that the minimisation of the total energy with respect to the virtual crack Δa_v is essentially unaffected by springs. Specifically, the energy release rate (ERR) at the tip of Δa_v propagating along Δa bridged by the spring distribution should be almost equal to the ERR of Δa_v propagating in pristine bulk material.

To eliminate the well-known spurious stress singularity at the tip of Δa , when the spring distribution bridging Δa has a finite stiffness at this tip, see [40–44], the novel idea introduced in the present study is to consider spring distributions with varying stiffness that have a singularity at the fictitious-crack tip, i.e. the spring stiffness diverges to infinity at this tip. The bridging laws considered in the present study exhibit power-law variation in spring stiffness with the distance from the fictitious-crack tip. The aim is to find an optimum spring-stiffness variation that strikes a balance between accuracy and computational costs. While a stiffer spring distribution yields more accurate predictions, it inherently incurs higher computational costs. Notably, although uniform stiffness of spring distribution is conventionally assumed, power-law variations have been explored by several authors in other mechanics contexts [45–49].

A suitable power-law for spring-stiffness variation along the bridged zone identified and tested in the present study is motivated by the recent mathematical results in [50], which determined the stress singularities at the apex of infinite corners with a spring distribution with a power-law stiffness variation along one corner face. Specifically, for the present study, it is relevant that [50]

¹ While the fictitious crack concept proposed here shares certain similarities with its counterpart in extrinsic CZMs, significant differences distinguish the two frameworks, as outlined below and discussed in detail in Section 6.

characterises stress singularity at a semi-infinite crack bridged by Winkler spring distribution with a power-law stiffness variation, considering antiplane shear.

Once a suitable power-law for spring-stiffness variation along the bridged zone of Δa is identified, it can be used in FEM implementations of PMTE-SC by applying AMA in each loading step to model competition of onset and propagation of multiple internal cracks in the bulk, as explained above, see Fig. 1(b). Such procedure needs to define suitable FEM mesh with discontinuous representation of cracks, either by re-meshing in each load step or taking advantage of modern FEM techniques as XFEM or unfitted FEM, avoiding the necessity of re-meshing, or VEM minimising remeshing just to the very neighbourhood of the fictitious crack Δa .

1.4. Present article

The present article is organised as follows: Section 2 introduces the fundamental parameters of a linear-elastic and perfectly brittle spring distribution, including damage variable, stiffness of spring distribution and ERR. Section 3 presents the results of numerical tests analysing the effect of spring distribution with constant stiffness on stress singularity at the fictitious crack tip, as well as the accuracy of the ERR calculation at the virtual crack tip, highlighting challenges and necessity for improvement. Section 4 extends the analysis to incorporating varying stiffness into the spring distribution, addressing limitations of constant stiffness assumptions and introducing a couple of parameters to enhance modelling accuracy and reliability. Section 5 summarises the findings of numerical tests, emphasising the effectiveness of varying stiffness of spring distribution in mitigating stress singularities and improving the ERR approximation. Section 6 establishes the relationship between the present approach and extrinsic Cohesive Zone Models (CZMs). Section 7 derives a novel path-independent J -integral formulation for a crack bridged by a Winkler spring distribution with varying stiffness. Finally, some concluding remarks are given in Section 8.

2. Spring distribution

2.1. Connecting LEFM, CC and PMTE-SC

In a 2D setting, assuming unit thickness, the classical Griffith-Irwin LEFM framework defines the energy criterion for an infinitesimal advance, da , of an existing crack of length a as

$$G = -\frac{d\Pi}{da} \geq G_c, \quad (1)$$

where G is the ERR, G_c is the fracture energy, and Π is the potential energy of the cracked solid under a prescribed load. Consequently, identifying the minimum load that triggers propagation requires satisfying this criterion in its alternative form

$$d\Pi + G_c da \leq 0. \quad (2)$$

Similarly, Leguillon's [15,16] Coupled Criterion (CC) expands upon this by requiring both a strength and an energy criterion to be satisfied for a (possibly) finite virtual crack advance $\Delta a_v \subset \Delta a$. First, the prospective crack path Δa must fulfil a strength criterion prior to propagation. In the simplest Mode I case, this is expressed as

$$\sigma \geq \sigma_c \quad \text{on } \Delta a, \quad (3)$$

where σ is the normal stress and σ_c is the tensile strength. Once the strength condition is met, the incremental energy criterion is applied

$$-\Delta\Pi \geq G_c |\Delta a_v|, \quad (4)$$

where $\Delta\Pi$ is the change of the potential energy due to the virtual crack advance Δa_v , and $|\Delta a_v|$ denotes the length of this crack. The PMTE-SC approach introduced in [33] reformulates this as a minimisation problem

$$\min_{\Delta a_v \subset \Delta a} (\Delta\Pi + G_c |\Delta a_v|) \leq 0, \quad (5)$$

which is a generalisation of the LEFM energy criterion in (2). Thus, determining the critical state involves identifying the minimum load that satisfies this inequality. This criterion for minimising the total energy can be straightforwardly generalised to consider n prospective crack paths Δa_i (each satisfying the strength criterion), alongside n virtual crack advances $\Delta a_{vi} \subset \Delta a_i$. This generalisation is achieved by defining the union of the paths as $\Delta a = \bigcup_{i=1}^n \Delta a_i$ and the overall virtual crack advance as $\Delta a_v = \bigcup_{i=1}^n \Delta a_{vi}$.

Consequently, the CC (and, equivalently, PMTE-SC) serves as a generalisation of classical LEFM. The CC reliably predicts the competition between crack nucleation and propagation under quasi-static increasing loading, extending predictive capabilities to configurations not covered by LEFM—such as stress concentrations, and weak stress singularities (V-notches), Fig. 1. Furthermore, for stable, infinitesimal crack propagation, the critical load predictions of CC and LEFM perfectly coincide [34,35]. In such cases, the strength criterion is inherently satisfied over an infinitesimal advance due to the square root stress singularity ahead of the crack tip.

As mentioned in the Introduction, the minimisation problem in (5) can be highly non-convex, especially for a competition between several virtual crack advances Δa_{vi} . To achieve a robust and efficient numerical procedure, we equip the prospective crack

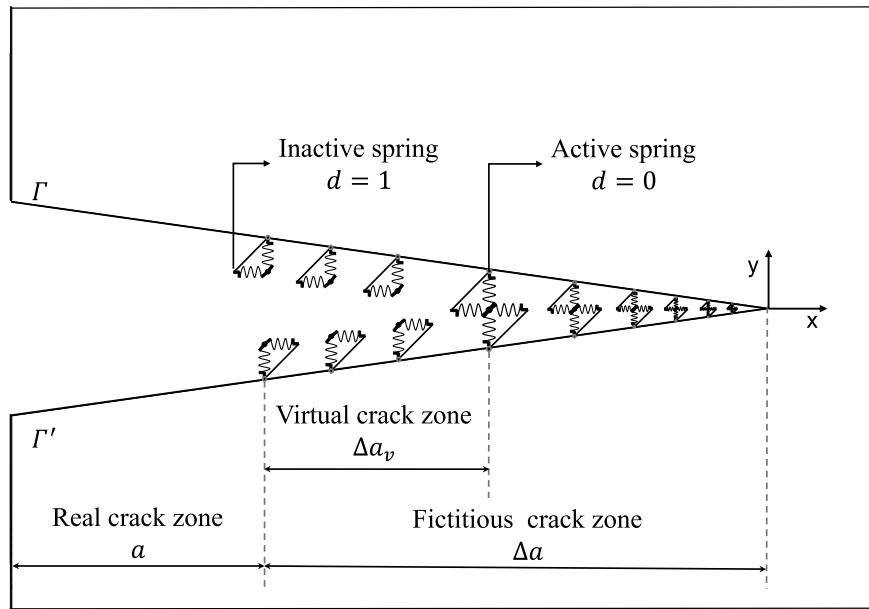


Fig. 2. Real crack, fictitious crack and virtual crack zones. Fictitious crack is bridged by a spring distribution with two possible spring states. Springs are inactive in the virtual crack zone and active elsewhere. The short solid lines between the crack face and the tangential springs indicate a rigid connection, enabling the generation of shear tractions.

paths Δa with a suitable Winkler spring distribution, bridging the crack faces along Δa , thereby introducing a continuous damage variable d on Δa , $0 \leq d \leq 1$. Thus, we replace the minimisation problem in (5) by the following minimisation problem

$$\min_{(u,d)} \left(\Pi(u,d) + \int_{\Delta a} d G_c ds \right), \quad (6)$$

where the total energy functional becomes separately convex with respect to the displacement u and the damage variable d . The potential energy $\Pi(u,d)$ now also includes the elastic strain energy of springs times the factor $(1-d)$ integrated along Δa , in addition to the bulk elastic strain energy and the potential energy of external forces. Since the total energy in (6) depends linearly on d , the minimisation with respect to d in an AMA iteration reduces to an elementary linear programming problem, where we just compare the elastic strain energy stored in a spring and the fracture energy G_c . This mathematical simplicity is the primary driver of the computational efficiency of the entire minimisation procedure. Ultimately, the damage variable acts strictly as a binary indicator; it adopts a value of either 0 or 1, representing an active or inactive spring, respectively.

In this context, the Winkler linear elastic spring distribution, bridging the crack faces along Δa , serves purely as a mathematical regularisation tool to ensure a robust and efficient minimisation procedure for the problem defined in (5) in a given load step. This spring distribution is not considered in subsequent load steps, where other prospective crack paths Δa will be defined. Notably, the inclusion of these springs does not directly alter the overall structural response – such as the force–displacement curve – resulting from a fracture process evolution. Strictly speaking, it only impacts the computational efficiency and numerical accuracy when solving (5). This marks a key distinction from CZMs, which can inherently modify the overall structural response by altering the global structural stiffness.

2.2. Three crack zones: Real, fictitious and virtual

In general, a crack bridging mechanism [40–44] is represented by a distribution of normal and shear tractions, which oppose the opening and the relative sliding of crack faces, along a part of crack adjacent to the crack tip. These tractions decrease the SIFs at the crack tip. In the present context, as described above, the crack bridging along the so-called *fictitious crack zone* Δa , situated ahead of the *real (physical) crack zone* a , is modelled by a continuous distribution of linear elastic but damageable (Winkler) springs with zero thickness. The real crack zone is characterised by stress-free crack surfaces with no springs connecting them. Fig. 2 illustrates the springs connecting the crack surfaces Γ and Γ' . The figure exaggerates the gap between the crack surfaces for clarity. In this section, parameters and constitutive law of the spring distribution are introduced and discussed.

The state of each spring, defined by the damage variable d , can be either active ($d = 0$) or inactive ($d = 1$), depending on the so-called *virtual crack extension* Δa_v , with $\Delta a_v \subset \Delta a$. The aim is to model virtual crack propagation (or healing) in a homogeneous material by changing d from 0 to 1 (or vice versa), for a sufficiently stiff spring distribution. Consequently, if all springs within Δa are inactive, then $\Delta a_v = \Delta a$. In contrast, Δa_v vanishes when all the springs in Δa are activated, i.e., $\Delta a_v = \emptyset$.

2.3. Constitutive law of the continuous spring distribution

The continuous distribution of springs in Δa is governed by a constitutive law formulated in terms of the normal and shear components of stresses (tractions) on the crack plane, $\sigma = [\sigma, \tau]^T$, and the corresponding relative displacements $\delta = [\delta_n, \delta_t]^T$, also called the displacement jump. The constitutive model is analogous to that used originally for LEBIM by Fig. 1 in [51],

$$\begin{aligned}\sigma(x) &= (1 - d(x)) k_n(x) \langle \delta_n(x) \rangle_+ + k_n(x) \langle \delta_n(x) \rangle_-, \\ \tau(x) &= (1 - d(x)) k_t(x) \delta_t(x),\end{aligned}\quad (7)$$

with $k_n(x)$ and $k_t(x)$ denoting the normal and tangential stiffness of the spring located at a point $x \in \Delta a$. Here, $\sigma(x)$ and $\tau(x)$ are the normal and shear tractions at x , while $\delta_n(x)$ and $\delta_t(x)$ are the normal (opening) and tangential (sliding) relative displacements between opposite points of the fictitious crack surfaces, respectively. The operator $\langle \cdot \rangle_{\pm}$ is the Macaulay bracket, or ramp function, and is defined as $\langle \cdot \rangle_{\pm} = \frac{(\cdot \pm |\cdot|)}{2}$.

Stress values for both active and inactive springs are expressed, by substituting the corresponding damage variable $d(x)$ into (7), as

$$\begin{aligned}\text{Active spring } (d(x) = 0) : \quad & \sigma(x) = k_n(x) \delta_n(x), \\ & \tau(x) = k_t(x) \delta_t(x), \\ \text{Inactive spring } (d(x) = 1) : \quad & \sigma(x) = k_n(x) \langle \delta_n(x) \rangle_-, \\ & \tau(x) = 0.\end{aligned}\quad (8)$$

A characteristic stiffness of the spring distribution is denoted by k_0 . To model perfect crack bridging, k_0 is set to a very high value to ensure a negligible gap between the fictitious crack faces.

To analyse the behaviour of compliant as well as stiffer springs, the following expression for k_0 , given in terms of Young's modulus E , the length of the spring distribution Δa , i.e., the fictitious crack length Δa , and a dimensionless parameter (multiplier) $M > 0$, is used

$$k_0 = M \frac{E'}{\Delta a}, \quad (9)$$

where E' is the reduced elastic modulus ($E' = E$ for plane stress; $E' = E/(1 - \nu^2)$ for plane strain, with ν denoting Poisson's ratio).

In the case of a spring distribution with uniform stiffness, the normal k_n and tangential k_t stiffness of the spring can be expressed in terms of k_0 as

$$\begin{aligned}k_n &= k_0, \\ k_t &= k_0 \xi,\end{aligned}\quad (10)$$

with $\xi \geq 0$ being a suitable dimensionless parameter.

In addition to traction and displacement jump values, the ERR can also be associated with this spring distribution, which is relevant for the application of a fracture criterion. In general, the ERR represents the potential energy released per unit of new crack area. In a continuous spring distribution, the ERR $G(x)$ is defined by the stored elastic strain energy (per unit area) within a spring at a specific point x ; see [51] and references therein for a proof. Consequently, the ERR for an active linear elastic spring under mixed mode is given by

$$G(x) = G_I(x) + G_{II}(x), \quad (11)$$

where

$$G_I(x) = \frac{\langle \sigma(x) \rangle_+ \langle \delta_n(x) \rangle_+}{2} = \frac{\langle \sigma(x) \rangle_+^2}{2 k_n(x)} = \frac{k_n(x) \langle \delta_n(x) \rangle_+^2}{2}. \quad (12a)$$

$$G_{II}(x) = \frac{\tau(x) \delta_t(x)}{2} = \frac{\tau^2(x)}{2 k_t(x)} = \frac{k_t(x) \delta_t^2(x)}{2}. \quad (12b)$$

Note that, as follows from the proof introduced in [51], the above expressions for ERR G are valid for any continuous and smooth variation of spring stiffnesses $k_n(x)$ and $k_t(x)$.

In a continuous spring distribution bridging crack faces, the stiffness is typically assumed to be constant over the crack faces, here denoted as k_0 ; a few exceptions are mentioned in Introduction. However, when such spring models are applied to cracks extending within a homogeneous linear elastic body, as shown in Fig. 2, the spring stiffness can be assumed to change abruptly from a finite value to infinity when moving along the fictitious crack path and crossing the fictitious crack tip towards the pristine material, as illustrated in Fig. 3. As is well known from the theory of bridged cracks (see [40–44]) and from a recent mathematical study [50], the stress state at the fictitious crack tip exhibits the classical $1/\sqrt{r}$ -type singularity, well known in linear elastic fracture mechanics (LEFM), independently of how large the finite value of k_0 is chosen, as illustrated in Fig. 3. The stress intensity factor (SIF) of this stress singularity decreases with increasing k_0 , and in the limit case where $k_0 \rightarrow \infty$, perfect bonding along the fictitious crack zone is recovered and the $1/\sqrt{r}$ singularity at the tip of the fictitious crack disappears.

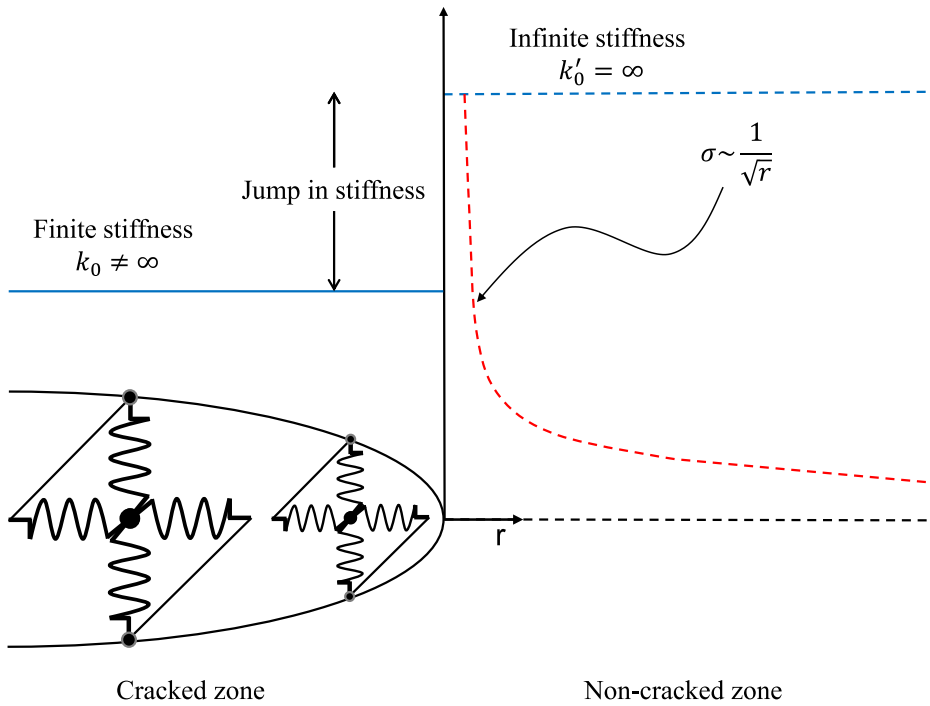


Fig. 3. Illustration of an infinite jump in the stiffness across the cracked zone. The short solid lines between the crack face and the tangential springs indicate a rigid connection, enabling the generation of shear tractions.

3. Numerical study: Spring distribution with constant stiffness k_0

This section presents numerical tests that illustrate the application of the proposed spring distribution to simulate crack advance zone bridging. The following analysis aims to determine the role of spring stiffness and FEM mesh refinement. The numerical implementation is developed using the user-subroutine UINTER in the FEM software Abaqus/Standard, which can be used to define normal and shear interactions between crack surfaces [52]. The formation or propagation of a crack is modelled by defining a damage variable able to activate or deactivate springs in the continuous spring distribution.

3.1. Case study 1: Stress singularity at the fictitious crack tip

The constant stiffness $k(x) = k_0$ of the spring distribution is defined by (9). Different values of the parameter M are considered to assess the influence of this stiffness on the stress singularity at the fictitious crack tip and the efficacy of approximating the ERR at the tip of the virtual crack Δa_v growing in a pristine material by the ERR at the tip of Δa_v growing in the spring distribution in the fictitious crack Δa .

The proposed spring distribution model is examined considering an infinite plate, with a Mode I crack, subjected to uniaxial tensile load in plane strain. Numerical computations are performed for centre-crack specimens, with a length of $L = 200$ mm, an initial width $2W = 200$ mm, and an initial crack of length $2\Delta a = 1$ mm situated at $y = 0$, with the origin of the Cartesian coordinate system (x, y) at the crack centre. The nominal applied load is $\sigma_{nom} = 100$ MPa. Young's modulus $E = 210$ GPa and Poisson's ratio $\nu = 0.22$. Fig. 4(a) shows that only half of the centre-cracked specimen is modelled, in view of the problem symmetry. To ensure sufficient accuracy in capturing the stress and displacement fields near the crack tip, a minimum element size of 7×10^{-7} m is adopted in the crack region, with the mesh gradually becoming coarser away from the crack zone. The corresponding mesh configuration and a magnified view of the crack region Δa , where the spring distribution is located, are illustrated in Fig. 5. Quadratic finite elements are used throughout all simulations. Since this study is a proof of concept, highly refined meshes are used to characterise the stress singularities at the virtual and fictitious crack tips. For practical applications, much coarser meshes can be used by deploying the enriched elements for logarithmic stress singularities along Δa [53,54].

Stress singularity variations at the fictitious crack tip obtained by FEA for various spring stiffnesses ranging from $M = 1$ to 10000, reveal a notable trend, as shown in Fig. 6. Note that all stress images are standardised with an identical zoom. With compliant springs with $M = 1$ and 10, pronounced stress singularities at the crack tip, associated to high values of SIFs, are evident. However, as the spring stiffness increases, the SIF gradually diminishes. For $M = 10000$, stress contour variations are barely discernible, indicating a significant reduction in the SIF value.

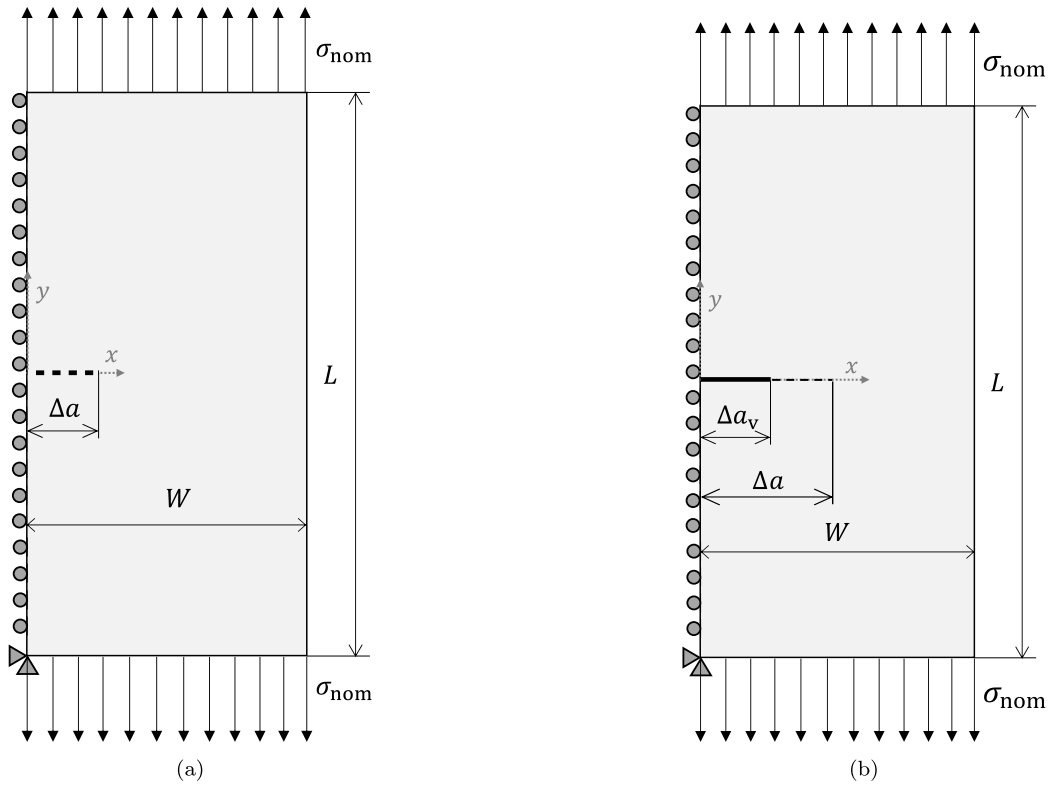


Fig. 4. Schematic representation of the specimen under uniform tension. (a) The spring distribution bridging the fictitious crack length Δa . (b) The spring distribution bridging the fictitious crack length Δa , with the springs inactive in the virtual crack zone Δa_v .

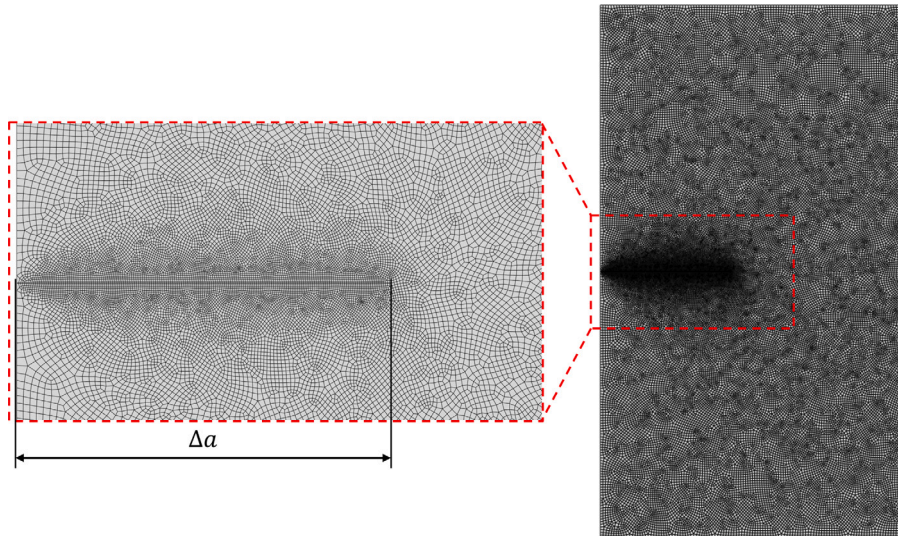


Fig. 5. FEM mesh of the specimen with a detailed view on the left showcasing the crack zone Δa , illustrating the concentration of elements along Δa where the spring distribution is applied.

Fig. 7 shows the variations of σ_{yy} along the fictitious crack Δa and slightly ahead of it, for spring distributions with $M = 1, 10, 100, 1000, 10000$. In these simulations, N denotes the number of nodes along the crack surface where the spring elements are attached. Since quadratic elements are employed, each spring is associated with a surface node rather than an individual finite element, representing a distributed spring network along the element side. The number of springs or nodes $N = 100$ in the Δa length is used in these simulations.

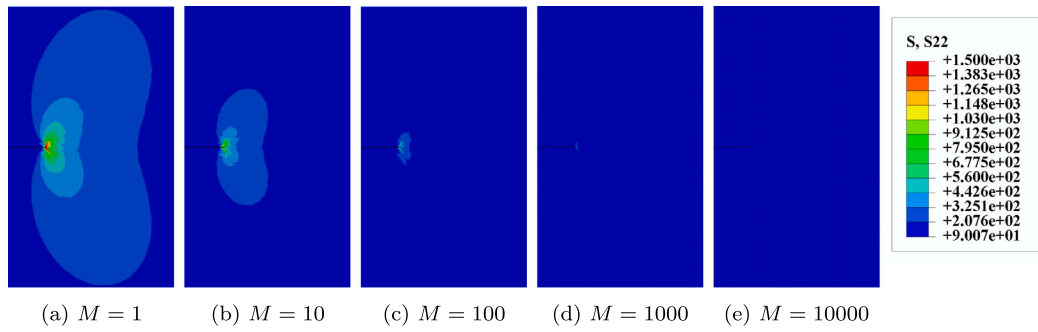


Fig. 6. Stress contours of σ_{yy} (in MPa) at the fictitious crack tip computed by FEM, considering constant spring stiffness along the fictitious crack path for different values of the multiplier M . The applied nominal stress used in the simulations is $\sigma_{nom} = 100$ MPa.

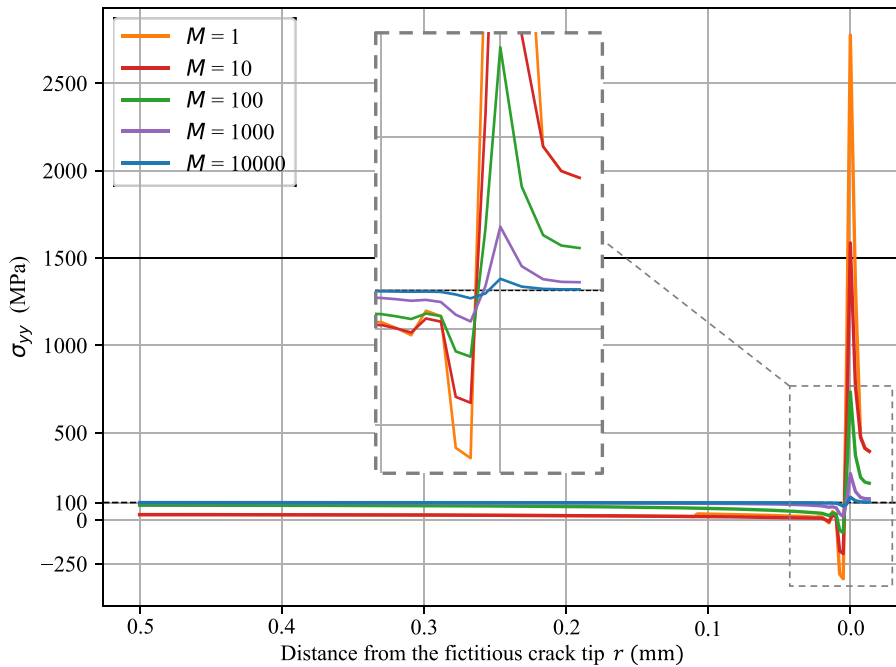


Fig. 7. Variations of stress σ_{yy} along the fictitious crack path Δa computed by FEM, showcasing the influence of the multiplier M in (9) on the stress singularity at the fictitious crack tip. The applied nominal stress used in the simulations is $\sigma_{nom} = 100$ MPa.

As can be seen from the figure, the stress distributions for stiff springs with $M \geq 100$ are almost identical when moving away from the fictitious crack tip. However, the peak values of all stress distributions at this tip are very different, indicating very different SIFs there. These peak values decrease rapidly as M increases. In this sense, stiff spring distributions with large M values significantly outperform compliant spring distributions.

A mesh sensitivity analysis is also performed for a stiff spring distribution with $M = 1000$, finding that the peak values increase with the number of nodes used in the vicinity of the crack tip. This indicates that the solution diverges with the mesh refinement, see Fig. 8. This observation holds true for all spring stiffnesses, as shown in Fig. 9, demonstrating that the relative error in stress σ_{yy} at the fictitious crack tip decreases for coarser meshes. Therefore, this proves that the spring distribution must be modified to address this fictitious stress singularity issue and mesh dependency.

3.2. Case study 2: Approximation of ERR for growing virtual crack Δa_v

As mentioned in Introduction, aside from stress distribution, accurate approximation of the ERR for the virtual crack Δa_v growing in the pristine material by means of the spring distribution is crucial for predicting crack propagation. Thus, assessing the accuracy of this approximation across different scenarios, such as Mode I and mixed mode crack problems, is essential. This assessment involves comparing the ERR derived from the spring distribution with the actual ERR values for the virtual crack Δa_v in the pristine material. Recall, that the elastic strain energy per unit area within a spring at a specific point x represents the ERR $G(x)$ as shown in (11).

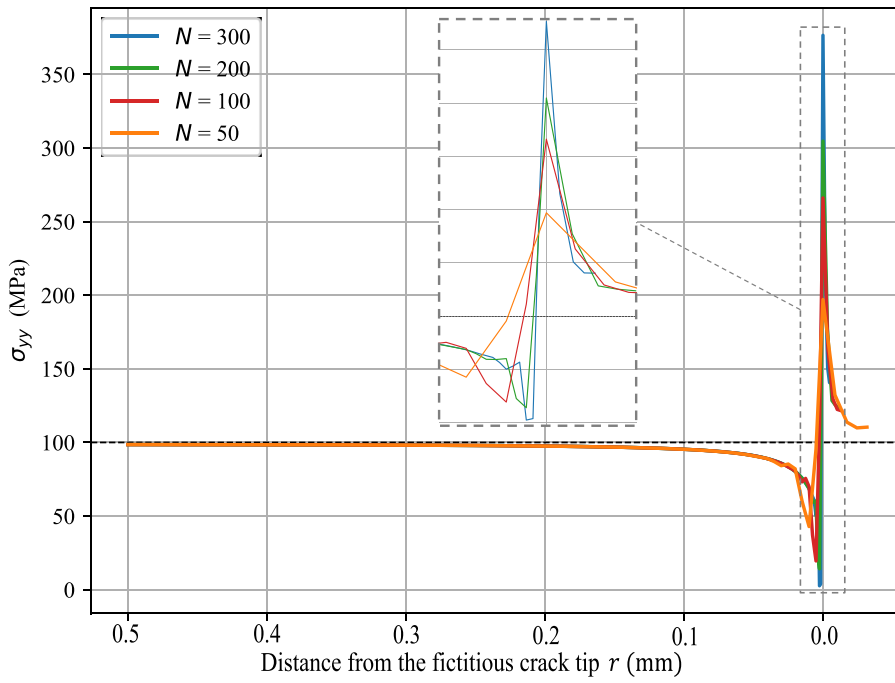


Fig. 8. Influence of the FEM mesh refinement on the stress distribution σ_{yy} by varying the number of nodes N along the fictitious crack path Δa , for a large value of $M = 1000$ in (9). The variations of these stresses near the fictitious crack tip at $x = \Delta a$ are highlighted. The applied nominal stress used in the simulations is $\sigma_{nom} = 100$ MPa.

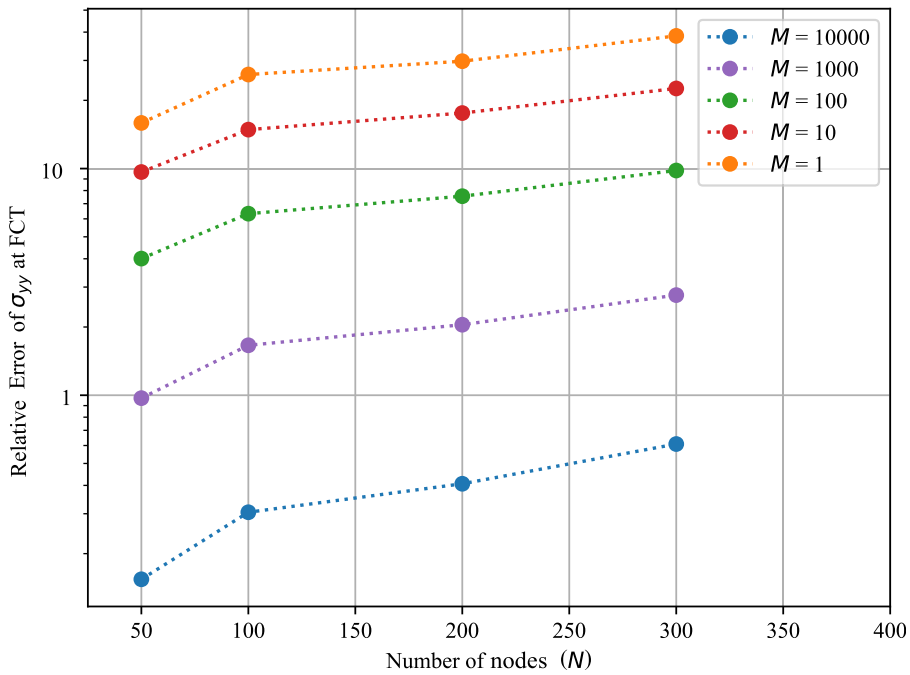


Fig. 9. Relative error in (peak) stress values at the fictitious crack tip (FCT) against the number of nodes N along the fictitious crack path Δa .

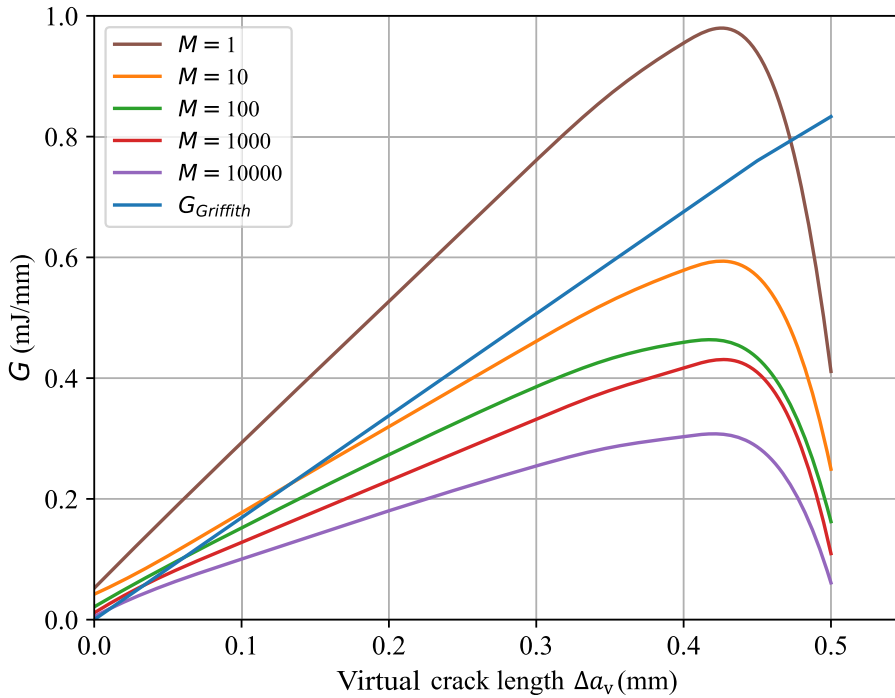


Fig. 10. ERR $G(\Delta a_v)$ values, defined by Eqs. (11), (12), computed by FEM for spring distributions with various stiffness values k_0 , defined in (9), given by the multiplier M , alongside the analytical G_{Griffith} values (13).

3.2.1. Mode I Griffith's problem

The problem of a central crack of increasing length Δa_v in a rectangular plate of dimensions $W \times L$ sufficiently large with respect to the crack length, i.e., $L \gg \Delta a_v$ and also $W \gg \Delta a_v$, and subjected to remote applied stress σ_{nom} , is considered. The right half of this problem is shown in Fig. 4(b). The plate dimensions and material properties considered remain the same as those in the previous problem studied in Section 3.1.

The above problem is modelled by introducing a fictitious crack zone of length Δa , represented by a row of springs. As the crack advances, the springs within the virtual crack extension Δa_v are progressively deactivated, following the procedure described in Section 2. The fictitious crack zone Δa is discretised using between 75 and 125 nodes, depending on the refinement level. These values were verified to be sufficient to obtain roughly mesh-independent results along the major part of crack.

The ERR obtained by [55] for this problem, but considering an infinite plate, is

$$G_{\text{Griffith}}(\Delta a_v) = \frac{\sigma_{\text{nom}}^2 \pi \Delta a_v}{E'}, \quad (13)$$

where E' denotes the reduced elasticity modulus.

The ERR derived from the spring distribution and computed at the tip of the virtual crack computed Δa_v , i.e. $G(x)$ for $x = \Delta a_v$, is compared with G_{Griffith} in (13). The aim is to determine whether, when the spring stiffness is sufficiently high, $G(\Delta a_v)$ approximates G_{Griffith} well.

The results plotted in Fig. 10 illustrate the influence of the spring stiffness k_0 , defined in (9), through the multiplier M , on the prediction of ERR as the virtual crack progresses. Upon examination, it is evident that the accuracy of ERR prediction is contingent upon the stiffness of the springs. Analysis of the most compliant ($M = 1$) and most stiff ($M = 1000$ and 10000) springs reveals significant deviations of $G(\Delta a_v)$ from G_{Griffith} as the virtual crack advances. Notably, the compliant spring distribution ($M = 1$) tends to overestimate ERR values, whereas the stiffer spring distributions tend to underestimate them.

Additionally, Fig. 11, provides supporting evidence, highlighting that the crack surface openings for compliant springs can be very large ($M = 1$), leading to overestimated ERR values. Notably, for the stiffest spring distributions ($M = 1000$ and 10000), crack opening is too minimal, resulting in significantly underestimated ERR values, as evident from Fig. 10. These observations underscore the importance of thoughtfully selecting multiplier M values, as they not only determine the fictitious singularity at the fictitious crack tip, but also significantly influence the accuracy of ERR value estimation at the virtual crack tip. However, as Δa_v approaches $\Delta a = 0.5$ mm, the numerical results for ERR drop down to a smaller value, possibly due to a shortage of springs along the remaining short segment between the virtual and fictitious crack tips.

For sufficiently high spring stiffness ($M = 1000$ and 10000) we can observe in Fig. 11 a typical stress distribution at the virtual crack tip Δa_v and essentially no alteration of the stress state at the fictitious crack tip Δa , similarly as in the pristine bulk material with no crack along the segment $\Delta a - \Delta a_v$.

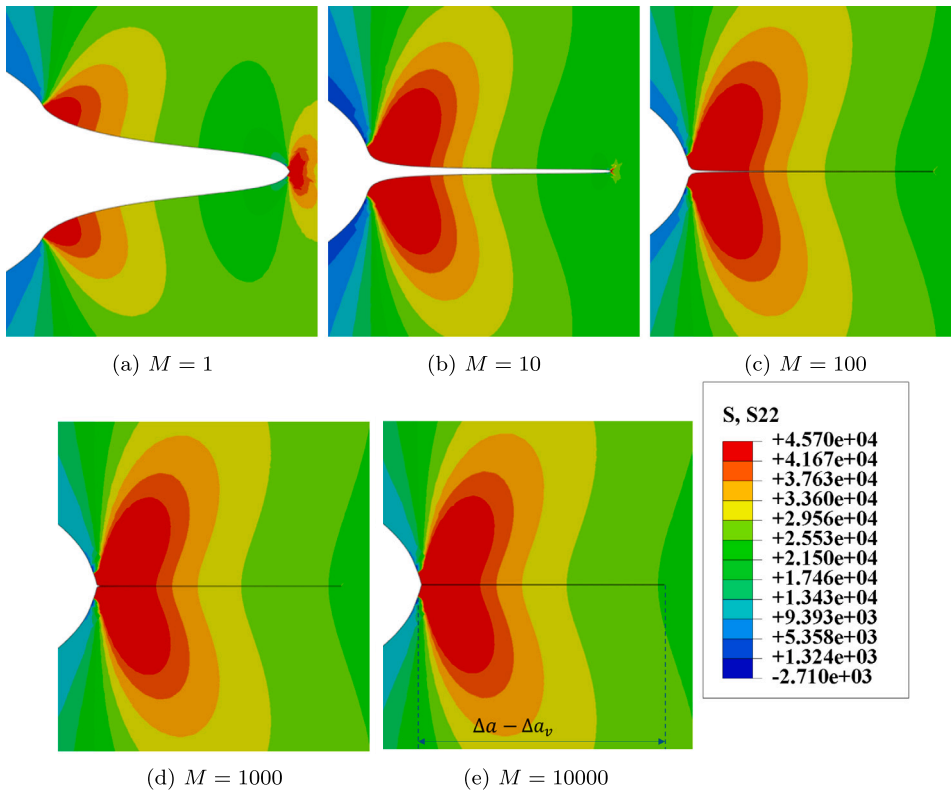


Fig. 11. Crack surface deformations and stress σ_{yy} contours computed by FEM under constant spring stiffness k_0 for different values of the multiplier M . Results are shown for $\Delta a_v = 0.36$ mm and $\Delta a = 0.5$ mm, using a consistent zoom and deformation scale factor for enhanced comparison.

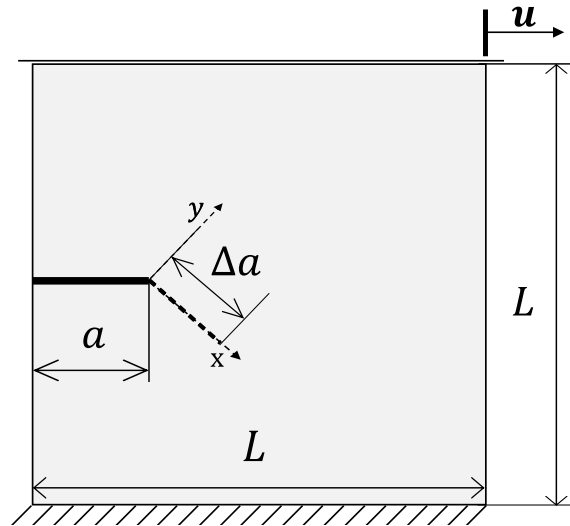


Fig. 12. Schematic representation of the specimen under pure shear loading conditions, highlighting the mother crack with length a , along with the fictitious daughter crack with length Δa .

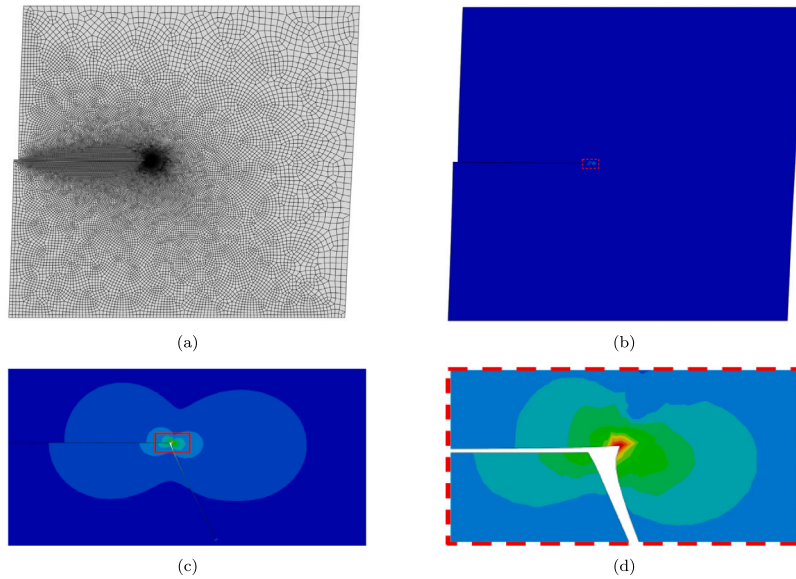


Fig. 13. (a) Deformed FEM mesh configuration, (b) deformation along the real and fictitious crack paths, (c) magnified view focusing on the fictitious crack with a uniform spring distribution, and (d) zoomed-in view of the bridged daughter kink crack, showing its characteristic curved deformed shape driven by the mixed-mode (shear-dominated) loading conditions at the kink point.

3.2.2. Mixed-mode crack problem with a kink crack

We consider a configuration in which a real crack of length a (mother crack) with a virtual crack Δa_v (daughter crack) growing along an inclined fictitious crack path of length Δa , undergoes shear loading, as shown in Fig. 12.

Amestoy and Leblond [56] derived an asymptotic analytic expression for the ERR at the tip of vanishing virtual (daughter) crack $\Delta a_v \rightarrow 0$, given in terms of the SIFs K_I and K_{II} of the real (mother) crack. In the present study these SIFs are computed by FEM using the path independent interaction integral, see [57].

The aim of the present study is to compute the ERR at the tip of a vanishing virtual crack using a uniform spring distribution that bridges the fixed fictitious crack of length Δa . This ERR value is obtained by taking the G value using (11)–(12) for the “first” spring on the fictitious crack, located in the kink point. The ERR computed in this way is compared to the ERR value G_{Leblond} predicted by the analytic expression from [56].

The dimensions used for this analysis are $L = 1$ mm, with the real crack length of $a = 0.4$ mm at $y = 0$, and a fictitious crack length $\Delta a = 0.01$ mm and kink angle -70.53° with respect to the horizontal line. Thus, the uniform spring distribution bridging the fictitious crack zone is subjected to a fracture mixed-mode. Numerical computations are conducted while maintaining the same material properties as in previous analyses.

Fig. 13(a) illustrates the mesh configuration employed, with a mesh size of 9×10^{-5} mm and around 100 nodes at the fictitious crack for ERR extraction. Fig. 13(b) showcases the deformed shape of the model, while Fig. 13(c) zooms into the fictitious crack length delineated by red rectangle, representing the stress distributions and crack openings at the kink point. As observed in Fig. 13(c) and (d), the daughter crack initiates at the kink point at an angle of -70.53° with respect to the mother crack direction, consistent with the maximum circumferential stress criterion under dominant Mode II loading. The white region in Fig. 13(d) represents the crack face opening, while the curved deformation of the bridged daughter crack reflects the asymmetric mixed-mode stress field at the kink point. The stress concentration (red region) is localised at the junction between the mother and daughter cracks, confirming that the spring distribution correctly captures the singular stress state at this critical point.

Fig. 14 illustrates the variation of G values (dots) derived from the energy stored in the spring positioned at the kink point compared to G_{Leblond} . The axis of abscissas represents the number of nodes in the fictitious crack zone. Significantly, the figure reveals notable disparities between predictions obtained from the spring model and G_{Leblond} , compliant springs ($M = 1$ and 10) overestimating and stiff springs ($M = 100, 1000$ and 10000) underestimating G_{Leblond} . The plot also highlights a mesh dependency, as the number of springs N increases the numerical G values decrease. This observation underscores the limitations of the current methodology and emphasises the need to suitably modify the spring distribution bridging the fictitious crack.

4. Spring distribution with power law stiffness variation

Based on the numerical results presented above, it is clear that the spring distribution, with a constant stiffness k_0 (9) bridging the fictitious crack, is unable to overcome the challenges posed to this study. Such a spring distribution has failed: (i) to remove the effects of the fictitious stress singularity at the fictitious crack tip Δa on the FEM solution (i.e., basically to provide a stress

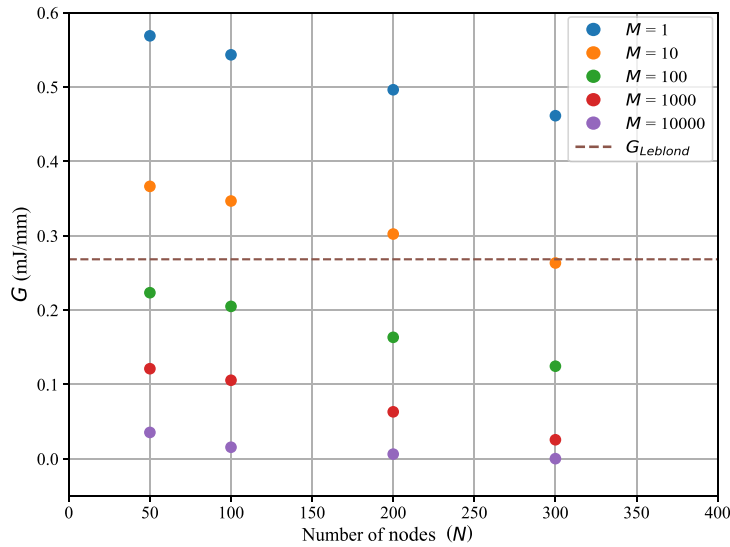


Fig. 14. ERR G values, for the vanishing virtual (daughter) crack $\Delta a_v \rightarrow 0$, derived from uniform spring distributions with constant stiffness k_0 (9), given by the multiplier M , alongside the G_{Leblond} value obtained by the analytical asymptotic formula from [56], employing the SIFs at the mother crack computed by FEM.

solution very similar to that obtained for perfect bonding along the still non-fractured zone $\Delta a - \Delta a_v$), except for extremely high spring stiffness k_0 ; and, more significantly, (ii) to provide accurate values for ERR $G(\Delta a_v)$ at the virtual-crack tip, resulting in very disparate values of $G(\Delta a_v)$, either overestimating (for small k_0) or underestimating (for large k_0) the expected analytical ERR value for Mode I and mixed Mode virtual cracks located along the fictitious crack path.

Thus, the constant stiffness of spring distribution that bridges the fictitious crack, as expressed in (9), must be reconsidered. In response to this challenge, we propose a new bridging law in which the spring stiffness varies along the fictitious crack path rather than being constant. This varying stiffness, denoted as $k(r)$, diverges to infinity at the fictitious crack tip for $r \rightarrow 0$, where $r > 0$ is the distance of a spring location to the fictitious crack tip (i.e., $r = -x$, see Fig. 2). In particular, a simple power law variation of $k(r)$ is examined.

The varying stiffness of the spring distribution can be defined in terms of the constant stiffness parameter k_0 as follows:

$$k(r) = M \left(\frac{E'}{\Delta a} \right) \left(\frac{\Delta a}{r} \right)^\alpha = k_0 \left(\frac{\Delta a}{r} \right)^\alpha, \quad (14)$$

where $\alpha \geq 0$ is the exponent of the power-law. According to (14), $k(r)$ tends to infinity when $r \rightarrow 0$ and the stiffness of the “first” spring located at $r = \Delta a$, i.e. at the real-crack tip with $x = -\Delta a$, is equal to $k(\Delta a) = k_0$.

We define the varying normal and tangential spring stiffnesses similarly as in (10)

$$\begin{aligned} k_n(r) &= k(r), \\ k_t(r) &= k(r) \xi, \end{aligned} \quad (15)$$

where we consider the same exponent α for both spring stiffnesses.

Whereas the exponent α of the power law in (14) will be discussed in the next section, in the following we will focus on the consistent value of stiffness k_0 of the “first” spring located at the real crack tip. k_0 must be appropriately defined to enhance the precision of the approximation of ERR $G(\Delta a_v)$, because k_0 plays a significant role in extracting this value, as was discussed in the preceding section. Drawing on insights from a pioneering work on cracks at spring interfaces by [58], differences from the traditional paradigms of LEFM becomes evident. By examining solutions around crack tips, Lenci [58] demonstrated logarithmically unbounded derivatives of displacements, suggesting the presence of logarithmic stress singularities at the crack tip that differ from the square root stress singularities in traditional LEFM [59, Section 5.3]. A complete double asymptotic series expansion of displacements and stresses in terms of the main terms and their associated shadow terms at the tip of cracks at spring interfaces has been derived in [60] for anti-plane Mode III and, more recently, for Mode I and Mode II cracks [61], thus confirming the results in [58].

Lenci [58] introduced a unique dimensionless parameter δ_{Lenci} , which accounts for the stiffness of the adherents, shear modulus of adherents μ , the stiffness of the interface k_0 , and the crack size a ,

$$\delta_{\text{Lenci}} = \frac{k_0 a (\kappa + 1)}{4 \mu \pi}, \quad (16)$$

where $\kappa = (3 - 4\nu)$ in the case of plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ in the case of generalised plane stress, ν denoting Poisson’s coefficient. Rewriting the above equation in terms of Young’s Modulus E' , gives

$$\delta_{\text{Lenci}} = \frac{2 k_0 a}{E' \pi}. \quad (17)$$

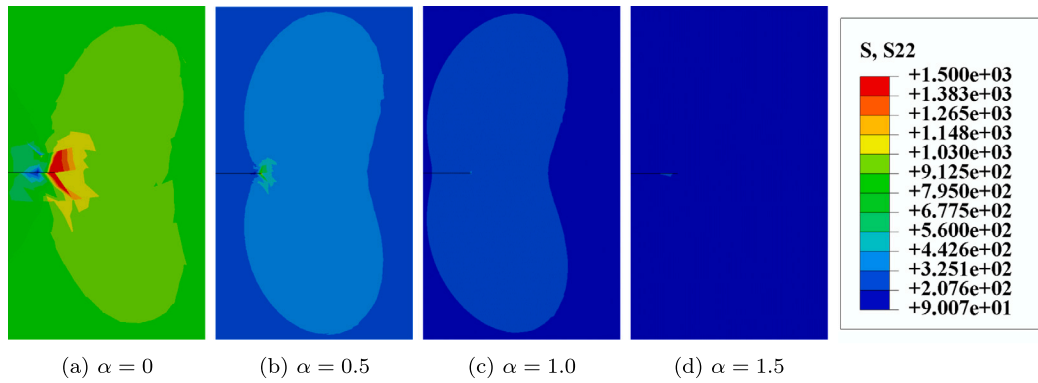


Fig. 15. Stress σ_{yy} contours computed by FEM for power-law variations of spring stiffness with different values of α . Results are shown with a consistent zoom. Stress values are in MPa.

The particular case of perfect bonding, i.e., the classical Griffith crack, can be recovered by assuming $\delta_{\text{Lenci}} \rightarrow \infty$.

Hereafter, based on (17), the stiffness parameter k_0 in (14) will be defined in terms of δ_{Lenci} parameter as follows

$$k_0 = \frac{E' \pi \delta_{\text{Lenci}}}{2a'}, \quad (18)$$

where, for the purpose of covering situations without a real crack, i.e. when $a = 0$, a' is suitably defined as $a' = a + \Delta a$. Once k_0 is defined, the varying stiffness $k(r)$ can be expressed in terms of Lenci's parameter as

$$k(r) = k_0 \left(\frac{\Delta a}{r} \right)^\alpha = \frac{E' \pi \delta_{\text{Lenci}}}{2a'} \left(\frac{\Delta a}{r} \right)^\alpha, \quad (19)$$

which shows to be very useful for applications of the present bridging law in computational modelling of crack onset and propagation.

5. Numerical study: Spring distribution with power law stiffness variation

The effect of power-law variation of stiffness of the spring distributions bridging fictitious crack surfaces on (i) the stress singularity at the fictitious crack tip and (ii) the approximation of ERR at the virtual crack tip is explored. For this aim, the analysis delves into the influence of the exponent α and Lenci's parameter δ_{Lenci} from (19). Following the results of the theoretical study of singularities in antiplane Mode III crack located on a spring interface with power-law variation of stiffness in [50], the α values ranging from 0 to 2 are considered, with 0 value representing a scenario of constant stiffness k_0 , considered as a benchmark. Following the results regarding the influence of the dimensionless parameter δ_{Lenci} (i) on the elastic solution for a crack located on a spring interface in [58], and also (ii) on the convergence of numerical results for ERR in [53,54], the δ_{Lenci} values ranging from 1 to 75 are considered.

5.1. Case study 1: Stress singularity at the fictitious crack tip

The same Mode-I crack problem analysed in Section 3.1 and shown in Fig. 4(a) is considered. The simulation results depicted in Fig. 15, encompassing four α values ranging from 0 to 1.5, exhibit a discernible trend. For springs with constant stiffness ($\alpha = 0$), a pronounced stress singularity is observable at the fictitious crack tip Δa , accompanied by notable variations in stress contours near the crack tip. However, as the exponent α of the varying stiffness increases, the intensity of stress singularity gradually diminishes. Particularly noteworthy is the case of $\alpha = 1.5$, where stress contour variations become minimal, indicating removal of stress singularity. All stress images are standardised with identical zoom and deformation scale factors to facilitate accurate comparison. The utilisation of red colour in stress contours, denoting stress values surpassing 1500 MPa, serves to emphasise the intensity of stress singularities. Fig. 16 shows the stress σ_{yy} variations along the fictitious crack length, $\Delta a = 0.5$ mm, for various spring distributions ($\alpha = 0, 0.5, 1, 1.5$ and 2).

For these simulations, the number of nodes N along the fictitious crack Δa length is set to 100. Values of σ_{yy} for a few neighbouring nodes ahead of the fictitious crack tip are also included in the plots. Fig. 16 shows that the stress distribution is nearly identical for all models away from the fictitious crack tip. However, the peak values differ greatly at this tip, as they diminish rapidly with increasing α . This behaviour could be explained by the different stress singularity character at the fictitious crack tip, as predicted by the theoretical analysis of the asymptotic expansion of the elastic solution at such a crack tip under anti-plane shear in [50], which shows that, for

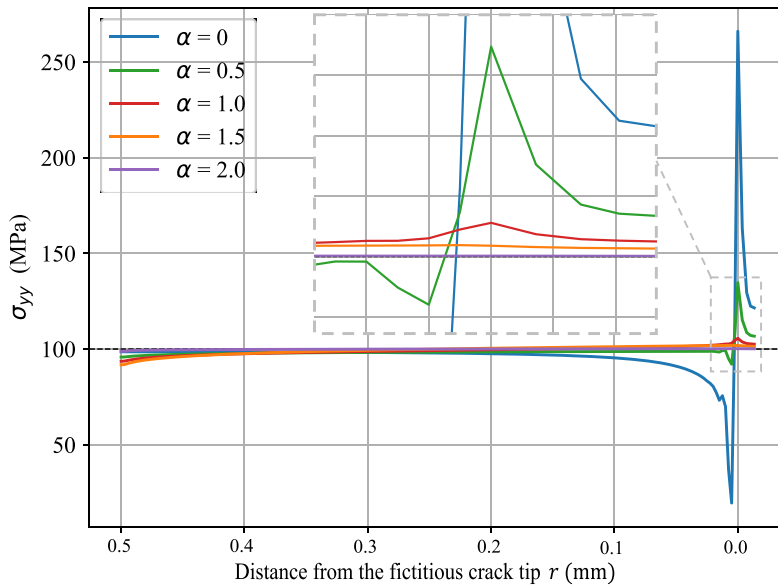


Fig. 16. Stress σ_{yy} distributions computed by FEM along the fictitious crack path Δa , for different values of the exponent α , illustrating the influence of α on the stress singularity at the fictitious crack tip.

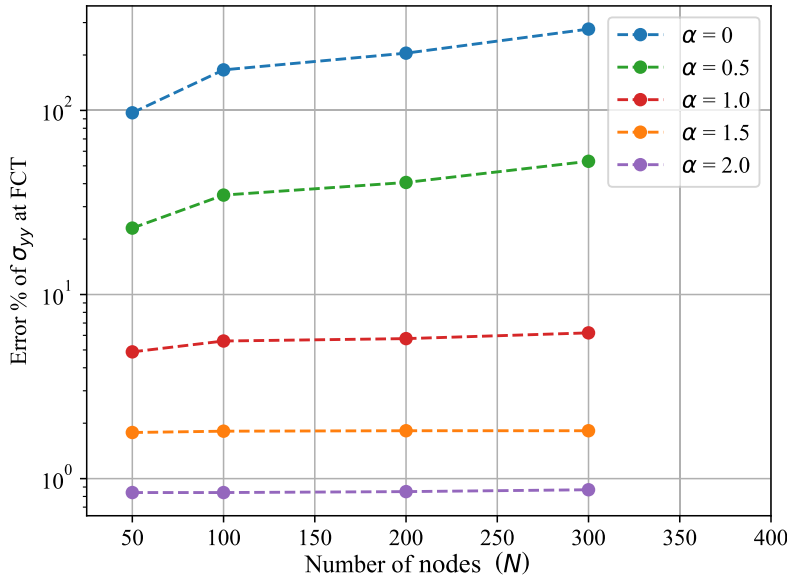


Fig. 17. FEM mesh-sensitivity analysis of the stress at the fictitious crack tip (FCT). Percentage error in stress versus the number of nodes N along the fictitious crack Δa , for spring stiffness distributions with different values of α .

- $\alpha < 1$, the fictitious crack-tip solution has a classical square root stress singularity, with the stress singularity exponent $\lambda = 1/2$,
- $\alpha = 1$ the fictitious crack-tip solution has a weak stress singularity, with the stress singularity exponent $1/2 < \lambda < 1$ whose value depends on the coefficient in the power law for spring stiffness,
- $\alpha > 1$, there is no stress singularity at the fictitious crack-tip, as $\lambda = 1$.

Therefore, spring distributions with $\alpha > 1$, which essentially have no peak values, outperform those with $\alpha < 1$ and also that with $\alpha = 1$, as they effectively mitigate numerical stress singularities at the fictitious crack tip.

Turning attention to mesh dependency of the spring distribution with power-law stiffness variation, $k(r)$, Fig. 17 plots the percentage error in stress value at the fictitious crack tip against the number N of springs along the fictitious crack Δa . While the error is high and increases with N for $\alpha < 1$ due to the square root stress-singularity, and similar behaviour is observed also for $\alpha = 1$ albeit with smaller errors because the stress singularity is weaker, the error is small and mesh independent for $\alpha > 1$, because there is no stress singularity there.

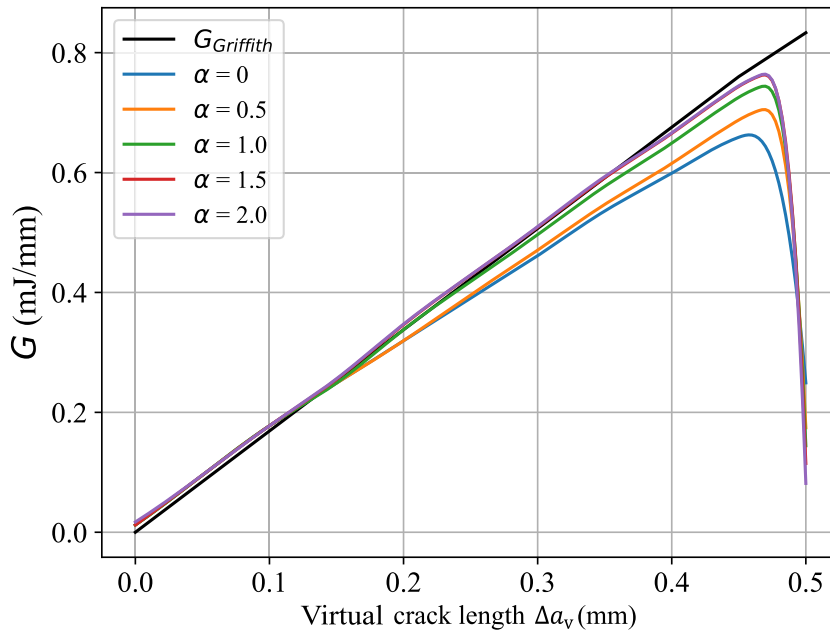


Fig. 18. $G(\Delta a_v)$ values computed by FEM for the problem in Fig. 4, using spring distributions with stiffness $k(r)$ defined by (19) along the fictitious crack Δa , for several values of α and $\delta_{\text{Lenci}} = 75$. These results are compared with the Griffith ERR $G_{\text{Griffith}}(\Delta a_v)$ obtained from (13).

The numerical results from these analyses, particularly stress distribution graphs and mesh dependency assessments, suggest that using spring distributions with $k(r)$ where $\alpha > 1$ leads to superior outcomes, advocating for its adoption for enhanced numerical simulations.

5.2. Case study 2: Approximation of ERR for growing virtual crack Δa_v

Similar to Section 3.2, it is crucial to assess the accuracy and potential enhancement of approximating the ERR derived from a spring distribution with spring stiffnesses varying according to (15) with $k(r)$ defined by (19), characterised by the exponent α and the coefficient determined by δ_{Lenci} , across different scenarios, including Mode I and mixed mode crack problems.

5.2.1. Mode I Griffith's problem

The Mode I crack problem depicted in Fig. 4 and studied in Section 3.2 is analysed again here, but this time using a spring distribution with varying stiffnesses as defined in (15) with $k(r)$ from (19). The propagation of the virtual crack Δa_v is modelled by progressively deactivating springs along the fictitious crack path Δa . The values of ERR $G(\Delta a_v)$ at the virtual crack tip are obtained from the elastic energy stored in the spring located at this tip. The resulting ERR values are then compared with the actual ERR values $G_{\text{Griffith}}(\Delta a_v)$ in (13) derived from theoretical Griffith crack analysis.

The results in Fig. 18 show the influence of the exponent α on the prediction of ERR $G(\Delta a_v)$ as the virtual crack propagates. The computations were performed using a Lenci parameter of $\delta_{\text{Lenci}} = 75$. It is evident that the accuracy of ERR prediction depends on the value of α . Notably, the spring distributions with $\alpha < 1$ provide significantly underestimated $G(\Delta a_v)$ values with respect to $G_{\text{Griffith}}(\Delta a_v)$, while those with $\alpha \geq 1$ provide values very close to the theoretical value. Nevertheless, near the fictitious crack tip, all numerical results for $G(\Delta a_v)$ drop significantly. This is probably due to the vanishing number of active springs between the virtual and fictitious crack tips, this effect was also observed in Section 3.2.1.

Fig. 19 plots deformations and stress contours at the fictitious crack for the case $\Delta a_v = 0.36$ mm, showing that the opening of the fictitious crack part with active springs of length $\Delta a - \Delta a_v$ significantly decreases with increasing values of α , and that spurious stress singularities at the fictitious crack tip are evident for $\alpha < 1$. These observations underscore the importance of thoughtfully selecting α values, as they not only determine the presence of stress singularity at the fictitious crack tip, but also significantly influence the accuracy of $G(\Delta a_v)$ estimation at the virtual crack tip.

Fig. 20 shows the influence of the parameter δ_{Lenci} on the predicted $G(\Delta a_v)$ values when $\alpha = 1.5$ and δ_{Lenci} varies from 1 to 75. The number of nodes, N , along Δa is around 100. Up to approximately half of the fictitious crack length, all numerical results show very good predictions of $G(\Delta a_v)$ compared to $G_{\text{Griffith}}(\Delta a_v)$. Even for larger values of Δa_v , all the numerical results are good, the best agreement with $G_{\text{Griffith}}(\Delta a_v)$ is obtained for $\delta_{\text{Lenci}} = 25$. The slight underestimation of the analytical value for the highest values of $\delta_{\text{Lenci}} = 50$ and 75 by the numerical values of $G(\Delta a_v)$ is due to the need for mesh refinement.

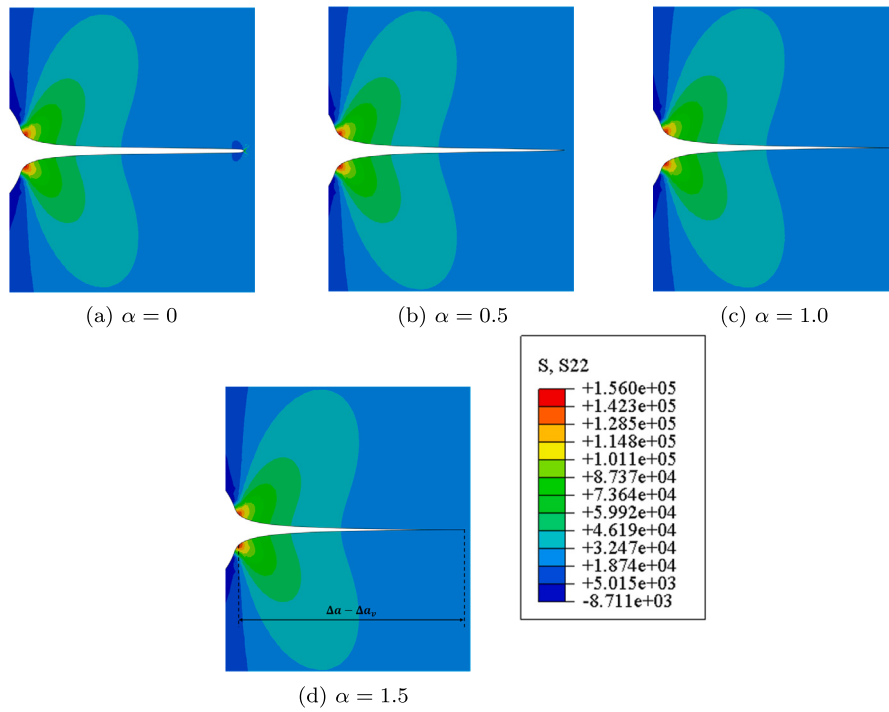


Fig. 19. Crack surface deformations and stress σ_{yy} contours computed by FEM using spring distributions with stiffness $k(r)$ defined by (19) along the fictitious crack $\Delta a = 0.5$ mm, for different values of α . Results are shown for $\Delta a_v = 0.36$ mm and $\delta_{Lenci} = 75$, with a consistent zoom and deformation scale factor, and stress contour range across all panels. Stress values are in MPa.

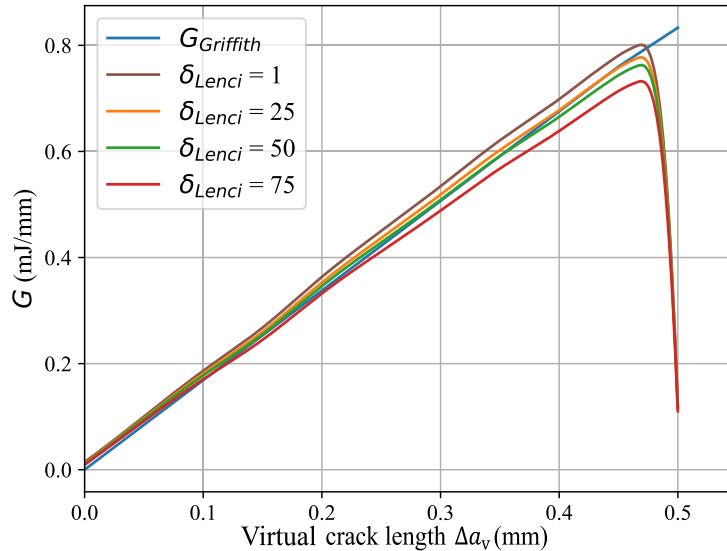


Fig. 20. $G(\Delta a_v)$ values computed by FEM for the problem in Fig. 4, using spring distributions with stiffness $k(r)$ defined by (19) along the fictitious crack Δa , for $\alpha = 1.5$ and different values of δ_{Lenci} . These results are compared with the Griffith ERR $G_{Griffith}(\Delta a_v)$ obtained from (13).

5.2.2. Mixed-mode crack problem with a kink crack

The same crack configuration and numerical model as described in Section 3.2.2 are considered, with a mother crack under pure shear loading conditions leading to an inclined daughter crack of length Δa_v growing along the fictitious crack path Δa . The only difference is that the spring distribution bridging this fictitious crack has varying stiffnesses as defined in (15) with $k(r)$ defined in (19). Fig. 21 plots the FEM results for ERR $G(\Delta a_v)$ at the tip of vanishing virtual (daughter) crack $\Delta a_v \rightarrow 0$ against the number of nodes N along Δa for several α values and $\delta_{Lenci} = 50$. Recall that the computed ERR is given by the elastic energy stored in

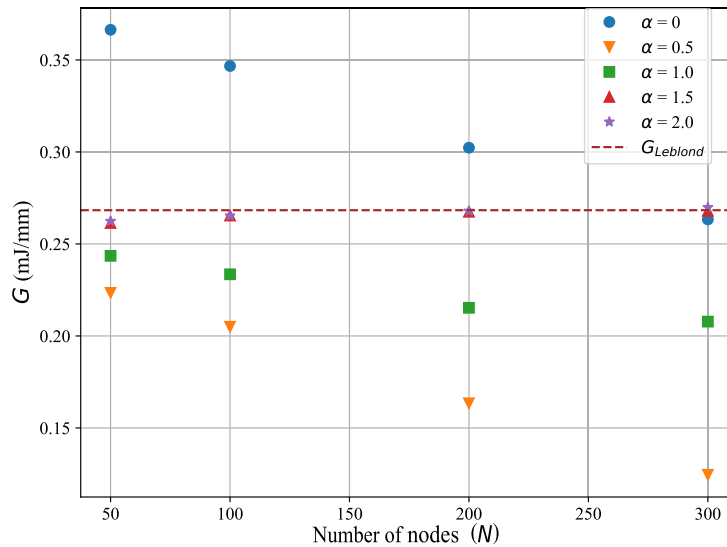


Fig. 21. ERR $G(\Delta a_v)$ for a vanishing virtual (daughter) crack $\Delta a_v \rightarrow 0$, computed by mesh refinement using spring distributions with stiffness $k(r)$ defined by (19), for $\delta_{Lenci} = 50$ and different values of the exponent α . The numerical results are compared with the analytical value $G_{Leblond}$ obtained from the asymptotic solution of [56].

the “first” spring on the fictitious crack path, located in the kink point. These ERR values are compared to the ERR value $G_{Leblond}$ predicted by the analytic expression from [56]. Taking the results for $\alpha = 0$ in Fig. 14 into account, it is evident from Fig. 21 that the ERR values for $\alpha \leq 1$ do not converge towards the analytical solution $G_{Leblond}$. Conversely, the ERR results for $\alpha > 1$ converge to the analytical solution $G_{Leblond}$, showing minimal dependency on mesh refinement, i.e. the number of nodes N .

The subsequent analysis explores the performance of the above spring distributions for various values of δ_{Lenci} from 1 to 75, with a fixed value of $\alpha = 1.5$, a value that showed convergence with mesh refinement in the previous analysis. The aim is to gain a deeper understanding of the predictive capabilities of spring distributions with this particular value of α . Fig. 22(a) shows that all the numerical values of ERR obtained for different δ_{Lenci} values and $\alpha = 1.5$ are very close to the analytical solution $G_{Leblond}$. Furthermore, Fig. 22(b) shows that they all converge quickly towards $G_{Leblond}$ except for those obtained for the very small value of $\delta_{Lenci} = 1$. In this study, the values of $\delta_{Lenci} = 50$ and 75 demonstrate superior accuracy compared to smaller values.

6. Relation to (extrinsic) cohesive zone models

For the sake of brevity, we focus on Mode I problems and consider extrinsic Cohesive Zone Models (CZMs) [62–68], where cohesive stress $\sigma(x)$ act on crack faces along the cohesive zone behind the fictitious crack tip. For linear softening, the relationship between the cohesive stress and crack opening displacement $\delta_n(x)$ is defined as [64,65]

$$\sigma(\delta_n) = \sigma_c - k_s \delta_n, \quad (20)$$

where σ_c is the tensile strength and k_s is the softening slope, see Fig. 23(a). For small values of $\delta_n \rightarrow 0_+$, the expression in (20) also represents the asymptotic behaviour for nonlinear softening curves. Note that the Dugdale model [69] is obtained by setting $k_s = 0$.

The crack faces in CZMs close smoothly at the fictitious crack tip, which is described by the following asymptotic law for crack opening displacement [64,65]

$$\delta_n(r) = c r^{3/2} + O(r^{5/2}), \quad \text{when } r \rightarrow 0, \quad (21)$$

where r is the distance to the fictitious crack tip and $c > 0$ a constant, see Fig. 23(b).

By defining the stiffness of the unloading-reloading branch in the cohesive law as usual, see Fig. 23(a),

$$k_n(r) = \frac{\sigma(\delta_n(r))}{\delta_n(r)}. \quad (22)$$

and substituting the relations (20) and (21) into this expression, we obtain the asymptotic law for stiffness in the cohesive zone,

$$\begin{aligned} k_n(r) &= \frac{\sigma_c}{c r^{3/2} + O(r^{5/2})} - k_s, \\ &= \frac{\sigma_c}{c} r^{-3/2} + O(r^{-1/2}) - k_s, \\ &= O(r^{-3/2}), \quad \text{when } r \rightarrow 0. \end{aligned} \quad (23)$$

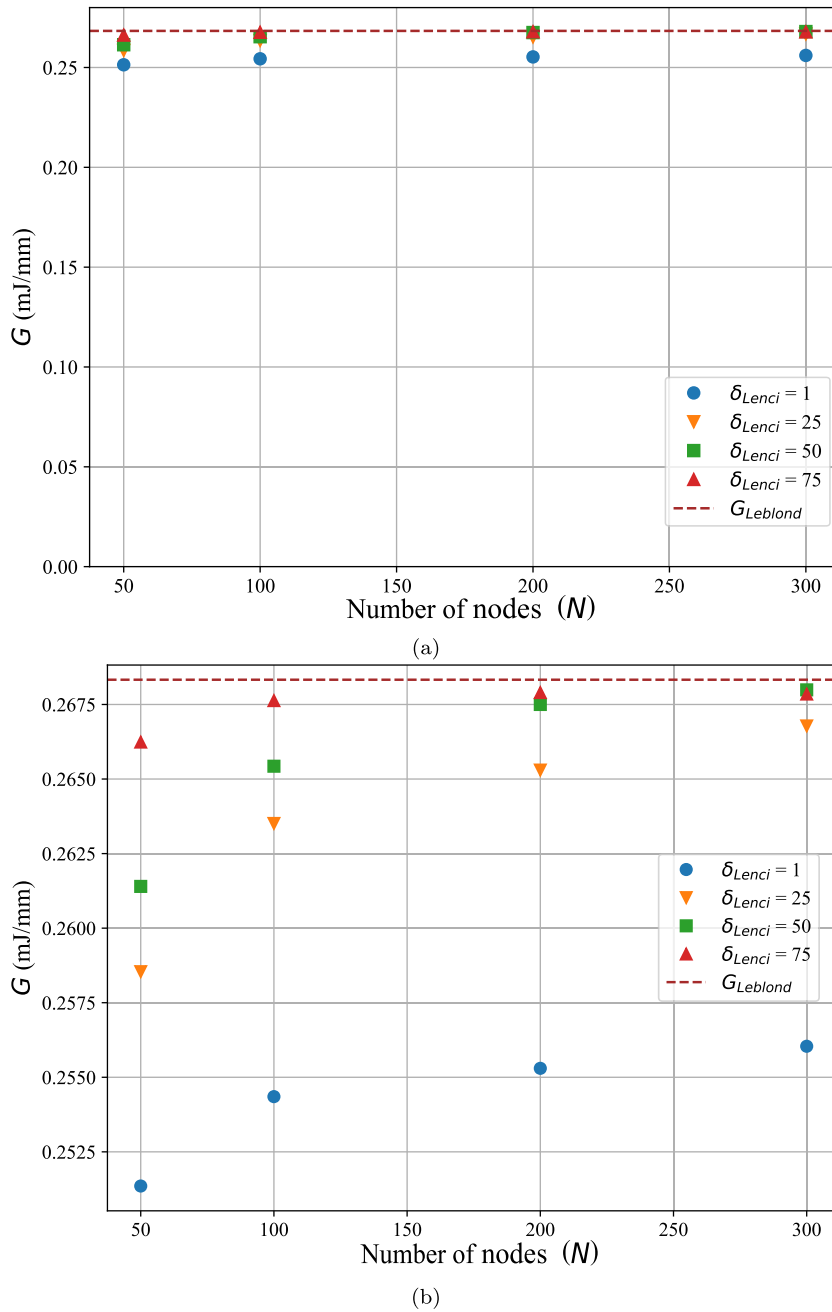


Fig. 22. ERR $G(\Delta a_v)$ for a vanishing virtual crack $\Delta a_v \rightarrow 0$, computed by mesh refinement using spring distributions with stiffness $k(r)$ defined by (19), for $\alpha = 1.5$ and various values of δ_{Lenci} : (a) full range of numerical results compared with the analytical asymptotic value $G_{Leblond}$ from [56], (b) zoomed-in view highlighting the numerical convergence.

This asymptotic law coincides with one of the power-law variations of stiffness in the Winkler spring distribution considered in the present study in (19), in particular when the exponent $\alpha = 3/2$ is taken. This exponent has been suggested as the most suitable when looking for a balance between the accuracy and computational costs.

Thus, we can expect that locally, near the fictitious crack tip, the solutions obtained in the present study for $\alpha = 3/2$ in (19) are similar to those obtained by extrinsic CZMs. In general, according to [50], for $\alpha > 1$ in (19), the tractions bridging the fictitious crack eliminate the square-root stress singularity at the fictitious crack tip (e.g., points F_i ($i = 1, 2, 3$) in Fig. 1(b)). In other words, they produce a negative SIF K_I^{bridge} , which annihilates the SIF due to the external load K_I^{load} , resulting in the vanishing total SIF $K_I^{load} + K_I^{bridge} = 0$. In this sense, the present work lies somewhere between the classical concept of bridged cracks with a positive

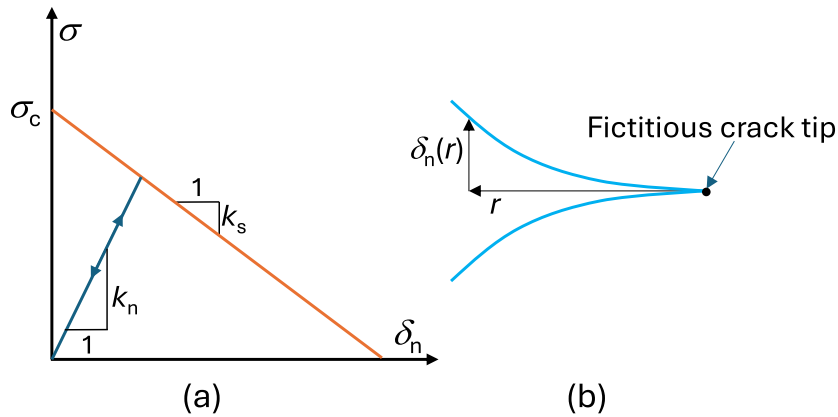


Fig. 23. (a) Cohesive law with linear softening and a unloading-reloading branch with stiffness k_n , (b) Smooth crack closing at the fictitious crack tip according to (21).

total SIF $K_I^{\text{load}} + K_I^{\text{bridge}} >$ and a square-root singularity at the crack tip [40–44], and the extrinsic CZMs with a vanishing total SIF, as discussed above.

However, there are also relevant differences in the overall stress distribution along the fictitious crack zone, e.g., the stresses and spring stiffness do not vanish at the opposite end-point of this zone (e.g., points P_i ($i = 1, 2, 3$) in Fig. 1(b)) as occurs in CZMs, neither the stress distribution is uniform in the fictitious crack zone as in the Dugdale model. Additionally, in CZMs σ_c is a priori given, while in the present study only the ratio $\frac{\sigma_c}{\epsilon_c}$ is defined by the coefficient in the power-law in (19).

Another key difference with extrinsic CZMs is that the present bridging law is linear elastic. This means that the resulting Boundary Value Problems (BVPs) to be solved remain entirely linear (provided the bulk material constitutive law and boundary conditions are also linear). This contrasts sharply with the nonlinear rigid-softening cohesive traction-separation laws, which inherently lead to highly nonlinear BVPs.

Furthermore, while both extrinsic cohesive elements and the proposed Winkler spring elements are inserted “on the fly” during a load step based on a strength criterion, their roles differ significantly. Extrinsic cohesive elements typically remain active across several subsequent load steps and considerably alter the local stress distribution near the cohesive zone. In contrast, the Winkler spring elements serve strictly as a numerical regularisation tool; they are not carried over to subsequent load steps, and their objective is to guide crack propagation modelling without significantly modifying the stress distribution near the fictitious crack path.

Ultimately, achieving an efficient and robust finite element implementation of extrinsic CZMs is quite difficult. Consequently, standard FEM packages rely almost exclusively on intrinsic CZMs. However, intrinsic formulations inherently introduce a spurious stress singularity at the cohesive zone tip. It is precisely this numerical artefact that our graded spring approach systematically eliminates.

7. J -integral

In this section we explore the possibility to compute the ERR at the virtual crack tip $G(\Delta a_v)$ by using a path-independent J -integral [70]. Although the expression of the path-independent J -integral for a crack in spring interface with uniform stiffness was published long time ago in [71], it seems that its generalisation to spring distributions with varying stiffness is not available. Related remarks for a non-homogeneous cohesive interface with varying CZM parameters can be found in [72]. As will be seen such generalisation has some similarities with the generalisation of J -integral to non-homogeneous materials, as e.g. functionally graded materials (FGMs) [73,74].

Fig. 24 shows two paths for computing a path-independent J -integral. The integration path in Fig. 24(a) crosses the fictitious crack zone at point C located between the virtual crack tip V and the fictitious crack tip F , while the integration path in Fig. 24(b) does not intersect this zone.

It can be shown that the J -integral for the path in Fig. 24(a) is given by the following expression

$$J = \frac{1}{2} (k_n(x_V)\delta_n^2(x_V) + k_t(x_V)\delta_t^2(x_V)) \quad (24a)$$

$$= \int_{\Gamma_0} (U n_x - T_i^n u_{i,x}) ds - \int_{\Gamma_1} \frac{1}{2} (k_{n,x}(x)\delta_n^2(x) + k_{t,x}(x)\delta_t^2(x)) dx + \frac{1}{2} (k_n(x_C)\delta_n^2(x_C) + k_t(x_C)\delta_t^2(x_C)), \quad (24b)$$

where U is the strain energy density, T_i^n is the traction vector, and $i = x, y$. This generalises the expression for uniform spring stiffness presented in [71], which lacks the integral over Γ_1 . The path independence of the J -integral in (24b) follows from the fact that the term in (24a) remains constant giving the ERR $G(\Delta a_v)$. A significant advantage of employing this J -integral to compute the

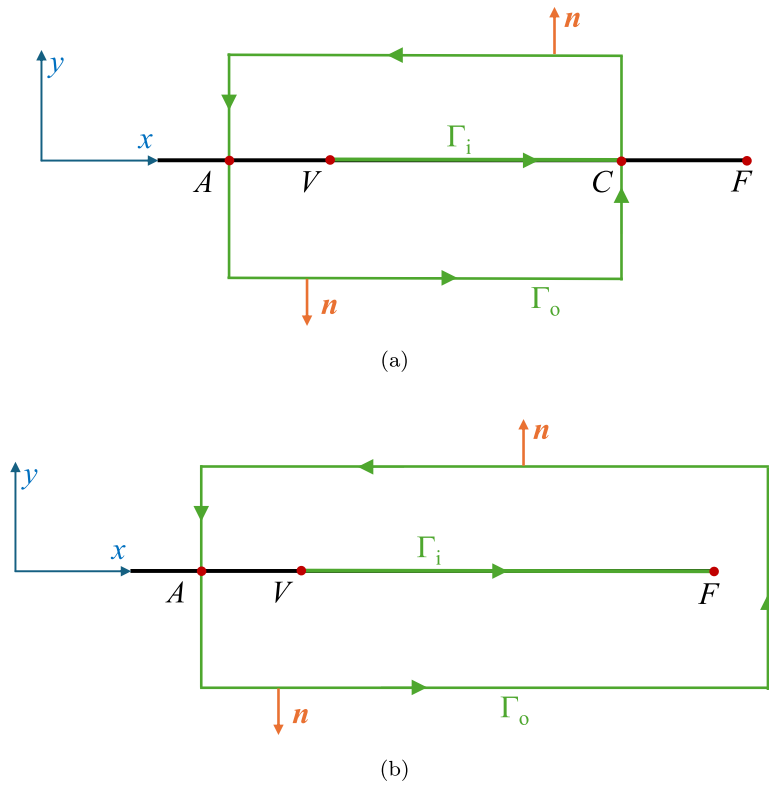


Fig. 24. Integration paths for the J -integral. (a) Path intersecting the fictitious crack zone at a point C between the virtual and fictitious crack tips. (b) Path bypassing the fictitious crack zone between the two crack tips.

ERR $G(\Delta a_v)$ is its numerical robustness; its accuracy is less sensitive to potential errors in calculating the crack-tip displacements at the virtual crack tip V .

A simple way how to prove the above expression is to consider two closed integration paths Γ^+ and Γ^- , respectively, located above and below the fictitious crack for which the J -integral is zero, as illustrated on Fig. 25. The upper integration path is split as follows $\Gamma^+ = \Gamma_o^+ \cup \Gamma_c^+ \cup \Gamma_{V\epsilon}^+ \cup \Gamma_i^+$, and similarly for Γ^- . The semicircular path $\Gamma_{V\epsilon}^+$ is used to bypass the logarithmic stress-singularity at V by considering a vanishing radius $\epsilon \rightarrow 0_+$. Note that $\Gamma_o = \Gamma_o^- \cup \Gamma_o^+$. Then, by adding the zero integrals over Γ^- and Γ^+ , and after some rearrangements we get, for $\epsilon \rightarrow 0_+$,

$$0 = \int_{\Gamma_o} (\mathcal{U} n_x - T_i^n u_{i,x}) ds + \int_{\Gamma_i} (k_n(x) \delta_n(x) \delta_{n,x}(x) + k_t(x) \delta_t(x) \delta_{t,x}(x)) dx, \quad (25)$$

where it has been considered that $n_x = 0$ on the crack faces, the traction vector $T_i^n = 0$ on $\Gamma_c^\pm$, the integrals along $\Gamma_{V\epsilon}^\pm$ vanish due to a very weak logarithmic stress singularity at V . Then, by integration by parts along Γ_i we easily get

$$0 = \int_{\Gamma_o} (\mathcal{U} n_x - T_i^n u_{i,x}) ds - \int_{\Gamma_i} \frac{1}{2} (k_{n,x}(x) \delta_n^2(x) + k_{t,x}(x) \delta_t^2(x)) dx + \frac{1}{2} (k_n(x_C) \delta_n^2(x_C) + k_t(x_C) \delta_t^2(x_C)) - \frac{1}{2} (k_n(x_V) \delta_n^2(x_V) + k_t(x_V) \delta_t^2(x_V)), \quad (26)$$

which leads to (24).

Similarly, it can be shown that the J -integral for the path in Fig. 24(b) is given by the following expression

$$J = \frac{1}{2} (k_n(x_V) \delta_n^2(x_V) + k_t(x_V) \delta_t^2(x_V)) \quad (27a)$$

$$= \int_{\Gamma_o} (\mathcal{U} n_x - T_i^n u_{i,x}) ds - \int_{\Gamma_i} \frac{1}{2} (k_{n,x}(x) \delta_n^2(x) + k_{t,x}(x) \delta_t^2(x)) dx. \quad (27b)$$

Similarly as above, the path independence of the J -integral in (27b) is inherent because the term in (27a) gives $G(\Delta a_v)$.

A simple way how to prove the above expression is to consider a closed integration path Γ bypassing the logarithmic stress singularity at V and a possible stress singularity at F , as illustrated on Fig. 26. This path is split similarly as in the previous case as $\Gamma = \Gamma_o \cup \Gamma_c^+ \cup \Gamma_{V\epsilon}^+ \cup \Gamma_i^+ \cup \Gamma_{F\epsilon}^- \cup \Gamma_i^- \cup \Gamma_{V\epsilon}^- \cup \Gamma_c^-$. Then, considering a power-law stiffness variation according to (19) for $\alpha > 1$, which

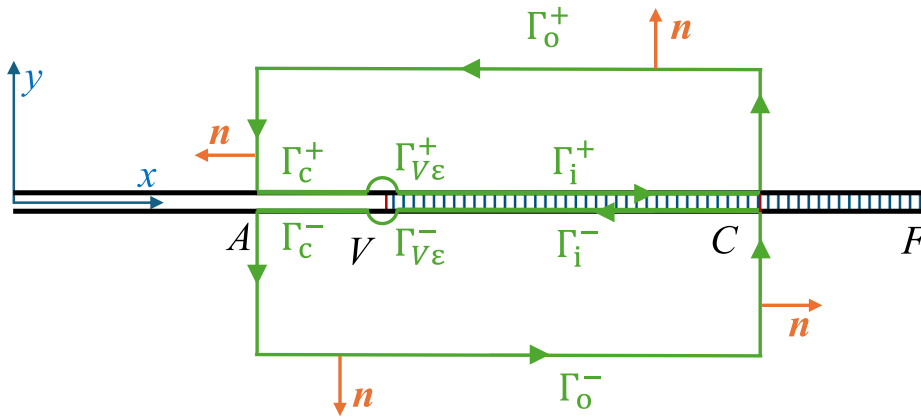


Fig. 25. Two closed integration paths Γ^+ and Γ^- along which the J -integral is zero.

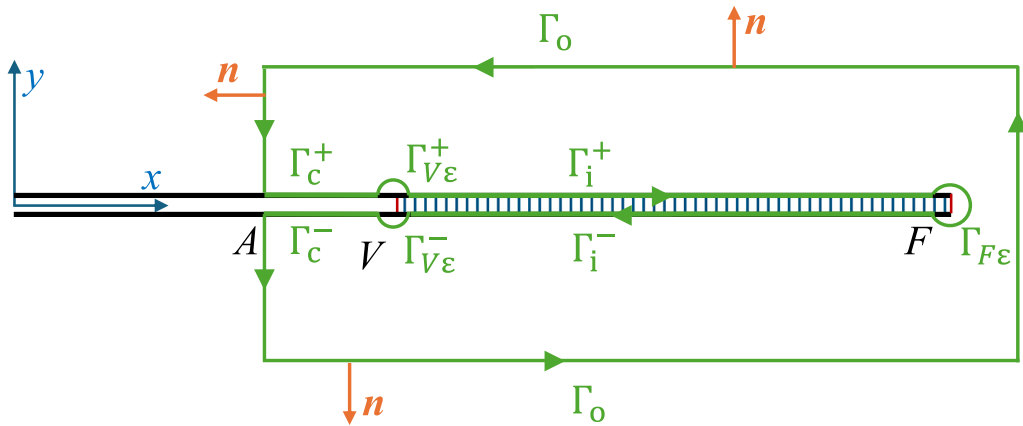


Fig. 26. A closed integration path Γ along which the J -integral is zero.

removes the stress singularity at F , and vanishing radius $\varepsilon \rightarrow 0_+$, we get similarly as in (25)

$$0 = \int_{\Gamma_0} (\mathcal{U} n_x - T_i^n u_{i,x}) ds + \int_{\Gamma_i} (k_n(x) \delta_n(x) \delta_{n,x}(x) + k_t(x) \delta_t(x) \delta_{t,x}(x)) dx, \quad (28)$$

where Γ_i is the bridged part of the fictitious crack between points V and F . Then, by integration by parts along Γ_i we easily get

$$0 = \int_{\Gamma_0} (\mathcal{U} n_x - T_i^n u_{i,x}) ds - \int_{\Gamma_i} \frac{1}{2} (k_{n,x}(x) \delta_n^2(x) + k_{t,x}(x) \delta_t^2(x)) dx - \frac{1}{2} (k_n(x_V) \delta_n^2(x_V) + k_t(x_V) \delta_t^2(x_V)), \quad (29)$$

which leads to (27). To get the last expression we have used the fact that $\lim_{x \rightarrow x_F} k_n(x) \delta_n^2(x) = 0$ and $\lim_{x \rightarrow x_F} k_t(x) \delta_t^2(x) = 0$ in view of the bounded tractions $k_n(x) \delta_n(x)$ and $k_t(x) \delta_t(x)$ at F , for $\alpha > 1$, and vanishing $\delta_n(x)$ and $\delta_t(x)$ at F . Unlike the J -integral in (24), the numerical application of (27) requires careful treatment of the strong singularity in the derivatives of spring stiffnesses at F within the integral over Γ_i to prevent numerical errors.

If $\alpha = 1$ in (19), the stress singularity at F is weaker than the square-root singularity [50]. Consequently, the integral over $\Gamma_{F\varepsilon}$ vanishes. However, the integral over Γ_i becomes weakly singular and should be computed as an improper integral.

Conversely, if $0 \leq \alpha < 1$ in (19), a square-root stress singularity is present at F [50]. Thus, the integral over $\Gamma_{F\varepsilon}$ does not vanish; instead, it yields a finite value that must be added to the J -integral expression in (27b). Furthermore, for $0 < \alpha < 1$, the integral over Γ_i becomes weakly singular and must also be evaluated as an improper integral. Note that this integral over Γ_i is zero when $\alpha = 0$.

As expected, the J -integral expressions in (24a) and (27a) are identical to the ERR G given in (11).

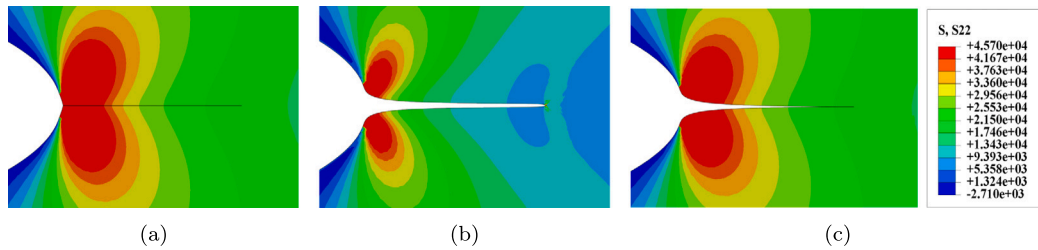


Fig. 27. Stress distribution contours in the vicinity of the fictitious crack path obtained using (a) tied spring constraints, (b) constant-stiffness spring distributions with $M = 10$ in (9), and (c) power-law graded spring distribution with $\alpha = 1.5$ and $\delta_{\text{Lenci}} = 75$ in (19). Stress values are in MPa.

8. Concluding remarks

8.1. Main contributions

Accurately modelling the competition between the simultaneous nucleation and propagation of multiple cracks under quasi-static loading remains a fundamental challenge in the field of LEFM. To contribute towards achieving this goal, the present paper addresses the challenging question of whether it is possible to model the propagation of an internal crack in pristine bulk material by:

- introducing a fictitious crack path, denoted as Δa , which is determined by a strength criterion, for example;
- introducing a linear elastic spring distribution between the faces of this fictitious crack; and
- modelling the propagation of a virtual crack, denoted as Δa_v , along this fictitious crack path by simply deactivating the springs. Notably, this spring distribution allows a damage variable to be introduced along the fictitious crack path in order to describe virtual crack propagation.

To predict crack propagation, it is essential to accurately approximate the ERR at the tip of the virtual crack by the elastic energy stored in the spring therein. This requires an assessment of various crack scenarios.

The results of a careful numerical study of elastic solutions at fictitious cracks Δa bridged by linear elastic spring distributions with uniform and power-law stiffness variations have been presented. The study also involved computing the ERR at the virtual crack Δa_v , growing along the fictitious crack path by simply deactivating springs on Δa_v , for both types of spring distribution with uniform and varying stiffness. This ERR is given by the elastic strain energy stored in the spring located at the tip of Δa_v .

The results clearly demonstrated that spring distributions with uniform stiffness are unsuitable for modelling virtual crack propagation, because the ERR results are affected by the pollution effect of the spurious square-root stress singularity at the fictitious crack tip.

The spring distributions with power-law stiffness variation with an exponent $\alpha > 1$ in (19) are very robust and suitable for modelling virtual crack propagation. In these distributions, the spring stiffness $k(r)$ diverges to infinity as one approaches the fictitious crack tip. This prevents a jump in spring stiffness as illustrated in Fig. 3, and restores consistency with the apparent infinite stiffness in the pristine bulk material ahead of the fictitious crack tip. Furthermore, inspired by the seminal work of [58], a dimensionless parameter δ_{Lenci} is introduced to the power-law for spring stiffness (19) whose proper adjustment is essential to increase the accuracy of the ERR estimation. The numerical study reveals that the ERR predictions are essentially insensitive to mesh refinement.

These spring distributions provide accurate ERR predictions at the virtual-crack tip, because the stress singularity at the fictitious-crack tip is eliminated. To illustrate this, the stress contours obtained for the Mode I centre-cracked plate under uniaxial tension, see Section 3.1 and Fig. 4, are plotted in Fig. 27. This figure shows that the stress field differs significantly when a spring distribution with uniform stiffness is used, owing to the emergence of a spurious square-root singularity at the fictitious crack tip, Fig. 27(b). In contrast, the power-law stiffness distribution employed in Fig. 27(c), defined by $\alpha = 1.5$, provides a higher effective stiffness near the fictitious crack tip and eliminates this artificial singularity, resulting in a stress field near the fictitious crack essentially identical to that obtained either without a fictitious crack or, equivalently, with a perfect bonding (tying) condition between the crack faces, Fig. 27(a).

In summary, the present approach allows virtual crack propagation to be modelled with uniform meshes and without traction singularities along the fictitious crack path, while maintaining the linearity of the fracture problem solution.

Furthermore, the relationship between the proposed framework and extrinsic CZMs has been clarified, and a new path independent J -integral formulation for graded spring distributions has been developed.

8.2. Enriched finite elements with logarithmic stress-singularity

Given the presence of a logarithmic stress-singularity at the virtual crack tip in the present spring model with a power-law variation of stiffness, the recently developed and successfully tested enriched crack-tip elements for logarithmic stress-singularity

in spring interfaces [53,54] can be applied to enhance the accuracy and computational efficiency of the ERR extraction at the virtual crack tip. This is particularly relevant for stiff spring interfaces, where conventional FEM approaches require excessive mesh refinement.

8.3. FEM implementation of the PMTE-SC procedure

The results obtained in this study open up new possibilities for the computational modelling of internal cracks. Indeed, the ultimate goal of this study is to simulate internal crack onset and propagation in bulk materials through the minimisation of the total energy – defined as the sum of the potential and dissipated energies – subject to a stress condition [33]. This methodology synthesises and extends foundational concepts introduced in the seminal works by Francfort and Marigo [2] and Leguillon [15,16]. The proposed approach has been integrated into a FEM software and successfully applied to predict the nucleation and growth of interface cracks in [34,35], and recently also of both internal and interface cracks in [75], where PMTE-SC predictions demonstrated strong agreement with experimental data, analytical solutions, and established numerical methods such as CZMs. In particular, the latter examples numerically confirm that choosing the same exponent $\alpha = 1.5$ for both normal and tangential stiffnesses in (15) is highly effective for mixed-mode conditions.

A key advantage of the proposed formulation is the separate convexity of the total energy functional with respect to the displacement field and the damage variable. This mathematical property is fundamentally enabled by the introduced power-law spring distribution along the fictitious cracks. Consequently, it allows for highly efficient and versatile finite element implementations using the AMA [3]. Notably, the PMTE-SC currently stands out as likely the most computationally efficient, general-purpose numerical procedure for implementing the CC of FFM, a framework originally introduced over 25 years ago in [15,16].

CRediT authorship contribution statement

K. Ambikakumari Sanalkumar: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Investigation. **V. Mantič:** Writing – review & editing, Validation, Supervision, Resources, Methodology, Investigation, Funding acquisition, Conceptualization. **M. Paggi:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

References

- [1] Mielke A, Roubíček T. Rate-independent systems: Theory and application. New York: Springer; 2015.
- [2] Francfort GA, Marigo J-J. Revisiting brittle fracture as an energy minimization problem. *J Mech Phys Solids* 1998;46(8):1319–42.
- [3] Bourdin B, Francfort GA, Marigo J-J. Numerical experiments in revisited brittle fracture. *J Mech Phys Solids* 2000;48(4):797–826.

- [4] Karma A, Kessler DA, Levine H. Phase-field model of mode III dynamic fracture. *Phys Rev Lett* 2001;87:045501.
- [5] Bourdin B, Francfort GA, Marigo J-J. The variational approach to fracture. *J Elasticity* 2008;91:5–148.
- [6] Miehe C, Hofacker M, Welschinger F. A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits. *Comput Methods Appl Mech Engrg* 2010;199(45):2765–78.
- [7] Bordas SPA, Menk A, Natarajan S, editors. Partition of unity methods. John Wiley; 2023.
- [8] Pommier S, Gravouil A, Combescure A, Moës N. Extended Finite Element Method for Crack Propagation. ISTE and Wiley; 2011.
- [9] Burman E, Claus S, Hansbo P, Larson MG, Massing A. CutFEM: Discretizing geometry and partial differential equations. *Internat J Numer Methods Engrg* 2015;104(7):472–501.
- [10] Bordas SPA, Burman E, Larson MG, Olshanskii MA, editors. Geometrically unfitted finite element methods and applications. Springer; 2017.
- [11] Badia S, Verdugo F, Martín AF. The aggregated unfitted finite element method for elliptic problems. *Comput Methods Appl Mech Engrg* 2018;336:533–53.
- [12] Antonietti PF, Beirão da Veiga L, Manzini G, editors. The virtual element method and its applications. Springer; 2022.
- [13] Wriggers P, Aldakheel F, Hudobivnik B. Virtual element methods in engineering sciences. Springer; 2024.
- [14] Hashin Z. Finite thermoelastic fracture criterion with application to laminate cracking analysis. *J Mech Phys Solids* 1996;44(7):1129–45.
- [15] Leguillon D. A criterion for crack nucleation at a notch in homogeneous materials. *C R Acad Sci Ser IIB - Mech* 2001;329(2):97–102.
- [16] Leguillon D. Strength or toughness? A criterion for crack onset at a notch. *Eur J Mech A Solids* 2002;21:61–72.
- [17] Parvizi A, Garrett KW, Bailey JE. Constrained cracking in glass fibre-reinforced epoxy cross-ply laminates. *Mater Sci* 1978;13:195–201.
- [18] Fisher JR, Gurland J. Void nucleation in spheroidized carbon steels, Part 2: Model. *Mater Sci* 1981;15(5):193–202.
- [19] Bažant ZP, Cedolin L. Stability of structures. elastic, inelastic, fracture and damage theories. Oxford University Press; 1991.
- [20] Tseng TC. A model of void nucleation from ellipsoidal inclusions in ductile fracture. *Scr Metall Mater* 1993;28(9):1065–70.
- [21] Cornetti P, Pugno N, Carpinteri A, Taylor D. Finite fracture mechanics: A coupled stress and energy failure criterion. *Engrg Frac Mech* 2006;73(14):2021–33.
- [22] Mantič V. Interface crack onset at a circular cylindrical inclusion under a remote transverse tension. Application of a coupled stress and energy criterion. *Int J Solids Struct* 2009;46(6):1287–304.
- [23] Weißgraeber P, Leguillon D, Becker W. A review of finite fracture mechanics: Crack initiation at singular and non-singular stress raisers. *Arch Appl Mech* 2016;86(1–2):375–401.
- [24] Doitrand A, Duminy T, Girard H, Chen X. A review of the coupled criterion. *J Theor Comput Appl Mech* 2024. jtcam:11072.
- [25] Wu J-Y. A unified phase-field theory for the mechanics of damage and quasi-brittle failure. *J Mech Phys Solids* 2017;103:72–99.
- [26] E. Tanné T Li, Bourdin B, Marigo J-J, Maurini C. Crack nucleation in variational phase-field models of brittle fracture. *J Mech Phys Solids* 2018;110:80–99.
- [27] Sargado JM, Keilegavlen E, Berre I, Nordbotten JM. High-accuracy phase-field models for brittle fracture based on a new family of degradation functions. *J Mech Phys Solids* 2018;111:458–89.
- [28] Kumar A, Bourdin B, Francfort GA, Lopez-Pamies O. Revisiting nucleation in the phase-field approach to brittle fracture. *J Mech Phys Solids* 2020;142:104027.
- [29] Larsen CJ, Dolbow JE, Lopez-Pamies O. A variational formulation of Griffith phase-field fracture with material strength. *Int J Fract* 2024;247:319–27.
- [30] Vicentini F, Zolesi C, Carrara P, Maurini C, Lorenzis L De. On the energy decomposition in variational phase-field models for brittle fracture under multi-axial stress states. *Int J Fract* 2024;247:291–317.
- [31] Vicentini F, Heinzmann J, Carrara P, Lorenzis L De. Variational phase-field modeling of cohesive fracture with flexibly tunable strength surface. *J Mech Phys Solids* 2026;207:106424.
- [32] Bourdin B, Marigo J-J, Maurini C, Zolesi C. A variational approach to fracture incorporating any convex strength criterion. 2025, arXiv preprint arXiv:2506.22558.
- [33] Mantič V. Prediction of initiation and growth of cracks in composites. Coupled stress and energy criterion of the finite fracture mechanics (Keynote lecture). In: Paris F, editor. Proceedings of the 16th European conference on composite materials (ECCM16). 2014, <https://hal.science/hal-04770340v1>.
- [34] Muñoz-Reja M. Analytical and numerical predictions by LEBIM and FFM for interface cracks in composites and their joints (Ph. D. Thesis), University of Seville; 2021.
- [35] Muñoz-Reja M, Mantič V, Távora L. Comparative analytical study of the coupled criterion and the principle of minimum total energy with stress condition applied to linear elastic interfaces. *Theor Appl Fract Mech* 2022;119:103274.
- [36] Winkler E. Die Lehre von der Elasticität und Festigkeit mit besondere Rücksicht auf ihre Anwendung in der Technik, für polytechnische Schulen, Bauakademien, Ingenieure, Maschinenbauer, Architekten, etc.. Prag: Verlag von H. Dominicus; 1867.
- [37] Dillard D, Mukherjee B, Karnal P, Batra R, Frechette J. A review of Winkler's foundation and its profound influence on adhesion and soft matter applications. *Soft Matter* 2018;14:3669–83.
- [38] Roubíček T, Kružík M, Zeman J. Delamination and adhesive contact models and their mathematical analysis and numerical treatment, chapter 9. In: Mantič V, editor. Mathematical methods and models in composites. London: Imperial College Press; 2014, p. 349–400.
- [39] Roubíček T, Kružík M, Zeman J, Panagiotopoulos CG, Vodička R, Mantič V. Delamination and adhesive contacts, their mathematical modeling and numerical treatment, chapter 11. In: Mantič V, editor. Mathematical methods and models in composites. 2nd ed.. London: World Scientific; 2023, p. 497–578.
- [40] Rose LRF. Crack reinforcement by distributed springs. *J Mech Phys Solids* 1987;35(4):383–405.
- [41] Bao G, Suo Z. Remarks on crack-bridging concepts. *Appl Mech Rev* 1992;45(8):355–66.
- [42] Cox BN, Marshall DB. Concepts for bridged cracks in fracture and fatigue. *Acta Metall* 1994;42(2):341–63.
- [43] Carpinteri A, Massabó R. Bridged versus cohesive crack in the flexural behavior of brittle-matrix composites. *Int J Fract* 1996;81:125–45.
- [44] Massabó R. The bridged-crack model, Chapter 4. In: Carpinteri A, editor. Nonlinear crack models for nonmetallic materials. Kluwer Academic Publishers; 1999, p. 141–208.
- [45] Mishuris G. Interface crack and nonideal interface concept (mode III). *Int J Fract* 107:2001.
- [46] Mishuris GS, Kuhn G. Asymptotic behaviour of the elastic solution near the tip of a crack situated at a nonideal interface. *ZAMM Z Angew Math Mech* 81:2001.
- [47] Ueda S, Biwa S, Watanabe K, Heuer R, Pecorari C. On the stiffness of spring model for closed crack. *Internat J Engrg Sci* 2006;44(13–14):874–88.
- [48] Watanabe K, Ueda S, Biwa S, Pecorari C, Heuer R. Closed interface crack with singular spring stiffness model. *Internat J Engrg Sci* 2007;45(2–8):210–26.
- [49] Wu J, Zhang L, Wan L. A mode-III crack under adhesion studied by non-uniform linear spring models. *Acta Mech* 2017;228:1621–9.
- [50] Piña-León N, Mantič V, Jiménez-Alfaro S. A recursive method for computing singular solutions in corners with homogeneous Dirichlet-Robin boundary condition with power-law coefficient variation, *ZAMM - J Appl Math Mech* (in press).
- [51] Mantič V, Távora L, Blázquez A, Graciani E, Paris F. A linear elastic-brittle interface model: application for the onset and propagation of a fibre-matrix interface crack under biaxial transverse loads. *Int J Fract* 2015;195:15–38.
- [52] Dassault Systèmes Simulia. Abaqus analysis user's guide. 2025, Version 2025, online documentation..
- [53] Mantič V, Vázquez-Sánchez A, Romero-Laborda M, Muñoz-Reja M, Jiménez-Alfaro S, Távora L. A new crack-tip element for the logarithmic stress-singularity of mode-III cracks in spring interfaces. *Comput Mech* 2024;74:641–60.
- [54] Herrera-Garrido MA, Távora L, Muñoz-Reja M, Vázquez-Sánchez A, Mantič V. Enriched crack-tip element for logarithmic stress singularities of mode I and mode II cracks in spring interfaces. *Engrg. Frac. Mech* 2026;336:111941.
- [55] Griffith AA. The phenomena of flow and rupture in solids. *Philos Trans A Math Phys Eng Sci* 1921;221:163–98.

- [56] Amestoy M, Leblond JB. Crack paths in plane situations–II Detailed form of the expansion of the stress intensity factors. *Int J Solids Struct* 1992;29(4):465–501.
- [57] Yu H, Kuna M. Interaction integral method for computation of crack parameters K-T – A review. *Engrg. Frac. Mech* 2021;249:107722.
- [58] Lenci S. Analysis of a crack at a weak interface. *Int J Fract* 2001;108:275–90.
- [59] Yosibash Z. Singularities in elliptic boundary value problems and elasticity and their connection with failure initiation. Springer; 2012.
- [60] Jiménez-Alfaro S, Mantič V. Crack tip solution for mode III cracks in spring interfaces. *Engrg. Frac. Mech* 2023;288:109293.
- [61] Jiménez-Alfaro S, Mantič V. Crack tip solution for mode I and mode II cracks in spring interfaces (to be submitted).
- [62] Barenblatt GI. The mathematical theory of equilibrium cracks in brittle fracture. *Adv Appl Mech* 1962;7:55–129.
- [63] Hillerborg A, Modéer M, Petersson P-E. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cem Concr Res* 1976;6(6):773–81.
- [64] Planas J, Elices M. Asymptotic analysis of a cohesive crack: 1. Theoretical background. *Int J Fract* 1992;55:153–77.
- [65] Planas J, Elices M. Asymptotic analysis of a cohesive crack: 2. Influence of the softening curve. *Int J Fract* 1993;64:221–37.
- [66] Bažant ZP, Planas J. Fracture and size effect in concrete and other quasibrittle materials. CRC Press; 1998.
- [67] Carpinteri A. Fracture and complexity. One century since Griffith's milestone. Springer; 2021.
- [68] Bažant ZP, Le J-L, Salviato M. Quasibrittle fracture mechanics and size effect: A first course. Oxford University Press; 2021.
- [69] Dugdale DS. Yielding of steel sheets containing slits. *J Mech Phys Solids* 1960;8:100–4.
- [70] Rice JR. A path independent integral and the approximate analysis of strain concentration by notches and cracks. *J Appl Mech* 1968;35:379–86.
- [71] de Montleau P, Rigolot A. Mécanique de la rupture fragile des couches fortement hétérogènes. *Comptes Rendus de L'Académie Des Sci - Ser IIB* 1997;324:83–9.
- [72] Škec L, Alfano G, Jelenić G. On G_c , J_c and the characterisation of the mode-I fracture resistance in delamination or adhesive debonding. *Int J Solids Struct* 2018;(100–122):144–5.
- [73] Eischen JW. Fracture of nonhomogeneous materials. *Int J Fract* 1987;34:3–22.
- [74] Paulino GH, Jin Z-H, Dodds RH. 2.13 - Failure of functionally graded materials. In: Milne I, Ritchie RO, Karihaloo B, editors. *Comprehensive structural integrity* (vol. 2). Pergamon: Elsevier, Oxford; 2003, p. 607–44.
- [75] Ambikakumari Sanalkumar K. Total energy minimization with stress conditions for mixed mode fracture in heterogeneous materials and structures (Ph. D. Thesis), University of Seville; 2026.