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Metastability for the degenerate Potts Model with positive external magnetic field under Glauber dynamics[☆]

Gianmarco Bet[a,*], Anna Gallo[b], F.R. Nardi[a,c]

[a] Department of Mathematics and Computer Science "U. Dini", University of Florence, Italy
[b] IMT School for Advanced Studies, Lucca, Italy
[c] Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands

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ABSTRACT

We consider the ferromagnetic q-state Potts model on a finite grid graph with non-zero external field and periodic boundary conditions. The system evolves according to Glauber-type dynamics described by the Metropolis algorithm, and we focus on the low temperature asymptotic regime. We analyze the case of positive external magnetic field associated to one spin value. In this energy landscape there is one stable configuration and $q-1$ metastable states. We study the asymptotic behavior of the first hitting time from any metastable state to the stable configuration as $\beta \to \infty$ in probability, in expectation, and in distribution. We also identify the exponent of the mixing time and find an upper and a lower bound for the spectral gap. Finally, we identify all the minimal gates and the tube of typical trajectories for the transition from any metastable state to the unique stable configuration by giving a geometric characterization.

1. Introduction

Metastability is a phenomenon that is observed when a physical system is close to a first order phase transition. More precisely, this phenomenon takes place when the physical system, for some specific values of the parameters, is imprisoned for a long time in a state which is different from the equilibrium state. The former is known as the *metastable state*, the latter is the *stable state*. After a long (random) time, the system may exhibit the so-called *metastable behavior* and this happens when the system performs a sudden transition from the metastable state to the stable state. On the other hand, when the system lies on the phase coexistence line, it is of interest to understand precisely the transition between two (or more) stable states. This is the so-called *tunneling behavior*.

The phenomenon of metastability occurs in several physical situations, such as supercooled liquids, supersaturated gases, ferromagnets in the hysteresis loop and wireless networks. For this reason, many models for the metastable behavior have been developed throughout the years. In these models a suitable stochastic dynamics is chosen and typically three main issues are investigated. The first is the study of the *first hitting time* at which the process starting from a metastable state visits some stable states. The second issue is the study of the so-called set of *critical configurations*, which are those configurations that the process visits during the transition from the metastable state to some stable states. The third issue is the study of the *tube of typical paths*, i.e., the

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* Corresponding author.
E-mail addresses: gianmarco.bet@unifi.it (G. Bet), anna.gallo@imtlucca.it (A. Gallo).

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set of the typical trajectories followed by the process during the transition from the metastable state to some stable states. On the other hand, when a system exhibits tunneling behavior the same three issues above are investigated for the transition between any two stable states.

In this paper we study the metastable behavior of the q-state Potts model with non-zero external magnetic field on a finite two-dimensional discrete torus Λ. We will refer to Λ as a *grid-graph*. The q-state Potts model is an extension of the classical Ising model from $q = 2$ to an arbitrary number q of spins with $q > 2$. The state space $\mathcal{X}$ is given by all possible configurations σ such that at each site i of Λ lies a spin with value $\sigma(i) \in \{1, \dots, q\}$. To each configuration $\sigma \in \mathcal{X}$ we associate an energy $H(\sigma)$ that depends on the local ferromagnetic interaction, $J = 1$, between nearest-neighbor spins, and on the external magnetic field h related only to a specific spin value. Without loss of generality, we choose this spin equal to the spin 1. We study the q-state ferromagnetic Potts model with Hamiltonian $H(\sigma)$ in the limit of large inverse temperature $\beta \to \infty$. The stochastic evolution is described by a *Glauber-type dynamics*, that is a Markov chain on the finite state space $\mathcal{X}$ with transition probabilities that allow single spin-flip updates and that is given by the Metropolis algorithm. This dynamics is reversible with respect to the stationary distribution that is the *Gibbs measure* μ_β, see (2.3).

Our analysis focuses on the model to which we will refer as *q-Potts model with positive external magnetic field*. In this energy landscape there are $q - 1$ degenerate-metastable states and only one stable state. Here, we use the term 'degenerate' to point out that we focus on a energy landscape which is different from the one of the typical Ising model, in which the uniqueness of the stable and metastable configuration is guaranteed. In the metastable configurations all spins are equal to some m, for $m \in \{2, \dots, q\}$, while the stable state is the configuration in which all spins are equal to 1. In this case, we focus our attention on the transition from one of the metastable states to the stable configuration.

The goal of this paper is to investigate all the three issues of metastability for the metastable behavior of the q-Potts model with positive external magnetic field. More precisely, we investigate the asymptotic behavior of the transition time, and identify the set of critical configurations and the tube of typical trajectories for the transition from a metastable state to the unique stable state. Furthermore, we also identify the union of all the critical configurations for the transition from a metastable configuration to the other metastable states.

Let us now briefly describe the strategy that we adopt. First we show that the metastable set contains the $q - 1$ configurations where all spins are equal to some $m \in \{2, \dots, q\}$. We give asymptotic bounds in probability for the first hitting time from any metastable state to the stable configuration and we identify the order of magnitude of the expected hitting time. Moreover, we characterize the behavior of the mixing time in the low-temperature regime and give a lower and an upper bound of the spectral gap. Finally, we find the set of all minimal gates for the transition from a metastable state to the stable configuration. For any $m \in \{2, \dots, q\}$, if the starting configuration is the one with all spins equal to m, we prove that the minimal gate contains those configurations in which all spins are m except those, which are 1, in a quasi-square with a unit protuberance on one of the longest sides. Furthermore, we prove that during the metastable transition the process almost surely does not visit any metastable state different from the initial one, and we exploit this result to identify the union of all minimal gates for the transition from a metastable state to the other metastable configurations. Finally, we identify geometrically those configurations that belong to the tube of typical paths for the transition from any metastable state to the stable state.

The Potts model is one of the most studied statistical physics models, as the vast literature on the subject attests, both on the mathematics side and the physics side. The study of the equilibrium properties of the Potts model and their dependence on q, have been investigated on the square lattice $\mathbb{Z}^d$ in [12,13], on the triangular lattice in [14,49] and on the Bethe lattice in [3,42,46]. The mean-field version of the Potts model has been studied in [40,47,48,52,70]. More recently, the Potts model has been further studied in different settings: in [17] it has been studied taking into account general coupling constants, in [39] the analysis has been carried out on random regular graphs an in [1] by considering asymmetrical external field. Furthermore, the tunneling behavior for the Potts model with zero external magnetic field has been studied in [18,57,62]. In this energy landscape there are q stable states and there is not any relevant metastable state. In [62], the authors derive the asymptotic behavior of the *first hitting time* for the transition between stable configurations, and give results in probability, in expectation and in distribution. They also characterize the behavior of the *mixing time* and give a lower and an upper bound for the *spectral gap*. In [18], the authors study the tunneling from a stable state to the other stable configurations and between two stable states. In both cases, they geometrically identify the union of all minimal gates and the tube of typical trajectories. Finally, in [57], the authors study the model in two and three dimensions. In both cases, they give a description of *gateway configurations* that is suitable to allow them to prove sharp estimate for the tunneling time by computing the so-called *prefactor*. These *gateway configurations* are quite different from the states belonging to the *minimal gates* identified by [18]. The q-Potts model with non-zero external magnetic field has been studied in [19], where the authors study the energy landscape defined by a Hamiltonian function with *negative* external magnetic field. In this scenario there are a unique metastable configuration and $q - 1$ stable states, and the authors answer to all the three issues of the metastability introduced above for the transition from the metastable state to the set of the stable states and also to any fixed stable state. Furthermore, they give sharp estimates on the expected transition time by computing the *prefactor*.

State of the art. In this paper we adopt the framework known as *pathwise approach*, that was initiated in 1984 by Cassandro, Galves, Olivieri, Vares in [27] and it was further developed in [66–68] and independtly in [28]. The pathwise approach is based on a detailed knowledge of the energy landscape and, thanks to ad hoc large deviations estimates it gives a quantitative answer to the three issues of metastability which we described above. This approach was further developed in [33,34,50,51,59,63] to distinguish the study of the transition time and of the critical configurations from the study of the third issue. This is achieved by proving the recurrence property and identifying the communication height between the metastable and the stable state that are the only two model-dependent inputs need for the results concerning the first issue of metastability. In particular, in [33,34,50,51,59,63] this method has been exploited to find answers valid increasing generality in order to reduce as much as possible the number of model dependent inputs necessary to study the metastable and tunneling behaviour, and to consider situations in which the energy landscapes has multiple stable and/or metastable states. For this reason, the pathwise approach has been used to study metastability in statistical mechanics lattice models. The pathwise approach has been also applied in [5,6,31,38,58,60,64,65,68] with the aim of answering the three issues for Ising-like models with Glauber dynamics. Moreover, it was also applied in [4,7–10,30,43,54,55,63,71] to study the transition time and the gates for Ising-like and hard-core models with Kawasaki and Glauber dynamics. Furthermore, this method was applied to probabilistic cellular automata (parallel dynamics) in [32,35,36,41,69]. See also [29] for a recent review.

The pathwise approach is not the only method which is applied to study the physical systems that approximate a phenomenon of metastability. For instance, the so-called *potential–theoretical approach* exploits a suitable Dirichlet form and spectral properties of the transition matrix to investigate the sharp asymptotic of the hitting time. An interesting aspect of this method is that it allows to estimate the expected value of the transition time including the so-called *prefactor*, i.e., the coefficient that multiplies the leading-order exponential factor. To find these results, it is necessary to prove the recurrence property, the communication height between the metastable and the stable state and a detailed knowledge of the critical configurations as well of those configurations connected with them by one step of the dynamics, see [22–24,37]. In particular, the potential–theoretical approach was applied to find the prefactor for Ising-like models and the hard-core model in [11,25,26,37,44,45,56] for Glauber and Kawasaki dynamics and in [20,61] for parallel dynamics. Recently, other approaches have been formulated in [15,16,53] and in [21] and they are particularly adapted to estimate the pre-factor when dealing with the tunneling between two or more stable states.

Outline. The outline of the paper is as follows. In Section 2 we define the ferromagnetic q-state Potts model and the Hamiltonian that we associate to each Potts configuration. In Section 3 we give a list of both model-independent and model dependent definitions that are required to state our main results in Section 4. In Section 5 we analyze the energy landscape and give the explicit proofs of the main results stated in Sections 4.1 and 4.2. Section 6.2 is devoted to the study on the transition from a metastable state to the other metastable configurations. In Sections 6.3 and 6.4 we give the explicit proofs of the main results on the critical configurations and on the tube of typical paths, respectively.

2. Model description

In the q-state Potts model each spin lies on a vertex of a finite two-dimensional rectangular lattice $\Lambda = (V, E)$, where $V = \{0, \dots, K-1\} \times \{0, \dots, L-1\}$ is the vertex set and E is the edge set, namely the set of the pairs of vertices whose spins interact with each other. We consider periodic boundary conditions. More precisely, we identify each pair of vertices lying on opposite sides of the rectangular lattice, so that we obtain a two-dimensional torus. Two vertices $v, w \in V$ are said to be nearest-neighbors when they share an edge of Λ. We denote by S the set of spin values, i.e., $S := \{1, \dots, q\}$ and assume $q > 2$. Each vertex $v \in V$ is associated to a spin value $\sigma(v) \in S$, and $\mathcal{X} := S^V$ denotes the set of spin configurations. We denote by $\mathbf{1}, \dots, \mathbf{q} \in \mathcal{X}$ those configurations in which all the vertices have spin value $1, \dots, q$, respectively.

To each configuration $\sigma \in \mathcal{X}$ we associate the energy $H(\sigma)$ given by

$$H(\sigma) := -J \sum_{(v,w)\in E} \mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - h \sum_{u\in V} \mathbb{1}_{\{\sigma(u)=1\}}, \quad \sigma \in \mathcal{X}, \tag{2.1}$$

where J is the *coupling* or *integration constant* and h is the *external magnetic field*. The function $H : \mathcal{X} \to \mathbb{R}$ is called *Hamiltonian* or *energy function*. The Potts model is said to be *ferromagnetic* when $J > 0$, and *antiferromagnetic* otherwise. In this paper we set $J = 1$ without loss of generality and, we focus on the ferromagnetic q-state Potts model with non-zero external magnetic field. More precisely, we study the model with positive external magnetic field, i.e., we rewrite (2.1) by considering the magnetic field $h_{\text{pos}} := h$,

$$H_{\text{pos}}(\sigma) := -\sum_{(v,w)\in E} \mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - h_{\text{pos}} \sum_{u\in V} \mathbb{1}_{\{\sigma(u)=1\}} = -\sum_{(v,w)\in E} \mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - h \sum_{u\in V} \mathbb{1}_{\{\sigma(u)=1\}}. \tag{2.2}$$

The *Gibbs measure* for the q-state Potts model on Λ is a probability distribution on the state space $\mathcal{X}$ given by

$$\mu_\beta(\sigma) := \frac{e^{-\beta H_{\text{pos}}(\sigma)}}{Z}, \tag{2.3}$$

where $\beta > 0$ is the inverse temperature and where $Z := \sum_{\sigma'\in\mathcal{X}} e^{-\beta H_{\text{pos}}(\sigma')}$.

The spin system evolves according to a Glauber-type dynamics. This dynamics is described by a single-spin update Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ on the state space $\mathcal{X}$ with the following transition probabilities: for $\sigma, \sigma' \in \mathcal{X}$,

$$P_\beta(\sigma, \sigma') := \begin{cases} Q(\sigma, \sigma') e^{-\beta [H_{\text{pos}}(\sigma') - H_{\text{pos}}(\sigma)]^+}, & \text{if } \sigma \neq \sigma', \\ 1 - \sum_{\eta\neq\sigma} P_\beta(\sigma, \eta), & \text{if } \sigma = \sigma', \end{cases} \tag{2.4}$$

where $[n] := \max\{0, n\}$ is the positive part of n and

$$Q(\sigma, \sigma') := \begin{cases} \frac{1}{q|V|}, & \text{if } |\{v \in V : \sigma(v) \neq \sigma'(v)\}| = 1, \\ 0, & \text{if } |\{v \in V : \sigma(v) \neq \sigma'(v)\}| > 1, \end{cases} \tag{2.5}$$

for any $\sigma, \sigma' \in \mathcal{X}$. Q is the so-called *connectivity matrix* and it is symmetric and irreducible, i.e., for all $\sigma, \sigma' \in \mathcal{X}$, there exists a finite sequence of configurations $\omega_1, \dots, \omega_n \in \mathcal{X}$ such that $\omega_1 = \sigma$, $\omega_n = \sigma'$ and $Q(\omega_i, \omega_{i+1}) > 0$ for $i = 1, \dots, n-1$. Hence, the resulting stochastic dynamics defined by (2.4) is reversible with respect to the Gibbs measure (2.3). The triplet $(\mathcal{X}, H_{\text{pos}}, Q)$ is the so-called *energy landscape*.

The dynamics defined above belongs to the class of Metropolis dynamics. Given a configuration σ in $\mathcal{X}$, at each step

1. a vertex $v \in V$ and a spin value $s \in S$ are selected independently and uniformly at random;
2. the spin at v is updated to spin s with probability

$$\begin{cases} 1, & \text{if } H_{\text{pos}}(\sigma^{v,s}) - H_{\text{pos}}(\sigma) \le 0, \\ e^{-\beta[H_{\text{pos}}(\sigma^{v,s}) - H_{\text{pos}}(\sigma)]}, & \text{if } H_{\text{pos}}(\sigma^{v,s}) - H_{\text{pos}}(\sigma) > 0, \end{cases} \tag{2.6}$$

where $\sigma^{v,s}$ is the configuration obtained from σ by updating the spin in the vertex v to s, i.e.,

$$\sigma^{v,s}(w) := \begin{cases} \sigma(w) & \text{if } w \neq v, \\ s & \text{if } w = v. \end{cases} \tag{2.7}$$

Hence, at each step the update of vertex v depends on the neighboring spins of v and on the energy difference

$$H_{\text{pos}}(\sigma^{v,s}) - H_{\text{pos}}(\sigma) = \begin{cases} \sum_{w \sim v} (\mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - \mathbb{1}_{\{\sigma(w)=s\}}) + h, & \text{if } \sigma(v) = 1, \ s \neq 1, \\ \sum_{w \sim v} (\mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - \mathbb{1}_{\{\sigma(w)=s\}}), & \text{if } \sigma(v) \neq 1, \ s \neq 1, \\ \sum_{w \sim v} (\mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - \mathbb{1}_{\{\sigma(w)=s\}}) - h, & \text{if } \sigma(v) \neq 1, \ s = 1. \end{cases} \tag{2.8}$$

3. Definitions and notations

In order to state our main results, we need to give some definitions and notations which are used throughout the next sections.

3.1. Model-independent definitions and notations

We now give a list of model-independent definitions and notations that will be useful in formulating our main results.

- We call *path* a finite sequence ω of configurations $\omega_0, \dots, \omega_n \in \mathcal{X}$, $n \in \mathbb{N}$, such that $Q(\omega_i, \omega_{i+1}) > 0$ for $i = 0, \dots, n-1$. Given $\sigma, \sigma' \in \mathcal{X}$, if $\omega_1 = \sigma$ and $\omega_n = \sigma'$, we denote a path from σ to σ' as $\omega : \sigma \to \sigma'$. Let $\Omega_{\sigma,\sigma'}$ be the set of all paths between σ and σ'.
- For any path $\omega = (\omega_0, \dots, \omega_n)$, we define the *height* of ω as

$$\Phi_\omega := \max_{i=0,\dots,n} H(\omega_i). \tag{3.1}$$

- A path $\omega = (\omega_0, \dots, \omega_n)$ is said to be *downhill* (*strictly downhill*) if $H(\omega_{i+1}) \le H(\omega_i)$ ($H(\omega_{i+1}) < H(\omega_i)$) for $i = 0, \dots, n-1$.
- For any pair $\sigma, \sigma' \in \mathcal{X}$, the *communication height* or *communication energy* $\Phi(\sigma, \sigma')$ between σ and σ' is the minimal energy across all paths $\omega : \sigma \to \sigma'$, i.e.,

$$\Phi(\sigma, \sigma') := \min_{\omega:\sigma\to\sigma'} \Phi_\omega = \min_{\omega:\sigma\to\sigma'} \max_{\eta\in\omega} H(\eta). \tag{3.2}$$

  More generally, the communication energy between any pair of non-empty disjoint subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ is $\Phi(\mathcal{A}, \mathcal{B}) := \min_{\sigma\in\mathcal{A},\ \sigma'\in\mathcal{B}} \Phi(\sigma, \sigma')$.
- We define *optimal paths* those paths that realize the min–max in (3.2) between σ and σ'. Formally, we define the set of *optimal paths* between $\sigma, \sigma' \in \mathcal{X}$ as

$$\Omega^{opt}_{\sigma,\sigma'} := \{\omega \in \Omega_{\sigma,\sigma'} : \max_{\eta\in\omega} H(\eta) = \Phi(\sigma, \sigma')\}. \tag{3.3}$$

- For any $\sigma \in \mathcal{X}$, let $\mathcal{I}_\sigma := \{\eta \in \mathcal{X} : H(\eta) < H(\sigma)\}$ be the set of states with energy strictly smaller than $H(\sigma)$. We define *stability level* of σ the energy barrier

$$V_\sigma := \Phi(\sigma, \mathcal{I}_\sigma) - H(\sigma). \tag{3.4}$$

  If $\mathcal{I}_\sigma = \emptyset$, we set $V_\sigma := \infty$.
- The bottom $\mathscr{F}(\mathcal{A})$ of a non-empty set $\mathcal{A} \subset \mathcal{X}$ is the set of the *global minima* of H in $\mathcal{A}$, i.e.,

$$\mathscr{F}(\mathcal{A}) := \{\eta \in \mathcal{A} : H(\eta) = \min_{\sigma\in\mathcal{A}} H(\sigma)\}. \tag{3.5}$$

  In particular, $\mathcal{X}^s := \mathscr{F}(\mathcal{X})$ is the set of the *stable states*.
- For any $\sigma \in \mathcal{X}$ and any $\mathcal{A} \subset \mathcal{X}$, $\mathcal{A} \neq \emptyset$, we set

$$\Gamma(\sigma, \mathcal{A}) := \Phi(\sigma, \mathcal{A}) - H(\sigma). \tag{3.6}$$

- We define the set of *metastable states* as

$$\mathcal{X}^m := \{\eta \in \mathcal{X} : V_\eta = \max_{\sigma\in\mathcal{X}\setminus\mathcal{X}^s} V_\sigma\}. \tag{3.7}$$

  We denote by Γ^m the stability level of a metastable state.

- We define *metastable set at level V* the set of all the configurations with stability level larger than V, i.e.,

$$\mathcal{X}_V := \{\sigma \in \mathcal{X} : V_\sigma > V\}. \tag{3.8}$$

- The set of *minimal saddles* between $\sigma, \sigma' \in \mathcal{X}$ is defined as

$$\mathcal{S}(\sigma,\sigma') := \{\xi \in \mathcal{X} : \exists \omega \in \Omega^{opt}_{\sigma,\sigma'},\ \xi \in \omega : \max_{\eta\in\omega} H(\eta) = H(\xi)\}. \tag{3.9}$$

- We say that $\eta \in \mathcal{S}(\sigma,\sigma')$ is an *essential saddle* if there exists $\omega \in \Omega^{opt}_{\sigma,\sigma'}$ such that either
  - $\{\arg\max_\omega H\} = \{\eta\}$ or
  - $\{\arg\max_\omega H\} \supset \{\eta\}$ and $\{\arg\max_{\omega'} H\} \not\subseteq \{\arg\max_\omega H\}\backslash\{\eta\}$ for all $\omega' \in \Omega^{opt}_{\sigma,\sigma'}$.
- A saddle $\eta \in \mathcal{S}(\sigma,\sigma')$ that is not essential is said to be *unessential*.
- Given $\sigma, \sigma' \in \mathcal{X}$, we say that $\mathcal{W}(\sigma,\sigma')$ is a *gate* for the transition from σ to σ' if $\mathcal{W}(\sigma,\sigma') \subseteq \mathcal{S}(\sigma,\sigma')$ and $\omega \cap \mathcal{W}(\sigma,\sigma') \neq \emptyset$ for all $\omega \in \Omega^{opt}_{\sigma,\sigma'}$.
- We say that $\mathcal{W}(\sigma,\sigma')$ is a *minimal gate* for the transition from σ to σ' if it is a minimal (by inclusion) subset of $\mathcal{S}(\sigma,\sigma')$ that is visited by all optimal paths. More in detail, it is a gate and for any $\mathcal{W}' \subset \mathcal{W}(\sigma,\sigma')$ there exists $\omega' \in \Omega^{opt}_{\sigma,\sigma'}$ such that $\omega' \cap \mathcal{W}' = \emptyset$. We denote by $\mathcal{G} = \mathcal{G}(\sigma,\sigma')$ the union of all minimal gates for the transition $\sigma \to \sigma'$.
- Given a non-empty subset $\mathcal{A} \subseteq \mathcal{X}$, it is said to be *connected* if for any $\sigma, \eta \in \mathcal{A}$ there exists a path $\omega : \sigma \to \eta$ totally contained in $\mathcal{A}$. Moreover, we define ∂A as the *external boundary* of $\mathcal{A}$, i.e., the set

$$\partial\mathcal{A} := \{\eta \notin \mathcal{A} :\ P(\sigma,\eta) > 0 \text{ for some } \sigma \in \mathcal{A}\}. \tag{3.10}$$

- A non-empty subset $C \subset \mathcal{X}$ is called *cycle* if it is either a singleton or a connected set such that

$$\max_{\sigma\in C} H(\sigma) < H(\mathscr{F}(\partial C)). \tag{3.11}$$

  When C is a singleton, it is said to be a *trivial cycle*. Let $\mathscr{C}(\mathcal{X})$ be the set of cycles of $\mathcal{X}$.
- The *depth* of a cycle C is given by

$$\Gamma(C) := H(\mathscr{F}(\partial C)) - H(\mathscr{F}(C)). \tag{3.12}$$

  If C is a trivial cycle we set $\Gamma(C) = 0$.
- Given a non-empty set $\mathcal{A} \subset \mathcal{X}$, we denote by $\mathcal{M}(\mathcal{A})$ the *collection of maximal cycles* $\mathcal{A}$, i.e.,

$$\mathcal{M}(\mathcal{A}) := \{C \in \mathscr{C}(\mathcal{X})|\ C \text{ maximal by inclusion under constraint } C \subseteq \mathcal{A}\}.$$

- For any $\sigma \in \mathcal{X}$, if $\mathcal{A}$ is a non-empty target set, we define the *initial cycle* for the transition from σ to $\mathcal{A}$ as

$$C^\sigma_\mathcal{A}(\Gamma) := \{\sigma\} \cup \{\eta \in \mathcal{X} :\ \Phi(\sigma,\eta) - H(\sigma) < \Gamma = \Phi(\sigma,\mathcal{A}) - H(\sigma)\}. \tag{3.13}$$

  If $\sigma \in \mathcal{A}$, then $C^\sigma_\mathcal{A}(\Gamma) = \{\sigma\}$ and it is a trivial cycle. Otherwise, $C^\sigma_\mathcal{A}(\Gamma)$ is either a trivial cycle (when $\Phi(\sigma,\mathcal{A}) = H(\sigma)$) or a non-trivial cycle containing σ (when $\Phi(\sigma,\mathcal{A}) > H(\sigma)$). In any case, if $\sigma \notin \mathcal{A}$, then $C^\sigma_\mathcal{A}(\Gamma) \cap \mathcal{A} = \emptyset$. Note that (3.13) coincides with [63, Equation (2.25)].

3.2. Model-dependent definitions and notations

In this section we give some further model-dependent notations, which hold for any fixed q-Potts configuration $\sigma \in \mathcal{X}$.

- For any $v, w \in V$, we write $w \sim v$ (and, equivalently, $v \sim w$) if there exists an edge $e \in E$ that links the vertices v and w.
- We denote the edge that links the vertices v and w as $(v,w) \in E$. Each $v \in V$ is identified by its coordinates (i,j), where i and j denote respectively the number of the row and of the column where v lies. Moreover, the collection of vertices with first coordinate equal to $i = 0,\dots,K-1$ is denoted as r_i, which is the ith row of Λ. The collection of those vertices with second coordinate equal to $j = 0,\dots,L-1$ is denoted as c_j, which is the jth column of Λ.
- We define the set $C^s(\sigma) \subseteq \mathbb{R}^2$ as the union of unit closed squares centered at the vertices $v \in V$ such that $\sigma(v) = s$. We define *s-clusters* the maximal connected components $C^s_1,\dots,C^s_n,\ n \in \mathbb{N}$, of $C^s(\sigma)$.
- For any $s \in S$, we say that a configuration $\sigma \in \mathcal{X}$ has an s-rectangle if it has a rectangular cluster (possibly wrapping around the grid-graph Λ in view of the periodic boundary conditions) in which all the vertices have spin s.
- Let R_1 an r-rectangle and R_2 an s-rectangle. They are said to be *interacting* if either they intersect (when $r = s$) or are disjoint but there exists a site $v \notin R_1 \cup R_2$ such that $\sigma(v) \neq r,s$ and v has two nearest-neighbor w,u lying inside R_1, R_2 respectively. Furthermore, we say that R_1 and R_2 are *adjacent* when they are at lattice distance one from each other.
- We set $R(C^s(\sigma))$ as the smallest rectangle containing $C^s(\sigma)$.
- Let $R_{\ell_1\times\ell_2}$ be a rectangle in $\mathbb{R}^2$ with sides of length ℓ_1 and ℓ_2.
- Let $s \in S$. If σ has a cluster of spins s which is a rectangle that wraps around Λ, we say that σ has an *s-strip*. For any $r,s \in S$, we say that an s-strip is *adjacent* to an r-strip if they are at lattice distance one from each other.

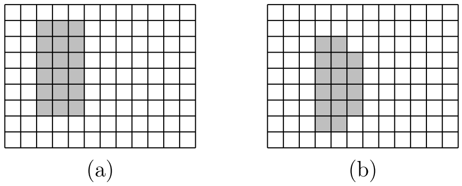


Fig. 1. Examples of configurations which belong to $\bar{R}_{3,6}(r,s)$ (a), $\bar{B}^4_{2,6}(r,s)$ (b). We color white the vertices whose spin is r and we color gray the vertices whose spin is s.

- $\bar{R}_{a,b}(r,s)$ denotes the set of those configurations in which all the vertices have spins equal to r, except those, which have spins s, in a rectangle $a \times b$, see Fig. 1(a). Note that when either $a = L$ or $b = K$, $\bar{R}_{a,b}(r,s)$ contains those configurations which have an r-strip and an s-strip.
- $\bar{B}^l_{a,b}(r,s)$ denotes the set of those configurations in which all the vertices have spins r, except those, which have spins s, in a rectangle $a \times b$ with a bar $1 \times l$ adjacent to one of the sides of length b, with $1 \le l \le b-1$, see Fig. 1(b).
- We define
$$\ell^* := \left\lceil \frac{2}{h} \right\rceil \tag{3.14}$$
as the *critical length*.

4. Main results on the q-state Potts model with positive external magnetic field

This section is devoted to the statement of our main results on the q-state Potts model with positive external magnetic field. Note that we give the proof of the main results by considering the condition $L \ge K \ge 3\ell^*$, where ℓ^* is defined in (3.14). It is possible to extend the results to the case $K > L$ by interchanging the role of rows and columns in the proof.

In this scenario related to the Hamiltonian H_{pos}, we add either a subscript or a superscript "pos" to the notation of the model-independent quantities (defined in general in Section 3.1) in order to remind the reader that these quantities are computed in the case of positive external magnetic field.

We assume the following conditions.

Assumption 4.1. We assume that the following conditions are verified:

(i) the magnetic field $h_{\text{pos}} := h$ is such that $0 < h < \frac{1}{2}$;

(ii) $2/h$ is not an integer.

4.1. Energy landscape

Using the definition (2.2) and by simple algebraic calculations, in the following proposition we identificate the set of the global minima of H_{pos}.

Proposition 4.2 (*Identification of the Stable Configuration*). *Consider the q-state Potts model on a $K \times L$ grid Λ, with periodic boundary conditions and with positive external magnetic field. Then, the set of global minima of the Hamiltonian (2.2) is given by $\mathcal{X}^s_{pos} := \{\mathbf{1}\}$.*

In the next theorem we define the configurations that belong to $\mathcal{X}^m_{\text{pos}}$ and give an estimate of the stability level Γ^m_{pos}. We refer to Fig. 2 for a pictorial representation of the 4-state Potts model related to the Hamiltonian H_{pos}.

Theorem 4.3 (*Identification of the Metastable States*). *Consider the q-state Potts model on a $K \times L$ grid Λ, with periodic boundary conditions and with positive external magnetic field. Then, $\mathcal{X}^m_{pos} = \{\mathbf{2}, \dots, \mathbf{q}\}$ and, for any $\mathbf{m} \in \mathcal{X}^m_{pos}$,*
$$\Gamma^m_{pos} = \Gamma_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}) = 4\ell^* - h(\ell^*(\ell^*-1)+1). \tag{4.1}$$

Proof. The theorem follows by [33, Theorem 2.4] since the first assumption follows by Propositions 5.7 and 5.8 and the second assumption is satisfied thanks to Proposition 4.5. □

For any $\mathcal{A} \subset \mathcal{X}$, we define the *maximum depth of* $\mathcal{A}$ as the maximum depth of a cycle contained in $\mathcal{A}$, i.e.,
$$\widetilde{\Gamma}(\mathcal{A}) := \max_{C \in \mathcal{M}(\mathcal{A})} \Gamma(C). \tag{4.2}$$
Note that in [63, Lemma 3.6] the authors give an alternative characterization of (4.2) as $\widetilde{\Gamma}(\mathcal{A}) = \max_{\eta \in \mathcal{A}} \Gamma(\eta, \mathcal{X} \backslash \mathcal{A})$.

Using (4.1), in Section 5.3 we prove the following corollary.

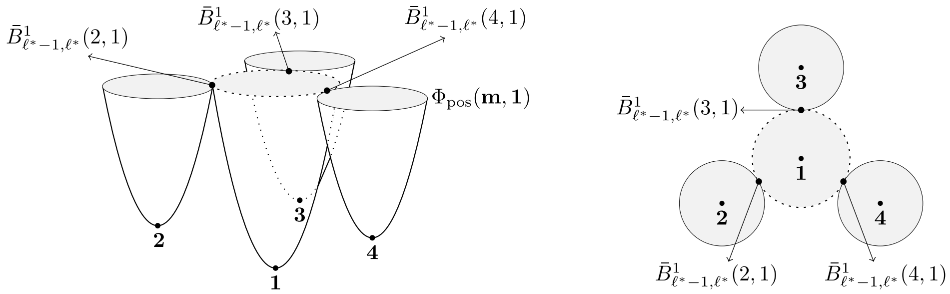


Fig. 2. On the left, energy landscape in the case of 4-state Potts model with positive external magnetic field around the unique stable state **1** cutting the configurations with the energy bigger than $\Phi_{\rm pos}(\mathbf{m},\mathbf{1})$, $\mathbf{m} \in \mathcal{X}^m_{\rm pos} = \{\mathbf{2},\mathbf{3},\mathbf{4}\}$. This picture is simplified since there are not represented the cycles (valleys) that contain configurations with stability level smaller than or equal to $\frac{4}{h}$ (see Proposition 4.5). On the right, viewpoint from above of the same energy landscape. For every $\mathbf{m} \in \mathcal{X}^m_{\rm pos}$, the cycle whose bottom is the stable state **1** is deeper than the initial cycles $C^{\mathbf{m}}_{\mathcal{X}^s_{\rm pos}}(\Gamma^m_{\rm pos})$. These last cycles are depicted with circles whose diameter is smaller than the one related to the stable state **1**.

Corollary 4.4 (*Maximum Depth of a Cycle in $\mathcal{X}\backslash\mathcal{X}^s_{pos}$*). *Consider the q-state Potts model on a $K \times L$ grid Λ, with periodic boundary conditions and with positive external magnetic field. Then,*

$$\widetilde{\Gamma}_{pos}(\mathcal{X}\backslash\mathcal{X}^s_{pos}) = \Gamma^m_{pos}. \tag{4.3}$$

In the following proposition, that we prove in Section 5.2, we investigate on the stability level of any configuration $\eta \in \mathcal{X}\backslash\{\mathbf{2},\dots,\mathbf{q}\}$.

Proposition 4.5 (*Estimate on the Stability Level*). *If the external magnetic field is positive, then for any $\eta \in \mathcal{X}\backslash\{\mathbf{1},\dots,\mathbf{q}\}$ and $\mathbf{m} \in \{\mathbf{2},\dots,\mathbf{q}\}$, $V^{pos}_\eta \le \Gamma_{pos}(\mathbf{m},\mathcal{X}^s_{pos})$.*

Exploiting the estimate of the stability level given in Proposition 4.5, we prove the following result on a recurrence property to set of the metastable and stable configurations, i.e., $\{\mathbf{1},\dots,\mathbf{q}\}$. Given a non-empty subset $\mathcal{A} \subset \mathcal{X}$ and a configuration $\sigma \in \mathcal{X}$, we define

$$\tau^\sigma_{\mathcal{A}} := \inf\{t > 0 : \ X^\beta_t \in \mathcal{A}\} \tag{4.4}$$

as the *first hitting time* of the subset $\mathcal{A}$ for the Markov chain $\{X^\beta_t\}_{t\in\mathbb{N}}$ starting from σ at time $t = 0$. Moreover, we recall that a function $\beta \to f(\beta)$ is said to be *super-exponentially small* (SES) if

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log f(\beta) = -\infty. \tag{4.5}$$

Theorem 4.6 (*Recurrence Property*). *Consider the q-state Potts model on a $K \times L$ grid Λ, with periodic boundary conditions and with positive external magnetic field. Let $V^* = \frac{4}{h}$. Then, for any $\sigma \in \mathcal{X}$ and any $\epsilon > 0$, there exists $k > 0$ such that for β sufficiently large*

$$\mathbb{P}(\tau^\sigma_{\{\mathbf{1},\dots,\mathbf{q}\}} > e^{\beta(V^*+\epsilon)}) \le e^{-e^{k\beta}} = SES. \tag{4.6}$$

Proof. Apply [59, Theorem 3.1] with $V = \frac{4}{h}$ and use (3.8) and Proposition 4.5 to get $\mathcal{X}_{V^*} = \{\mathbf{1},\dots,\mathbf{q}\} = \mathcal{X}^m_{\rm pos} \cup \mathcal{X}^s_{\rm pos}$, where the last equality follows by Proposition 4.2 and Theorem 4.3. $\square$

4.2. Asymptotic behavior of the first hitting time to the stable state and mixing time

Let $\{X^\beta_t\}_{t\in\mathbb{N}}$ be the Markov chain with transition probabilities (2.4) and stationary distribution (2.3). For every $\epsilon \in (0,1)$, we define the *mixing time* $t^{\rm mix}_\beta(\epsilon)$ by

$$t^{\rm mix}_\beta(\epsilon) := \min\{n \ge 0 : \ \max_{\sigma\in\mathcal{X}} \|P^n_\beta(\sigma,\cdot) - \mu_\beta(\cdot)\|_{\rm TV} \le \epsilon\}, \tag{4.7}$$

where the total variance distance is defined by $\|\nu - \nu'\|_{\rm TV} := \frac{1}{2}\sum_{\sigma\in\mathcal{X}} |\nu(\sigma) - \nu'(\sigma)|$ for every two probability distribution ν, ν' on $\mathcal{X}$. Furthermore, we define *spectral gap* as

$$\rho_\beta := 1 - \lambda^{(2)}_\beta, \tag{4.8}$$

where $1 = \lambda^{(1)}_\beta > \lambda^{(2)}_\beta \ge \dots \ge \lambda^{(|\mathcal{X}|)}_\beta \ge -1$ are the eigenvalues of the matrix $P_\beta(\sigma,\eta)_{\sigma,\eta\in\mathcal{X}}$.

In the following theorem we give asymptotic bounds in probability and identify the order of magnitude of the expected value of $\tau^{\mathbf{m}}_{\mathcal{X}^s_{\rm pos}}$ (see (4.4)). Moreover, we identify the exponent at which the mixing time of the Markov chain $\{X^\beta_t\}_{t\in\mathbb{N}}$ asymptotically grows as β and give an upper and a lower bound for the spectral gap, see (4.7) and (4.8).

Theorem 4.7 (*Asymptotic Behavior of* $\tau^{\mathbf{m}}_{\mathcal{X}^s_{pos}}$ *and Mixing Time*)**.** *Consider the* q*-state Potts model on a* $K \times L$ *grid* Λ*, with periodic boundary conditions and with positive external magnetic field. Then, for any* $\mathbf{m} \in \mathcal{X}^m_{pos}$*, the following statements hold:*

(a) *for every* $\epsilon > 0$, $\lim_{\beta\to\infty} \mathbb{P}(e^{\beta(\Gamma^m_{pos}-\epsilon)} < \tau^{\mathbf{m}}_{\mathcal{X}^s_{pos}} < e^{\beta(\Gamma^m_{pos}+\epsilon)}) = 1$;

(b) $\lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}[\tau^{\mathbf{m}}_{\mathcal{X}^s_{pos}}] = \Gamma^m_{pos}$;

(c) *for every* $\epsilon \in (0,1)$, $\lim_{\beta\to\infty} \frac{1}{\beta} \log t^{mix}_{\beta}(\epsilon) = \Gamma^m_{pos}$ *and there exist two constants* $0 < c_1 \le c_2 < \infty$ *independent of* β *such that, for any* $\beta > 0$, $c_1 e^{-\beta\Gamma^m_{pos}} \le \rho_\beta \le c_2 e^{-\beta\Gamma^m_{pos}}$.

Proof. Items (a) and (b) follow by [59, Theorem 4.1] and by [59, Theorem 4.9], respectively, with $\eta_0 = \mathbf{m}$ and $\Gamma = \Gamma^m_{\text{pos}}$ (using Theorem 4.3). Item (c) follows by Corollary 4.4 and by [63, Proposition 3.24]. □

In the literature there exist some model-independent results on the asymptotic rescaled distribution of the first hitting time from some $\eta \in \mathcal{X}$ to a certain target $G \subset \mathcal{X}$, see for instance [59, Theorem 4.15], [50, Theorem 2.3], [63, Theorem 3.19]. Unfortunately none of these results are suitable for our scenario when $\eta \in \mathcal{X}^m_{\text{pos}}$ and $G = \mathcal{X}^s_{\text{pos}}$. This fact follows by the presence of multiple degenerate metastable states that implies the presence of other deep wells in $\mathcal{X}$ different from the initial cycle $C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$. Hence, we consider a different target and we investigate the asymptotic rescaled distribution of the first hitting time from a metastable state to the subset $G \subset \mathcal{X}$ setting

$$G = \mathcal{X} \backslash C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}}). \tag{4.9}$$

We defer to Section 5.3 for the proof of the following theorem.

Theorem 4.8. *Consider the* q*-state Potts model on a* $K \times L$ *grid* Λ*, with periodic boundary conditions and with positive external magnetic field. Let* $\mathbf{m} \in \mathcal{X}^m_{pos}$ *and let* G *as defined in* (4.9)*. Then,*

$$\frac{\tau^{\mathbf{m}}_G}{\mathbb{E}[\tau^{\mathbf{m}}_G]} \xrightarrow{d} Exp(1), \ \ as \ \beta \to \infty. \tag{4.10}$$

Note that by definition (4.9), by Proposition 4.2 and by Theorem 4.3, we have $\mathscr{F}(G) = \mathcal{X}^s_{\text{pos}}$ and that the maximal stability level is $V(G) = \Gamma^m_{\text{pos}}$.

4.3. Minimal gates for the metastable transition

A further goal is to identify the union of all minimal gates for the transition from any metastable state to the unique stable state $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$. In order to do this, for any $m \in S \backslash \{1\}$, let us define

$$\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) := \bar{B}^1_{\ell^*-1,\ell^*}(m,1) \ \text{ and } \ \mathcal{W}'_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) := \bar{B}^1_{\ell^*,\ell^*-1}(m,1). \tag{4.11}$$

We refer to Fig. 15(b)–(c) for an example of configurations belonging respectively to $\mathcal{W}'_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ and to $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. These sets are investigated in Section 6.1. In particular, in Proposition 6.3 we show that $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ is a gate for the transition from any $\mathbf{m} \in X^m_{\text{pos}}$ to $\mathcal{X}^s_{\text{pos}}$.

Furthermore, in Section 6.3 we prove the following result.

Theorem 4.9 (*Minimal Gates for the Transition from* $\mathbf{m} \in \mathcal{X}^m_{pos}$ *to* $\mathcal{X}^s_{pos}$)**.** *Consider the* q*-state Potts model on a* $K \times L$ *grid* Λ*, with periodic boundary conditions and with positive external magnetic field. Then,* $\mathcal{W}_{pos}(\mathbf{m}, \mathcal{X}^s_{pos})$ *is a minimal gate for the transition from any metastable state* $\mathbf{m} \in \mathcal{X}^m_{pos}$ *to* $\mathcal{X}^s_{pos} = \{\mathbf{1}\}$*. Moreover,*

$$\mathcal{G}_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}) = \mathcal{W}_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}). \tag{4.12}$$

We remark that in [19, Theorems 4.5 and 4.6] the authors identify the union of all minimal gates for the metastable transitions for the q-Potts model with *negative* external magnetic fields. These minimal gates have the same geometric definition of those of our scenario, the main difference is that in the negative case there are $q-1$ possible "colors" for the vertices inside the quasi-square with a unit protuberance.

In the next corollary we show that the process typically intersects $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ during the transition $\mathbf{m} \in \mathcal{X}^m_{\text{pos}} \to \mathcal{X}^s_{\text{pos}}$.

Corollary 4.10. *Consider the* q*-state Potts model on a* $K \times L$ *grid* Λ*, with periodic boundary conditions and with positive external magnetic field. Then, for any* $\mathbf{m} \in \mathcal{X}^m_{pos}$

$$\lim_{\beta\to\infty} \mathbb{P}(\tau^{\mathbf{m}}_{\mathcal{W}_{pos}(\mathbf{m}, \mathcal{X}^s_{pos})} < \tau^{\mathbf{m}}_{\mathcal{X}^s_{pos}}) = 1. \tag{4.13}$$

Proof. The corollary follows from Proposition 6.3 and from [59, Theorem 5.4]. □

4.4. Minimal gates for the transition from a metastable state to the other metastable configurations

In Section 6.2 we study the transition from a metastable state to the set of the other metastable states. We prove that during the transition from any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$ to $\mathcal{X}^s_{\text{pos}}$ almost surely the process does not intersect $\mathcal{X}^m_{\text{pos}} \backslash \{\mathbf{m}\}$, and we exploit this result to identificate the union of all minimal gates for this type of transition.

Theorem 4.11 (*Minimal Gates for the Transition from* $\mathbf{m}$ *to* $\mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}$)**.** *Let* $\mathbf{m} \in \mathcal{X}^m_{pos}$*. If the external magnetic field is positive, then the following sets are minimal gates for the transition from* $\mathbf{m}$ *to* $\mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}$

$$\text{(a) } \mathcal{W}_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}), \tag{4.14}$$

$$\text{(b) } \bigcup_{\mathbf{z}\in\mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}} \mathcal{W}_{pos}(\mathbf{z}, \mathcal{X}^s_{pos}). \tag{4.15}$$

Furthermore,

$$\mathcal{G}_{pos}(\mathbf{m}, \mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}) = \bigcup_{\mathbf{z}\in\mathcal{X}^m_{pos}} \mathcal{W}_{pos}(\mathbf{z}, \mathcal{X}^s_{pos}). \tag{4.16}$$

We defer the proof of the above theorem in Section 6.3. Note that in the negative scenario [19] the theorem corresponding to Theorem 4.11 is not present since there is only one metastable state.

Corollary 4.12. *If the external magnetic field is positive, then for any* $\mathbf{m} \in \mathcal{X}^m_{pos}$*,*

(a) $\lim_{\beta\to\infty} \mathbb{P}(\tau^{\mathbf{m}}_{\mathcal{W}_{pos}(\mathbf{m},\mathcal{X}^s_{pos})} < \tau^{\mathbf{m}}_{\mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}}) = 1,$

(b) $\lim_{\beta\to\infty} \mathbb{P}(\tau^{\mathbf{m}}_{\bigcup_{\mathbf{z}\in\mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}} \mathcal{W}_{pos}(\mathbf{z},\mathcal{X}^s_{pos})}) < \tau^{\mathbf{m}}_{\mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}} = 1.$

Proof. The corollary follows from Propositions 6.7 and 6.10 and from [59, Theorem 5.4]. □

4.5. Tube of typical trajectories of the metastable transition

4.5.1. Further model-independent and model-dependent definitions

In addition to the list of Section 3.1, in order state the main result concerning the tube of the typical trajectories we give some further definitions that are taken from [34,63,68].

- We call *cycle-path* a finite sequence $(C_1, \dots, C_m)$ of trivial or non-trivial cycles $C_1, \dots, C_m \in \mathscr{C}(\mathcal{X})$, such that $C_i \cap C_{i+1} = \varnothing$ and $\partial C_i \cap C_{i+1} \neq \varnothing$, for every $i = 1, \dots, m-1$.
- A cycle-path $(C_1, \dots, C_m)$ is said to be *downhill* (*strictly downhill*) if the cycles $C_1, \dots, C_m$ are pairwise connected with decreasing height, i.e., when $H(\mathscr{F}(\partial C_i)) \geq H(\mathscr{F}(\partial C_{i+1}))$ $(H(\mathscr{F}(\partial C_i)) > H(\mathscr{F}(\partial C_{i+1})))$ for any $i = 0, \dots, m-1$.
- For any $C \in \mathscr{C}(\mathcal{X})$, we define as

$$\mathcal{B}(C) := \begin{cases} \mathscr{F}(\partial C) & \text{if } C \text{ is a non-trivial cycle,} \\ \{\eta \in \partial C \,:\, H(\eta) < H(\sigma)\} & \text{if } C = \{\sigma\} \text{ is a trivial cycle,} \end{cases} \tag{4.17}$$

  the *principal boundary* of C. Furthermore, let $\partial^{np}C$ be the *non-principal boundary* of C, i.e., $\partial^{np}C := \partial C \backslash \mathcal{B}(C)$.
- The *relevant cycle* $C^+_{\mathcal{A}}(\sigma)$ is

$$C^+_{\mathcal{A}}(\sigma) := \{\eta \in \mathcal{X} :\ \Phi(\sigma, \eta) < \Phi(\sigma, \mathcal{A}) + \delta/2\}, \tag{4.18}$$

  where δ is the minimum energy gap between any optimal and any non-optimal path from σ to $\mathcal{A}$.
- We denote the set of cycle-paths that lead from σ to $\mathcal{A}$ and consist of maximal cycles in $\mathcal{X}\backslash\mathcal{A}$ as

$$\mathcal{P}_{\sigma,\mathcal{A}} := \{\text{cycle-path } (C_1, \dots, C_m) :\ C_1, \dots, C_m \in \mathcal{M}(C^+_{\mathcal{A}}(\sigma)\backslash\mathcal{A}), \sigma \in C_1, \partial C_m \cap \mathcal{A} \neq \varnothing\}.$$

- Given a non-empty set $\mathcal{A} \subset \mathcal{X}$ and $\sigma \in \mathcal{X}$, we constructively define a mapping $G : \Omega_{\sigma,\mathcal{A}} \to \mathcal{P}_{\sigma,\mathcal{A}}$ in the following way. Given $\omega = (\omega_1, \dots, \omega_n) \in \Omega_{\sigma,\mathcal{A}}$, we set $m_0 = 1$, $C_1 = C_{\mathcal{A}}(\sigma)$ and define recursively $m_i := \min\{k > m_{i-1} :\ \omega_k \notin C_i\}$ and $C_{i+1} := C_{\mathcal{A}}(\omega_{m_i})$. We note that ω is a finite sequence and $\omega_n \in \mathcal{A}$, so there exists an index $n(\omega) \in \mathbb{N}$ such that $\omega_{m_{n(\omega)}} = \omega_n \in \mathcal{A}$ and there the procedure stops. By $(C_1, \dots, C_{m_{n(\omega)}})$ is a cycle-path with $C_1, \dots, C_{m_{n(\omega)}} \subset \mathcal{M}(\mathcal{X}\backslash\mathcal{A})$. Moreover, the fact that $\omega \in \Omega_{\sigma,\mathcal{A}}$ implies that $\sigma \in C_1$ and that $\partial C_{n(\omega)} \cap \mathcal{A} \neq \varnothing$, hence $G(\omega) \in \mathcal{P}_{\sigma,\mathcal{A}}$ and the mapping is well-defined.
- We say that a cycle-path $(C_1, \dots, C_m)$ is *connected via typical jumps* to $\mathcal{A} \subset \mathcal{X}$ or simply *vtj–connected* to $\mathcal{A}$ if

$$\mathcal{B}(C_i) \cap C_{i+1} \neq \varnothing, \quad \forall i = 1, \dots, m-1, \quad \text{and} \quad \mathcal{B}(C_m) \cap \mathcal{A} \neq \varnothing. \tag{4.19}$$

  Let $J_{C,\mathcal{A}}$ be the collection of all cycle-paths $(C_1, \dots, C_m)$ that are vtj-connected to $\mathcal{A}$ and such that $C_1 = C$.

- Given a non-empty set $\mathcal{A}$ and $\sigma \in \mathcal{X}$, we define $\omega \in \Omega_{\sigma,\mathcal{A}}$ as a *typical path* from σ to $\mathcal{A}$ if its corresponding cycle-path $G(\omega)$ is vtj-connected to $\mathcal{A}$ and we denote by $\Omega^{\text{vtj}}_{\sigma,\mathcal{A}}$ the collection of all typical paths from σ to $\mathcal{A}$, i.e.,

$$\Omega^{\text{vtj}}_{\sigma,\mathcal{A}} := \{\omega \in \Omega_{\sigma,\mathcal{A}} : \; G(\omega) \in J_{\mathcal{C}_{\mathcal{A}}(\sigma),\mathcal{A}}\}. \tag{4.20}$$

- We define the *tube of typical paths* $T_{\mathcal{A}}(\sigma)$ from σ to $\mathcal{A}$ as the subset of states $\eta \in \mathcal{X}$ that can be reached from σ by means of a typical path which does not enter $\mathcal{A}$ before visiting η, i.e.,

$$T_{\mathcal{A}}(\sigma) := \{\eta \in \mathcal{X} : \; \exists \omega \in \Omega^{\text{vtj}}_{\sigma,\mathcal{A}} : \; \eta \in \omega\}. \tag{4.21}$$

  Moreover, we define $\mathfrak{T}_{\mathcal{A}}(\sigma)$ as the set of all maximal cycles that belong to at least one vtj-connected path from $\mathcal{C}^{\sigma}_{\mathcal{A}}(\Gamma)$ to $\mathcal{A}$, i.e.,

$$\mathfrak{T}_{\mathcal{A}}(\sigma) := \{\mathcal{C} \in \mathcal{M}(\mathcal{C}^{+}_{\mathcal{A}}(\sigma)\backslash\mathcal{A}) : \; \exists (\mathcal{C}_1,\dots,\mathcal{C}_n) \in J_{\mathcal{C}^{\sigma}_{\mathcal{A}}(\Gamma),\mathcal{A}} \text{ and} \exists j \in \{1,\dots,n\} : \; \mathcal{C}_j = \mathcal{C}\}. \tag{4.22}$$

  Note that

$$\mathfrak{T}_{\mathcal{A}}(\sigma) = \mathcal{M}(T_{\mathcal{A}}(\sigma)\backslash\mathcal{A}) \tag{4.23}$$

  and that the boundary of $T_{\mathcal{A}}(\sigma)$ consists of states either in $\mathcal{A}$ or in the non-principal part of the boundary of some $\mathcal{C} \in \mathfrak{T}_{\mathcal{A}}(\sigma)$:

$$\partial T_{\mathcal{A}}(\sigma)\backslash\mathcal{A} \subseteq \bigcup_{\mathcal{C}\in\mathfrak{T}_{\mathcal{A}}(\sigma)} (\partial\mathcal{C}\backslash\mathcal{B}(\mathcal{C})) =: \partial^{np}\mathfrak{T}_{\mathcal{A}}(\sigma). \tag{4.24}$$

  For the sake of simplicity, we will also refer to $\mathfrak{T}_{\mathcal{A}}(\sigma)$ as tube of typical paths from σ to $\mathcal{A}$.

Furthermore, in addition to the list given in Section 3.2, we give some further model-dependent definitions.

- For any $m, s \in S, m \neq s$, we define $\mathcal{X}(m,s) = \{\sigma \in \mathcal{X} : \sigma(v) \in \{m,s\}$ for any $v \in V\}$.
- For any $m \in S\backslash\{1\}$, we define

$$\begin{aligned}
\mathcal{S}^{v}_{\text{pos}}(m,1) &:= \{\sigma \in \mathcal{X}(m,1) : \sigma \text{ has a vertical 1-strip of thickness at least } \ell^* \text{ with} \\
&\qquad \text{possibly a bar of length } l = 1,\dots,K \text{ on one of the two vertical edges}\}, && (4.25)\\
\mathcal{S}^{h}_{\text{pos}}(m,1) &:= \{\sigma \in \mathcal{X}(m,1) : \sigma \text{ has a horizontal 1-strip of thickness at least } \ell^* \text{ with} \\
&\qquad \text{possibly a bar of length } l = 1,\dots,L \text{ on one of the two horizontal edges}\}. && (4.26)
\end{aligned}$$

4.5.2. Main results on the tube of typical trajectories

In this subsection we give our main result concerning the tube of the typical trajectories for the transition $\mathbf{m} \to \mathcal{X}^s_{\text{pos}}$ for any fixed $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$. The tube of typical paths for this transition turns out to be

$$\begin{aligned}
\mathfrak{T}_{\mathcal{X}^s_{\text{pos}}}(\mathbf{m}) :=& \bigcup_{\ell=1}^{\ell^*-1} \bar{R}_{\ell-1,\ell}(m,1) \cup \bigcup_{\ell=1}^{\ell^*} \bar{R}_{\ell-1,\ell-1}(m,1) \cup \bigcup_{\ell=1}^{\ell^*-1}\bigcup_{l=1}^{\ell-1} \bar{B}^{l}_{\ell-1,\ell}(m,1) \\
&\cup \bigcup_{\ell=1}^{\ell^*}\bigcup_{l=1}^{\ell-2} \bar{B}^{l}_{\ell-1,\ell-1}(m,1) \cup \bar{B}^{1}_{\ell^*-1,\ell^*}(m,1) \cup \bigcup_{\ell_1=\ell^*}^{K-1}\bigcup_{\ell_2=\ell^*}^{K-1} \bar{R}_{\ell_1,\ell_2}(m,1) \cup \bigcup_{\ell_1=\ell^*}^{K-1}\bigcup_{\ell_2=\ell^*}^{K-1}\bigcup_{l=1}^{\ell_2-1} \bar{B}^{l}_{\ell_1,\ell_2}(m,1) \\
&\cup \bigcup_{\ell_1=\ell^*}^{L-1}\bigcup_{\ell_2=\ell^*}^{L-1} \bar{R}_{\ell_1,\ell_2}(m,1) \cup \bigcup_{\ell_1=\ell^*}^{L-1}\bigcup_{\ell_2=\ell^*}^{L-1}\bigcup_{l=1}^{\ell_2-1} \bar{B}^{l}_{\ell_1,\ell_2}(m,1) \cup \mathcal{S}^{v}_{\text{pos}}(m,1) \cup \mathcal{S}^{h}_{\text{pos}}(m,1).
\end{aligned} \tag{4.27}$$

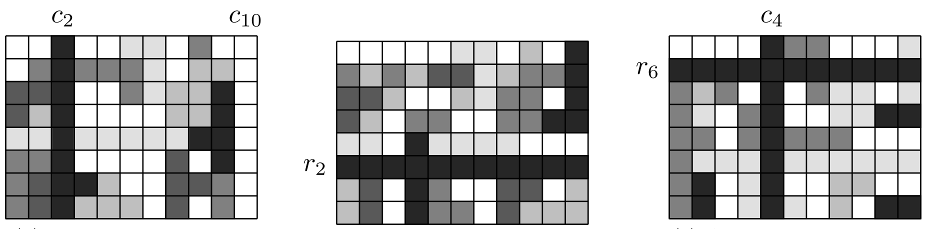
As illustrated in the next result, which we prove in Section 6.4, $\mathfrak{T}_{\mathcal{X}^s_{\text{pos}}}(\mathbf{m})$ includes those configurations with a positive probability of being visited by the Markov chain $\{X_t\}^{\beta}_{t\in\mathbb{N}}$ started in $\mathbf{m}$ before hitting $\mathcal{X}^s_{\text{pos}}$ in the limit $\beta \to \infty$. Note that the relation between $T_{\mathcal{X}^s_{\text{pos}}}(\mathbf{m})$ and $\mathfrak{T}_{\mathcal{X}^s_{\text{pos}}}(\mathbf{m})$ is given by (4.23).

Theorem 4.13. *If the external magnetic field is positive, then for any $\mathbf{m} \in \mathcal{X}^m_{pos}$ the tube of typical trajectories for the transition $\mathbf{m} \to \mathcal{X}^s_{pos}$ is (4.27) and there exists $k > 0$ such that for β sufficiently large*

$$\mathbb{P}_{\beta}(\tau^{\mathbf{m}}_{\partial^{np}\mathfrak{T}_{\mathcal{X}^s_{pos}}(\mathbf{m})} \leq \tau^{\mathbf{m}}_{\mathcal{X}^s_{pos}}) \leq e^{-k\beta}. \tag{4.28}$$

5. Energy landscape analysis and asymptotic behavior

In this section we analyze the energy landscape of the q-state Potts model with positive external magnetic field. First we recall some useful definitions and lemmas from [62].



(a) Two vertical bridges on columns c_2 and c_{10}.

(b) An horizontal bridge on row r_2.

(c) A cross on column c_4 and on row r_6.

Fig. 3. Example of configurations on a 8 × 11 grid graph displaying a vertical s-bridge (a), a horizontal s-bridge (b) and a s-cross (c). We color black the spins s.

5.1. Known local geometric properties

In the following list we introduce the notions of *disagreeing edges*, *bridges* and *crosses* of a Potts configuration on a grid-graph Λ.

- We call $e = (v, w) \in E$ a *disagreeing edge* if it connects two vertices with different spin values, i.e., $\sigma(v) \neq \sigma(w)$.
- For any $i = 0, \dots, K-1$, let

$$d_{r_i}(\sigma) := \sum_{(v,w)\in r_i} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} \tag{5.1}$$

  be the total number of disagreeing edges that σ has on row r_i. Furthermore, for any $j = 0, \dots, L-1$ let

$$d_{c_j}(\sigma) := \sum_{(v,w)\in c_j} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}}, \tag{5.2}$$

  be the total number of disagreeing edges that σ has on column c_j.
- We define $d_h(\sigma)$ as the total number of disagreeing horizontal edges and $d_v(\sigma)$ as the total number of disagreeing vertical edges, i.e.,

$$d_h(\sigma) := \sum_{i=0}^{K-1} d_{r_i}(\sigma), \text{ and } d_v(\sigma) := \sum_{j=0}^{L-1} d_{c_j}(\sigma). \tag{5.3}$$

  Since we may partition the edge set E in the two subsets of horizontal edges E_h and of vertical edges E_v, such that $E_h \cap E_v = \emptyset$, the total number of disagreeing edges is given by

$$\sum_{(v,w)\in E} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} = \sum_{(v,w)\in E_v} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} + \sum_{(v,w)\in E_h} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} = d_v(\sigma) + d_h(\sigma). \tag{5.4}$$

- We say that σ has a *horizontal bridge* on row r if $\sigma(v) = \sigma(w)$, for all $v, w \in r$.
- We say that σ has a *vertical bridge* on column c if $\sigma(v) = \sigma(w)$, for all $v, w \in c$.
- We say that $\sigma \in \mathcal{X}$ has a *cross* if it has at least one vertical and one horizontal bridge.

For sake of simplicity, if σ has a bridge of spins $s \in S$, then we say that σ has an s-bridge. Similarly, if σ has a cross of spins s, we say that σ has an s-cross (see Fig. 3).

- For any $s \in S$, the total number of s-bridges of the configuration σ is denoted by $B_s(\sigma)$.

Note that if a configuration $\sigma \in \mathcal{X}$ has an s-cross, then $B_s(\sigma)$ is at least 2 since the presence of an s-cross implies the presence of two s-bridges, i.e., of a horizontal s-bridge and of a vertical s-bridge.

We conclude this section by recalling the following three useful lemmas from [62]. These results give us some geometric properties for the q-state Potts model on a grid-graph and they are verified regardless of the definition of the external magnetic field.

Lemma 5.1 (*[62, Lemma 2.2]*). *A Potts configuration on a grid-graph Λ does not have simultaneously a horizontal bridge and a vertical bridge of different spins.*

Lemma 5.2 (*[62, Lemma 2.6]*). *Let $\sigma, \sigma' \in \mathcal{X}$ be two Potts configurations which differ by a single-spin update, that is $|\{v \in V : \sigma(v) \neq \sigma'(v)\}| = 1$. Then for every $s \in S$ we have that*

(i) $B_s(\sigma') - B_s(\sigma) \in \{-2, -1, 0, 1, 2\}$,
(ii) $B_s(\sigma') - B_s(\sigma) = 2$ *if and only if σ' has an s-cross that σ does not have.*

Lemma 5.3 (*[62, Lemma 2.3]*). *The following properties hold for every Potts configuration $\sigma \in \mathcal{X}$ on a grid graph Λ with periodic boundary conditions:*

(i) *$d_r(\sigma)$=0 if and only if σ has a horizontal bridge on row r;*
(ii) *$d_c(\sigma)$=0 if and only if σ has a vertical bridge on column c;*
(iii) *if σ has no horizontal bridge on row r, then $d_r(\sigma) \geq 2$;*
(iv) *if σ has no vertical bridge on column c, then $d_c(\sigma) \geq 2$.*

5.2. Metastable states and stability level of the metastable configurations

By Proposition 4.2, H_{pos} has only one global minimum, $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$. Furthermore, the configurations $\mathbf{2}, \dots, \mathbf{q}$ are such that $H_{\text{pos}}(\mathbf{2}) = \cdots = H_{\text{pos}}(\mathbf{q})$. In this subsection, our aim is to prove that the metastable set $\mathcal{X}^m_{\text{pos}}$ is the union of these configurations. We are going to prove this claim by steps. We begin by obtaining an upper bound for the stability level of the states $\mathbf{2}, \dots, \mathbf{q}$.

Given $\mathbf{m} \in \{\mathbf{2}, \dots, \mathbf{q}\}$, let us compute the following energy gap between any $\sigma \in \mathcal{X}$ and $\mathbf{m}$,

$$\begin{aligned} H_{\text{pos}}(\sigma) - H_{\text{pos}}(\mathbf{m}) &= \sum_{(v,w)\in E} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} - h \sum_{u \in V} \mathbb{1}_{\{\sigma(u)=1\}} \\ &= d_v(\sigma) + d_h(\sigma) - h \sum_{u\in V} \mathbb{1}_{\{\sigma(u)=1\}}, \end{aligned} \tag{5.5}$$

where in the last equality we used (5.4).

We say that a path $\omega \in \Omega_{\sigma,\sigma'}$ is the *concatenation* of the L paths $\omega^{(i)} = (\omega^{(i)}_0, \dots, \omega^{(i)}_{n_i})$, for some $n_i \in \mathbb{N}$, $i = 1, \dots, L$ if $\omega = (\omega^{(1)}_0 = \sigma, \dots, \omega^{(1)}_{n_1}, \omega^{(2)}_0, \dots, \omega^{(2)}_{n_2}, \dots, \omega^{(L)}_0, \dots, \omega^{(L)}_{n_L} = \sigma')$.

Definition 5.4. For any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$, we define a *reference path* $\tilde{\omega} : \mathbf{m} \to \mathbf{1}$, $\tilde{\omega} = (\tilde{\omega}_0, \dots, \tilde{\omega}_{KL})$ as the concatenation of the two paths $\tilde{\omega}^{(1)} := (\mathbf{1} = \tilde{\omega}_0, \dots, \tilde{\omega}_{(K-1)^2})$ and $\tilde{\omega}^{(2)} := (\tilde{\omega}_{(K-1)^2}, \dots, \mathbf{m} = \tilde{\omega}_{KL})$. The paths $\tilde{\omega}^{(1)}$ and $\tilde{\omega}^{(2)}$ are obtained by replacing $\mathbf{1}$ with $\mathbf{m}$ and $\mathbf{s}$ with $\mathbf{1}$ in the paths $\hat{\omega}^{(1)}$ and $\hat{\omega}^{(2)}$ of [19, Definition 5.1]. See Appendix A.1.1 for the explicit definition.

For any fixed $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$, let us focus on the transition from $\mathbf{m}$ to $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$. Given $m \in S$, let

$$N_m(\sigma) := |\{v \in V : \ \sigma(v) = m\}| \tag{5.6}$$

be the number of vertices with spin m in $\sigma \in \mathcal{X}$.

Lemma 5.5. *Let $\mathbf{m} \in \mathcal{X}^m_{pos}$. For any $\sigma \in \bar{R}_{\ell^*-1,\ell^*}(m,1)$ there exists a path $\gamma : \sigma \to \mathbf{m}$ such that the maximum energy along γ is bounded as*

$$\max_{\xi\in\gamma} H_{pos}(\xi) < 4\ell^* - h(\ell^*(\ell^* - 1) + 1) + H_{pos}(\mathbf{m}). \tag{5.7}$$

Proof. The proof proceeds analogously to the proof of [19, Lemma 5.4] by replacing $\mathbf{1}$ with $\mathbf{m}$, $\mathbf{s}$ with $\mathbf{1}$ and $\hat{\omega}$ with $\tilde{\omega}$. See Appendix A.1.2 for the explicit proof. □

In the next lemma we show that for any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$, $\bar{B}^2_{\ell^*-1,\ell^*}(m,1)$ is connected to the stable set $\mathcal{X}^s_{\text{pos}}$ by a path that does not overcome the energy value $4\ell^* - h(\ell^*(\ell^* - 1) + 1) + H_{\text{pos}}(\mathbf{m})$.

Lemma 5.6. *Let $\mathbf{m} \in \mathcal{X}^m_{pos}$. For any $\sigma \in \bar{B}^2_{\ell^*-1,\ell^*}(m,1)$ there exists a path $\gamma : \sigma \to \mathbf{m}$ such that the maximum energy along γ is bounded as*

$$\max_{\xi\in\gamma} H_{pos}(\xi) < 4\ell^* - h(\ell^*(\ell^* - 1) + 1) + H_{pos}(\mathbf{m}). \tag{5.8}$$

Proof. The proof proceeds analogously to the proof of [19, Lemma 5.5] by replacing $\mathbf{1}$ with $\mathbf{m}$, $\mathbf{s}$ with $\mathbf{1}$ and $\hat{\omega}$ with $\tilde{\omega}$. See Appendix A.1.3 for the explicit proof. □

We are now able to prove the following propositions, in which we give an upper bound and a lower bound for $\Gamma_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) := \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) - H_{\text{pos}}(\mathbf{m})$, for any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$.

Proposition 5.7 (*Upper Bound for the Communication Height*). *For every $\mathbf{m} \in \mathcal{X}^m_{pos}$,*

$$\Phi_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}) - H_{pos}(\mathbf{m}) \leq 4\ell^* - h(\ell^*(\ell^* - 1) + 1). \tag{5.9}$$

Proof. The upper bound (5.9) follows by the proof of Lemma 5.6, where we proved that $\max_{\xi\in\tilde{\omega}} H_{\text{pos}}(\xi) = H_{\text{pos}}(\tilde{\omega}_{k^*}) = 4\ell^* - h(\ell^*(\ell^* - 1) + 1) + H_{\text{pos}}(\mathbf{m})$. □

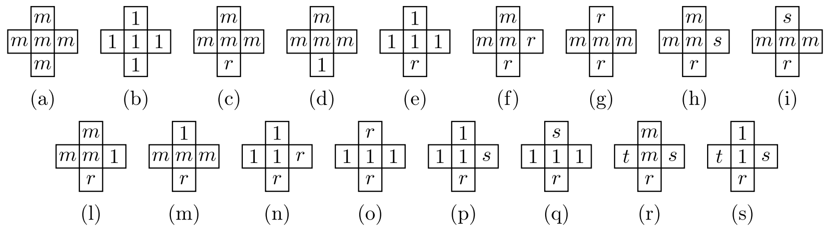


Fig. 4. Stable tiles centered in any $v \in V$ for a q-Potts configuration on Λ for any $m, r, s, t \in S\backslash\{1\}$ different from each other. The tiles are depicted up to a rotation of $\alpha\frac{\pi}{2}$, $\alpha \in \mathbb{Z}$.

Proposition 5.8 (*Lower Bound for the Communication Height*)**.** *For every* $\mathbf{m} \in \mathcal{X}^m_{pos}$,

$$\Phi_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}) - H_{pos}(\mathbf{m}) \geq 4\ell^* - h(\ell^*(\ell^* - 1) + 1). \tag{5.10}$$

Proof. The proof proceeds analogously to the proof of [19, Proposition 5.2] by replacing **1** with **m**, **s** with **1** and $\hat{\omega}$ with $\tilde{\omega}$. See Appendix A.1.4 for the explicit proof. □

The above Propositions 5.7 and 5.8 are used to prove (4.1) in Theorem 4.3. Note that (4.1) is the min–max energy value reached by any optimal path $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\text{pos}}}$ for every $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$.

Lemma 5.9. *Any* $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{pos}}$ *is such that* $\omega \cap \bar{R}_{\ell^*-1,\ell^*}(m, 1) \neq \emptyset$.

Proof. The proof proceeds analogously to the proof of [19, Lemma 5.6] by replacing **1** with **m**, **s** with **1** and $\hat{\omega}$ with $\tilde{\omega}$. See Appendix A.1.5 for the explicit proof. □

Let $\sigma \in \mathcal{X}$ and let $v \in V$. We define the *tile* centered in v, denoted by v-tile, as the set of five sites consisting of v and its four nearest neighbors. See for instance Fig. 4. A v-tile is said to be *stable* for σ if by flipping the spin on vertex v from $\sigma(v)$ to any $s \in S$ the energy difference $H_{\text{pos}}(\sigma^{v,s}) - H_{\text{pos}}(\sigma)$ is greater than or equal to zero. In Lemma 5.10 we define the set of all possible stable tiles induced by the Hamiltonian (2.2). For any $\sigma \in \mathcal{X}$, $v \in V$ and $s \in S$, we define $n_s(v)$ as the number of nearest neighbors to v with spin s in σ, i.e., $n_s(v) := |\{w \in V : w \sim v, \sigma(w) = s\}|$.

Lemma 5.10 (*Characterization of Stable v-Tiles in a Configuration σ*)**.** *Let $\sigma \in \mathcal{X}$ and let $v \in V$. The tile centered in v is stable for σ if and only if it satisfies one of the following conditions.*

(1) For any $m \in S$, $m \neq 1$, if $\sigma(v) = m$, v has either at least three nearest neighbors with spin m or two nearest neighbors with spin m and two nearest neighbors with spin $r, t \in S\backslash\{m\}$ such that they may be not both equal to 1, see Fig. 4(a),(c),(d),(f)–(m), or one nearest neighbor m and three nearest neighbors with spin $r, s, t \in S\backslash\{1\}$ different from each other, see Fig. 4(r).

(2) If $\sigma(v) = 1$, v has either at least two nearest neighbors with spin 1, see Fig. 4(b),(e),(n)–(q) or it has one nearest neighbor 1 and three nearest neighbors with spin $r, s, t \in S\backslash\{1\}$ different from each other, see Fig. 4(s).

In particular, if $\sigma(v) = m$, then

$$H_{pos}(\sigma^{v,r}) - H_{pos}(\sigma) = n_m(v) - n_r(v) + h\mathbb{1}_{\{m=1\}} - h\mathbb{1}_{\{r=1\}}. \tag{5.11}$$

Proof. Let $\sigma \in \mathcal{X}$ and let $v \in V$. To find if a v-tile is stable for σ we reduce ourselves to flip the spin on vertex v from $\sigma(v) = m$ to a spin r such that $n_r(v) > 1$. Indeed, otherwise the energy difference (2.8) is for sure strictly positive. Let us divide the proof in several cases.
Case 1. Assume that $n_m(v) = 0$ in σ. Then the corresponding v-tile is not stable for σ. Indeed, for any $m \in S$ and $r \notin \{1, m\}$, by flipping the spin on vertex v from m to r we get

$$H_{\text{pos}}(\sigma^{v,r}) - H_{\text{pos}}(\sigma) = -n_r(v) + h\mathbb{1}_{\{m=1\}}. \tag{5.12}$$

Moreover, by flipping the spin on vertex v from $m \neq 1$ to 1 we have

$$H_{\text{pos}}(\sigma^{v,1}) - H_{\text{pos}}(\sigma) = -n_1(v) - h. \tag{5.13}$$

Hence, for any $m \in S$, if v has spin m and it has four nearest neighbors with spins different from m, then the tile centered in v is not stable for σ since the energy difference (2.8) is always strictly negative.
Case 2. Assume that $v \in V$ has three nearest neighbors with spin value different from m in σ, i.e., $n_m(v) = 1$. Then, in view of the energy difference (2.8), for any $m \in S$ and $r \notin \{1, m\}$, by flipping the spin on vertex v from m to r we have

$$H_{\text{pos}}(\sigma^{v,r}) - H_{\text{pos}}(\sigma) = 1 - n_r(v) + h\mathbb{1}_{\{m=1\}}. \tag{5.14}$$

Furthermore, for any $m \neq 1$, if $r = 1$ by flipping the spin on vertex v from m to 1 we have

$$H_{\text{pos}}(\sigma^{v,1}) - H_{\text{pos}}(\sigma) = 1 - n_1(v) - h. \tag{5.15}$$

Hence, for any $m \in S$, $m \neq 1$, if v has only one nearest neighbor with spin m, a tile centered in v is stable for σ only if v has nearest neighbors with spins different from each other and from 1, see Fig. 4(r). While, if $m = 1$, if v has only one nearest neighbor with spin 1, a tile centered in v is stable for σ only if v has nearest neighbors with spins different from each other, see Fig. 4(s).

Case 3. Assume that $v \in V$ has two nearest neighbors with spin m in σ, i.e., $n_m(v) = 2$. Then, in view of the energy difference (2.8), for any $m \in S$ and $r \notin \{1, m\}$, by flipping the spin on vertex v from m to r we have

$$H_{\text{pos}}(\sigma^{v,r}) - H_{\text{pos}}(\sigma) = 2 - n_r(v) + h\mathbb{1}_{\{m=1\}}. \tag{5.16}$$

Furthermore, by flipping the spin on vertex v from $m \neq 1$ to 1 we get

$$H_{\text{pos}}(\sigma^{v,1}) - H_{\text{pos}}(\sigma) = 2 - n_1(v) - h. \tag{5.17}$$

Hence, for any $m \in S$, if v has two nearest neighbors with spin m and two nearest neighbors with spin 1, then the corresponding v-tile is not stable. In all the other cases, for any $m \in S$, if v has two nearest neighbors with spin m, the corresponding v-tile is stable for σ, see Fig. 4(f)–(q).

Case 4. Assume that $v \in V$ has three nearest neighbors with spin m and one nearest neighbor r in σ, i.e., $n_m(v) = 3$ and $n_r(v) = 1$. Then, for any $m \in S$ and $r \notin \{1, m\}$, by flipping the spin on vertex v from m to r we have

$$H_{\text{pos}}(\sigma^{v,r}) - H_{\text{pos}}(\sigma) = 2 + h\mathbb{1}_{\{m=1\}}. \tag{5.18}$$

Moreover, by flipping the spin on vertex v from $m \neq 1$ to 1 we get

$$H_{\text{pos}}(\sigma^{v,1}) - H_{\text{pos}}(\sigma) = 2 - h. \tag{5.19}$$

Case 5. Assume that $v \in V$ has four nearest neighbors with spin m, i.e., $n_m(v) = 4$ in σ. Then, we have $n_r(v) = 0$ and

$$H_{\text{pos}}(\sigma^{v,r}) - H_{\text{pos}}(\sigma) = 4 + h\mathbb{1}_{\{m=1\}}. \tag{5.20}$$

Furthermore, by flipping the spin on vertex v from $m \neq 1$ to 1 we get

$$H_{\text{pos}}(\sigma^{v,1}) - H_{\text{pos}}(\sigma) = 4 - h. \tag{5.21}$$

From Case 4 and Case 5, for any $m \in S$, we get that a v-tile is always stable for σ if v has at least three nearest neighbors with spin m, see Fig. 4(a)–(e). In particular, in all the cases $1 - 5$ we verify that (5.11) is satisfied by (5.12)–(5.21). □

Let $\mathscr{M}_{\text{pos}}$ be the set of the local minima of the Hamiltonian H_{pos}, that is

$$\mathscr{M}_{\text{pos}} := \{\sigma \in \mathcal{X} : \; H_{\text{pos}}(\sigma) < H_{\text{pos}}(\eta) \text{ for any } \eta \in \mathcal{X}\backslash\{\sigma\} \text{ such that } P_\beta(\sigma, \eta) > 0\}. \tag{5.22}$$

A subset $D \subset \mathcal{X}$ is said to be a *plateau* if it is a maximal connected set of equal energy configurations and it is said to be *stable* if $H_{\text{pos}}(\mathscr{F}(\partial D)) > H_{\text{pos}}(D)$. Let $\bar{\mathscr{M}}_{\text{pos}} := \bigcup_{D \text{ stable plateau}} D$ be the set the stable plateaux of H_{pos}.

Remark 5.11. Note that a configuration $\sigma \in \mathcal{X}$ is a local minimum for H_{pos}, respectively a stable plateau, when for any $v \in V$ and $s \in S$ the energy difference (2.8) is strictly positive, respectively null. Then if σ has at least one unstable v-tile, for some $v \in V$, it does not belong to $\mathscr{M}_{\text{pos}} \cup \bar{\mathscr{M}}_{\text{pos}}$.

Lemma 5.12 (*Characterization of the* 1*-Clusters in Local Minima and Stable Plateaux*)**.** *Let $\sigma \in \mathscr{M}_{pos} \cup \bar{\mathscr{M}}_{pos}$. Any* 1*-cluster of σ is a rectangle.*

Proof. Let $C^1(\sigma)$ be a 1-cluster of σ and let $\partial C^1(\sigma) = \{v \in V : \; v \notin C^1(\sigma) \text{ and } \exists u \in C^1(\sigma) \text{ s.t. } \{u, v\} \in E\}$ be the boundary of $C^1(\sigma)$. Assume by contradiction that $C^1(\sigma)$ is not a rectangle. This means that there exist at least a $v \notin C^1(\sigma)$ and two $u_1, u_2 \in C^1(\sigma)$ such that the edges $\{v, u_1\}$ and $\{v, u_2\}$ form an internal angle of $\frac{3}{2}\pi$ on the border of $C^1(\sigma)$. Since $\sigma(v) \neq 1$ and $\sigma(u_1) = \sigma(u_2) = 1$, by Lemma 5.10 the tile centered at vertex v is not stable and this is a contradiction in view of Remark 5.11. □

We are now able to prove Proposition 4.5.

Proof of Proposition 4.5. In order to estimate the stability level of any $\sigma \in \mathcal{X}\backslash\{\mathbf{1}, \dots, \mathbf{q}\}$ it is enough to focus on $\sigma \in \mathscr{M}_{\text{pos}} \cup \bar{\mathscr{M}}_{\text{pos}}\backslash\{\mathbf{1}, \dots, \mathbf{q}\}$. Our goal is to prove that for any $\sigma \in \mathscr{M}_{\text{pos}} \cup \bar{\mathscr{M}}_{\text{pos}}\backslash\{\mathbf{1}, \dots, \mathbf{q}\}$, $V_\sigma^{\text{pos}} < V^* := \frac{4}{h}$. Indeed, given $\delta \in (0, 1)$ such that $\ell^* = \lceil \frac{2}{h} \rceil = \frac{2}{h} + \delta$,

$$\begin{aligned} V^* - \Gamma_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) &\leq \frac{4}{h} - 4\ell^* + h(\ell^*)^2 - h\ell^* + h \\ &= \frac{4}{h} - 4\left(\frac{2}{h} + \delta\right) + h\left(\frac{4}{h^2} + \frac{4}{h}\delta + \delta^2\right) - h\left(\frac{2}{h} + \delta\right) + h \\ &= h - 2 + h\delta(\delta - 1) < 0, \end{aligned} \tag{5.23}$$

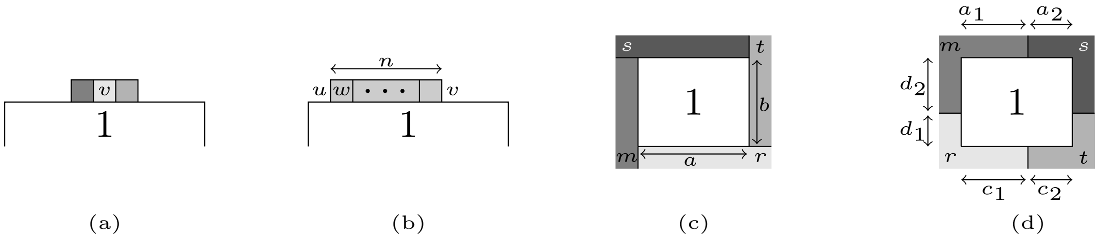


Fig. 5. Pictorial illustrations of the proof of Proposition 4.5. We color white the 1-cluster and with a color different from white each m-cluster for any $m \in S$, $m \neq 1$. Notice that, in (d) we assume $a_i, b_i, c_i, d_i \geq 1$ for $i = 1, 2$; indeed, if (at least) one of them is zero, then we would retrieve a configuration for which a proof based on a similar construction of the one given for the case depicted in (c) applies.

where we used $0 < h, \delta < 1$. We divide the proof into three cases depending on the geometry of the configuration $\eta \in \mathcal{X} \setminus \{\mathbf{1}, \ldots, \mathbf{q}\}$, as follows. In the first case, we focus on those configurations in which there is a super-critical 1-rectangle. This part of the proof is independent from what surrounds such a rectangle. In the second case, we consider those configurations in which there is a sub-critical 1-rectangle and we divide the proof into five sub-cases, depending on the spin configuration in the external perimeter of the 1-rectangle. Finally, in the third case we consider those configurations in which there are no 1-rectangles, and analyze them differently depending on the geometry of their perimeter. First, we take into account the case in which the configurations are characterized by clusters with only angles of π; second, we consider those configurations composed by clusters with only angles of π or $\frac{\pi}{2}$; finally, we study the scenario in which a configuration has at least one cluster with angles of $\frac{3}{2}\pi$. We further subdivide the analysis of this last class of configurations in three cases, depending on the spin configuration in the external perimeter, similarly as in the second case above. We encourage the reader to refer to the figures throughout the proof.

Case 1. Assume that σ has a 1-cluster, i.e., a 1-rectangle in view of Lemma 5.12. Let $a, b \in \mathbb{N}$ be the side lengths of this 1-rectangle and assume $a = \max\{a, b\} \geq \ell^*$. We construct a path $\omega = (\omega_0, \ldots, \omega_a)$, where $\omega_0 = \sigma$ and $\omega_a = \bar{\sigma}$, that flips to 1 consecutively those spins adjacent to a side of length a of the 1-rectangle. Using (2.8), we have

$$H_{\text{pos}}(\omega_1) - H_{\text{pos}}(\sigma) \leq 2 - h \quad \text{and} \quad H_{\text{pos}}(\omega_i) - H_{\text{pos}}(\omega_{i-1}) \leq -h, \text{ for } i = 2, \ldots, a. \tag{5.24}$$

From (3.14) and (5.24), if $a > \ell^*$, then $H_{\text{pos}}(\bar{\sigma}) - H_{\text{pos}}(\sigma) \leq 2 - ha < 2 - h\ell^* < 0$. Since the maximum energy is reached at the first step, we conclude $V_\sigma^{\text{pos}} = 2 - h < V^*$.

Otherwise, if σ has only 1-rectangles $\ell^* \times \ell^*$, then $\bar{\sigma}$ has a 1-rectangle $\ell^* \times (\ell^* + 1)$, say $\bar{R}$. Either $\bar{R}$ does not interact with the other 1-rectangles of $\bar{\sigma}$, or $\bar{R}$ interacts with another 1-rectangle $\hat{R}$. In the former case, it is enough to repeat the above construction along the side of length $\ell^* + 1 > \ell^*$. In the latter case, there exists in $\bar{\sigma}$ a vertex w that is connected to both $\bar{R}$ and $\hat{R}$. This vertex w has at least two nearest neighbors with spin 1 inside the 1-rectangles $\bar{R}$ and $\hat{R}$, respectively. Hence, set $\hat{\sigma} := \bar{\sigma}^{(w,1)}$ and using (2.8), we get $H_{\text{pos}}(\hat{\sigma}) - H_{\text{pos}}(\bar{\sigma}) \leq -h < 0$. Using (5.24), it follows that along the path $(\sigma, \omega_1, \ldots, \omega_{\ell-2}, \bar{\sigma}, \hat{\sigma})$ the maximum energy is reached at the first step. Thus, we conclude that $V_\sigma^{\text{pos}} = 2 - h < V^*$.

Case 2. Now assume that in σ each 1-rectangle has sides of lengths $a, b < \ell^*$, $a, b \in \mathbb{N}$.

Case 2.1 Assume that along a side of the boundary of 1-rectangle there are three adjacent vertices that have a spin value different from each other, see Fig. 5(a). Flipping to 1 the spin on the central vertex v, the energy decreases by at least $-h$. Hence, $V_\sigma^{\text{pos}} = 0$.

Case 2.2 Assume that along a side, say of length a, of the boundary of 1-rectangle there are $2 \leq n \leq a$ adjacent vertices having the same spin value $m \in S$, $m \neq 1$, see Fig. 5(b). Note that we assume $\sigma(u), \sigma(v) \neq m, 1$. We construct a path $\omega = (\omega_0 = \sigma, \ldots, \omega_n)$ that starting from the vertex w flips to 1 the n spins m. We have

$$H_{\text{pos}}(\omega_i) - H_{\text{pos}}(\omega_{i-1}) \leq \begin{cases} 1 - h, & \text{if } i = 1, \\ -h, & \text{if } i = 2, \ldots, n-1, \\ -1 - h, & \text{if } i = n. \end{cases} \tag{5.25}$$

From (5.25), we conclude that $H_{\text{pos}}(\omega_n) - H_{\text{pos}}(\sigma) = -hn < 0$ and $V_\sigma^{\text{pos}} = 1 - h < V^*$. Note that the proof of this case and of the previous one holds also for the case in which the 1-rectangle is a 1-strip that is an admissible case in view of the stable tile as in Fig. 4(o) and (q).

Case 2.3 Assume that the 1-rectangle is surrounded by four different blocks as in Fig. 5(c). We assume that s, m, r, t are such that $r, t \in S \setminus \{m, s\}$ but the cases $s = m$ and/or $r = t$ are admissible. Conversely to the previous case, we are now allowing to have along each side of length a, b of the 1-rectangle a sequence of at least either $a + 1$ or $b + 1$, respectively, spins of the same value such that this sequence starts from the vertex above to a corner of the 1-rectangle and ends beyond the opposite corner of the same side. Note that on each corner of the 1-rectangle there is a v-tile as in Fig. 4(p) and also that we are imposing the above conditions along all the four sides of the 1-rectangle. Otherwise, if there is at least a side as in Fig. 5(a)–(b), we retrieve the proof of Case 2.1 and Case 2.2.

First assume that $a + b \geq \ell^*$. We construct a path $\omega : \sigma \to \bar{\sigma}$ that first updates to 1 the spins on the vertices v_1, v_2, v_3, v_4, see Fig. 6(a), then flips to 1 the remaining spins different from 1 on the external boundary of the 1-rectangle. It starts from w and follows

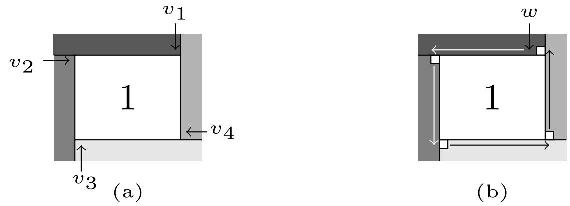


Fig. 6. Pictorial illustration of the first part of the proof of Case 2.3 of Proposition 4.5 when $a + a \geq \ell^*$.

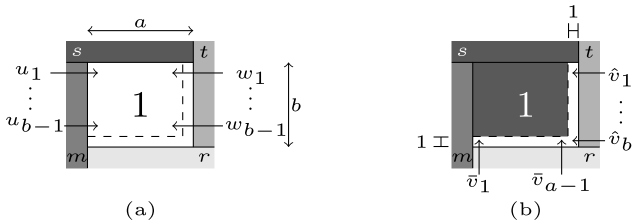


Fig. 7. Pictorial illustration of the proof of Case 2.3 of Proposition 4.5 when $a + b \leq \ell^* - 1$.

a clockwise order, see Fig. 6(b). Using (2.8), the assumption $a + b > \ell^*$ and $\ell^* = \lceil \frac{2}{h} \rceil > \frac{2}{h}$,

$$\begin{aligned} H_{\text{pos}}(\bar{\sigma}) - H_{\text{pos}}(\sigma) &\leq 4(1-h) - ha - hb - ha - hb \\ &= -2h(a+b) - 4h + 4 \leq -2h\ell^* - 4h + 4 < -4h < 0. \end{aligned} \tag{5.26}$$

The maximum of the energy along ω is reached at the fourth step, so $V_\sigma^{\text{pos}} \leq \Phi_\omega \leq 4 - 4h$ and $4 - 4h < V^*$ from $0 < h < \frac{1}{2}$.

Next assume $a + b \leq \ell^* - 1$. Without loss of generality, let $b \leq \frac{\ell^*-1}{2}$. Otherwise it is enough to replace the role of spin s with the one of spin m in the following, see Fig. 5(c). We define a path ω as the concatenation of b paths $\omega^{(i)}$ for $i = 1, \dots, b$. Let $\sigma_0 = \sigma$ and, for $i = 1, \dots, b-1$, let $\omega^{(i)} : \sigma_{i-1} \to \sigma_i$ be the path that flips to s the spins 1 from vertex u_i to vertex w_i, see Fig. 7(a). Hence, for any $i = 1, \dots, b-1$, we have

$$H_{\text{pos}}(\sigma_i) - H_{\text{pos}}(\sigma_{i-1}) = 1 + h + h(a-2) = h(a-1) + 1. \tag{5.27}$$

Finally, the path $\omega^{(b)}$ flips to s the spins 1 first from $\bar{v}_1$ to $\bar{v}_{a-1}$ and then from $\hat{v}_1$ to $\hat{v}_b$, see Fig. 7(b). Now we have

$$\begin{aligned} H_{\text{pos}}(\sigma_b) - H_{\text{pos}}(\sigma_{b-1}) &= h + (-1+h)(a-2) + (-1+h)(b-1) - 2 + h \\ &= 1 - h - a(1-h) - b(1-h). \end{aligned} \tag{5.28}$$

Thus,

$$H_{\text{pos}}(\sigma_b) - H_{\text{pos}}(\sigma) = hab - a \leq \begin{cases} a(2h-1), & \text{if } b = 2, \\ a\left(h\frac{\ell^*-1}{2} - 1\right), & \text{if } b > 2, \end{cases} \tag{5.29}$$

and in both cases the two quantities are strictly lower than 0 since $0 < h < \frac{1}{2}$ and $\ell^* = \lceil \frac{2}{h} \rceil$ implies $h(\ell^* - 1) < 2$.

The maximum along ω is reached at the first step of $\omega^{(b)}$, i.e.,

$$\begin{aligned} V_\sigma^{\text{pos}} \leq \Phi_\omega &= (ha + 1 - h)(b-1) + h = ha(b-1) + b - 1 + 2h - hb \\ &< b - 1 + ha(b-1) \leq \frac{\ell^*-1}{2} + h\frac{(\ell^*-1)^2}{2} \\ &< \frac{1}{2}\left(\frac{2}{h} + 1 - 1\right) + \frac{h}{2}\left(\frac{2}{h} + 1 - 1\right)^2 = \frac{3}{h}. \end{aligned} \tag{5.30}$$

where we used $h(2-b) < 0$ since $b \geq 2$ in view of Lemma 5.10 and $\ell^* = \lceil \frac{2}{h} \rceil < \frac{2}{h} + 1$. Since $\frac{3}{h} < V^*$, we conclude.

Note that the proof of this case holds for all those scenarios in which at least a side of the 1-rectangle satisfies the above conditions.

Case 2.4 Assume that the 1-rectangle is surrounded by four different blocks as in Fig. 5(d). In this case $r, s \in S \backslash \{m, t\}$ and it is admissible that $r = s$ and/or $m = t$. Let $a_1 + a_2 = c_1 + c_2 = a$ and $b_1 + b_2 = d_1 + d_2 = b$. In order to complete the enumeration of all possible cases we require that $a_i, b_i, c_i, d_i \geq 1$ for $i = 1, 2$. Indeed, note that if at least one among them is 0, then we retrieve the case of Fig. 5(c) on that side. Note that here the novelty is that we allow to have each corner of the 1-rectangle surrounded by the same spin value, i.e., to have on each external corner three spins that form an angle of $\frac{3}{2}\pi$ and that have the same spin value among them but different from the one on the opposite external corner on the same side. The case in which at least a side of the 1-rectangle is

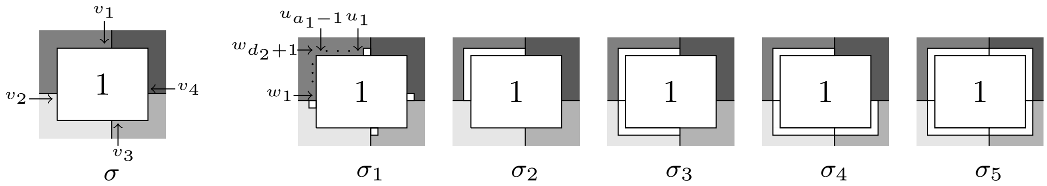


Fig. 8. Illustrations of the configurations visited by the path defined in the proof of Case 2.4 under the condition $a + b \geq \ell^*$.

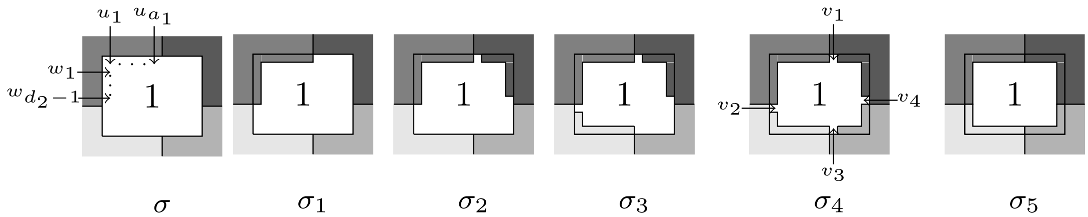


Fig. 9. Configurations visited by the path ω defined in the case $a + b < \ell^*$ of Case 2.4.

completely surrounded by the same spin value is considered in the next Case 2.5. Once again we impose these conditions on each side of the 1-rectangle in order to not retrieve the Cases 2.1, 2.2, 2.3.

First assume that $a + b \geq \ell^*$. We refer to Figs. 5(d) and 8 for a pictorial illustration of the proof. We define a path ω as the concatenation of the subpaths $\omega^{(i)}$ for $i = 1, \ldots, 5$ that are constructed as follows. The path $\omega^{(1)} : \sigma \to \sigma_1$ flips to 1 the spins on the vertices v_1, v_2, v_3, v_4. Hence, $H_{\text{pos}}(\sigma_1) - H_{\text{pos}}(\sigma) = 4 - 4h$.

The path $\omega^{(2)} : \sigma_1 \to \sigma_2$ flips to 1 the spins s first from the vertex u_1 to the vertex u_{a_1}, then from w_1 to w_{d_2+1}. Thus,

$$H_{\text{pos}}(\sigma_2) - H_{\text{pos}}(\sigma_1) = -ha_1 - hd_2. \tag{5.31}$$

For $i = 2, 3, 4$, the path $\omega^{(i)} : \sigma_i \to \sigma_{i+1}$ is defined in the same way on one of the other three corners of the 1-rectangle. Thus,

$$H_{\text{pos}}(\sigma_3) - H_{\text{pos}}(\sigma_2) = -hd_1 - hc_2, \tag{5.32}$$

$$H_{\text{pos}}(\sigma_4) - H_{\text{pos}}(\sigma_3) = -hc_1 - hb_2, \tag{5.33}$$

$$H_{\text{pos}}(\sigma_5) - H_{\text{pos}}(\sigma_4) = -hb_1 - ha_2. \tag{5.34}$$

Then, given $\delta \in (0, 1)$ such that $\ell^* = \frac{2}{h} + \delta$, note that

$$\begin{aligned} H_{\text{pos}}(\sigma_5) - H_{\text{pos}}(\sigma) &= -2h(a+b) - 4h + 4 \leq -2h\ell^* - 4h + 4 \\ &= -2h\left(\frac{2}{h} + \delta\right) - 4h + 4 = -2h\delta - 4h < 0. \end{aligned} \tag{5.35}$$

The maximum of the energy along ω is reached at the fourth step, i.e., $V_\sigma^{\text{pos}} \leq \Phi_\omega^{\text{pos}} = 4 - 4h$, and $4 - 4h < V^*$ from $0 < h < \frac{1}{2}$.

Next assume $a + b < \ell^*$. If $b = \min\{a, b\} = 2$, then we flip to m and to r the spins 1 adjacent to the left side of length 2 of the 1-rectangle, see Fig. 5(d). Let $\bar{\sigma}$ be the configuration that we obtain from σ after these flips. Note that the first spin-update increases the energy by h, the second one decreases it by $-1 + h$. Hence, $H_{\text{pos}}(\bar{\sigma}) - H_{\text{pos}}(\sigma) = 2h - 1 < 0$ since $0 < h < \frac{1}{2}$ and $V_\sigma^{\text{pos}} = h < V^*$

Let now assume that $a, b > 2$. We refer to Figs. 5(d) and 9 for a pictorial illustration of the proof. We define a path ω as the concatenation of the subpaths $\omega^{(i)}$ for $i = 1, \ldots, 5$ that are constructed as follows. The path $\omega^{(1)} : \sigma \to \sigma_1$ flips to s the spins 1 first from the vertex u_1 to the vertex u_{a_1}, then from w_1 to w_{d_2-1}. Thus,

$$H_{\text{pos}}(\sigma_1) - H_{\text{pos}}(\sigma) = ha_1 + h(d_2 - 1). \tag{5.36}$$

Similarly we define the path $\omega^{(i)} : \sigma_i \to \sigma_{i+1}$ for $i = 2, 3, 4$ on one of the other three internal corners of the 1-rectangle. In particular,

$$H_{\text{pos}}(\sigma_2) - H_{\text{pos}}(\sigma_1) = h(d_1 - 1) + h(c_2 - 1), \tag{5.37}$$

$$H_{\text{pos}}(\sigma_3) - H_{\text{pos}}(\sigma_2) = h(c_1 - 1) + h(b_2 - 1), \tag{5.38}$$

$$H_{\text{pos}}(\sigma_4) - H_{\text{pos}}(\sigma_3) = h(b_1 - 1) + h(a_2 - 2). \tag{5.39}$$

Finally the path $\omega^{(5)} : \sigma_4 \to \sigma_5$ flips to s, m, t, r the spins on the vertices v_1, v_2, v_3, v_4. Hence,

$$H_{\text{pos}}(\sigma_5) - H_{\text{pos}}(\sigma_4) = -4 + 4h. \tag{5.40}$$

Then, since $\ell^* = \lceil \frac{2}{h} \rceil < \frac{2}{h} + 1$,

$$H_{\text{pos}}(\sigma_5) - H_{\text{pos}}(\sigma) = 2h(a+b) - 4 - 4h \leq 2h(\ell^* - 1) - 4h - 4 < 4 - 4h - 4 = -4h < 0.$$

The maximum of the energy along ω is reached at the end of the subpath $\omega^{(4)}$, i.e., $V_\sigma^{\text{pos}} \le \Phi_\omega^{\text{pos}} = 2h(a+b)-8h \le 2h(\ell^*-1)-8h < 4-8h$ where we used $\ell^* = \lceil \frac{2}{h} \rceil < \frac{2}{h}+1$. Note that $4-8h < V^*$ in view of the condition $0 < h < \frac{1}{2}$.

Case 2.5 Assume that the 1-rectangle is surrounded by spins of the same value. We define a path $\omega = (\omega_0, \dots, \omega_a)$ that flips consecutively, from 1 to m, those spins next to a side of length $a < \ell^*$. According to (2.8), we have

$$H_{\text{pos}}(\omega_i) - H_{\text{pos}}(\omega_{i-1}) = \begin{cases} h, & \text{for } i = 1, \dots, a-1; \\ -(2-h), & \text{for } i = a. \end{cases} \tag{5.41}$$

Thus, the maximum energy along ω is reached at the step $a-1$ and, $V_\sigma^{\text{pos}} = h(a-1)$. To conclude note that $a < \ell^*$ and $\ell^* = \lceil \frac{2}{h} \rceil < \frac{2}{h}+1$ imply $h(a-1) < h(\ell^*-1) < 2 < V^*$.

Case 3. Finally, assume that σ does not have an 1-cluster. We divide the proof in three cases.

First, consider the scenario in which for any $m \in S$, $m \neq 1$, the boundary of an m-cluster has only angles of π. This means that each m-cluster is a strip. Without loss of generality let σ have an m-strip $a \times K$ adjacent to an r-strip $b \times K$ with $a, b \in \mathbb{Z}$, $a, b \ge 1$. Let $\bar{\sigma}$ be the configuration obtained from σ by flipping all the spins m belonging to the m-strip from m to r. Let $\sigma_0 = \sigma$ and let σ_i be the configuration in which the initial m-strip is reduced to a strip $(a-i) \times K$ and the r-strip to a strip $(b+i) \times K$. We define a path $\omega : \sigma \to \bar{\sigma}$ as the concatenation of a paths $\omega^{(1)}, \dots, \omega^{(a)}$ such that $\omega^{(i)} := (\omega_0^{(i)} = \sigma_{i-1}, \omega_1^{(i)}, \dots, \omega_K^{(i)} = \sigma_i)$ is the path $\omega^{(i)}$ flips consecutively from m to r those spins m belonging to the column next to the r-strip. Thus, using (2.8), we have

$$H_{\text{pos}}(\omega_j^{(i)}) - H_{\text{pos}}(\omega_{j-1}^{(i)}) = \begin{cases} 2, & \text{if } j = 1, \\ 0, & \text{if } j = 2, \dots, K-1, \\ -2, & \text{if } j = K. \end{cases} \tag{5.42}$$

Hence, for any $i = 1, \dots, a-1$, the maximum energy value along $\omega^{(i)}$ is reached at the first step. Finally, we construct a path $\omega^{(a)} := (\omega_0^{(a)} = \sigma_{a-1}, \dots, \omega_K^{(a)} = \bar{\sigma})$ that flips consecutively from m to r the spins m of the remaining column of the initial m-strip in σ. Note that for the energy difference there are two possible values depending on whether the strips next to the initial m strip $a \times K$ have the same value r or one has value r and the other $s \neq r, m$. Hence, using (2.8), we have

$$H_{\text{pos}}(\omega_1^{(a)}) - H_{\text{pos}}(\sigma_{a-1}) \le 1, \quad \text{and} \quad H_{\text{pos}}(\omega_i^{(a)}) - H_{\text{pos}}(\omega_{i-1}^{(a)}) \le -2 \text{ for } i = 2, \dots, K. \tag{5.43}$$

In view of 5.12–(5.43), we get $H_{\text{pos}}(\sigma) > H_{\text{pos}}(\bar{\sigma})$. Furthermore, since the maximum energy value is reached at the first step, we get $V_\sigma^{\text{pos}} \le 2 < V^*$.

Second, we consider the case in which for any $m \in S$, $m \neq 1$, each m-cluster is σ has angles of either $\frac{\pi}{2}$ or π. This means that each m-cluster is a rectangle. Without loss of generality let σ contain an m-rectangle $\bar{R} := R_{a\times b}$ and an r-rectangle $\tilde{R} := R_{c\times d}$ such that the m-rectangle $\bar{R}$ has a side of length a adjacent to a side of the r-rectangle $\tilde{R}$ of length $c \ge a$. The case $c < a$ may be studied by interchanging the role of the spins m and r. Given $\bar{\sigma}$ the configuration obtained from σ by flipping from m to r all the spins m belonging to $\bar{R}$, we construct a path $\omega : \sigma \to \bar{\sigma}$ as the concatenation of b paths $\omega^{(1)}, \dots, \omega^{(b)}$. Let $\sigma_0 \equiv \sigma$ and for any $i = 1, \dots, b$ let σ_i be the configuration in which the initial r-rectangle $\tilde{R}$ is reduced to a rectangle $c \times d$ with a protuberance $a \times i$ and the initial m-rectangle $\bar{R}$ is reduced to a rectangle $a\times(b-i)$. We define $\omega^{(i)} := (\omega_0^{(i)} = \sigma_{i-1}, \omega_1^{(i)}, \dots, \omega_a^{(i)} = \sigma_i)$ as the path which flips consecutively to r those spins m adjacent to the side of length a of the m-rectangle $a \times (b-i)$. Thus, using (2.8),

$$H_{\text{pos}}(\omega_j^{(i)}) - H_{\text{pos}}(\omega_{j-1}^{(i)}) = \begin{cases} 1, & \text{if } j = 1, \\ 0, & \text{if } j = 2, \dots, a-1, \\ -1, & \text{if } j = a. \end{cases} \tag{5.44}$$

Then, for any $i = 1, \dots, b-1$, the maximum energy value along $\omega^{(i)}$ is reached at the first step. Finally, we define a path $\omega^{(b)} := (\omega_0^{(b)} = \sigma_{b-1}, \dots, \omega_\ell^{(b)} = \bar{\sigma})$ that flips consecutively from m to r the spins m belonging to the remaining m-rectangle $a \times 1$. In particular, using (2.8), we get

$$H_{\text{pos}}(\omega_1^{(b)}) - H_{\text{pos}}(\sigma_{b-1}) \le -1, \quad \text{and} \quad H_{\text{pos}}(\omega_i^{(b)}) - H_{\text{pos}}(\omega_{i-1}^{(b)}) \le -2. \tag{5.45}$$

Thanks to 5.12–(5.45), we get $H_{\text{pos}}(\sigma) > H_{\text{pos}}(\bar{\sigma})$. Moreover, since the maximum along ω is reached at the first step, by 5.12 we get $V_\sigma^{\text{pos}} = 1 < V^*$.

Finally, assume that for some $m \in S$, $m \neq 1$, σ has an m-cluster with an angle of $\frac{3}{2}\pi$ and none of the previous cases are applicable. Then the only admissible stable v-tiles appearing in σ are the ones depicted in Fig. 4(a), (c), (f), and (h). First, let us focus on the stable v-tiles (a), (c) and (f). Upon fixing a spin value, say r, such that there exists at least an r-cluster having an angle of $\frac{3}{2}\pi$, we prove that there exists a procedure defined on the plateaux of configurations where σ belongs and that leads either to decrease the energy or to a configuration where the r-clusters are only rectangles. We often refer to Fig. 10 for a pictorial illustration of the proof.

First we focus on a stable v-tile as the one depicted in Fig. 4(f) where $m \in S$, $m \neq 1, r$. Flipping from m to r the spin on the central vertex leads to a configuration having the same energy of σ, see Fig. 10(a). This spin flip changes four other tiles. We focus our attention solely on the tile centered in $\hat{v}$, which corresponds to the unit square highlighted by a thick border in Fig. 10(a), as the others are treated similarly. If in σ the stable $\hat{v}$-tile is as the one depicted in Fig. 4(f), then we flip from m to r its spin central m and we conclude since the energy decreases by 2, see Fig. 10(b). Otherwise, if in σ the stable $\hat{v}$-tile is as the one depicted in Fig. 4(c), then in the new configuration it is the center of a stable tile as depicted in Fig. 4(f), see Fig. 10(c). Finally, if in σ the stable $\hat{v}$-tile is

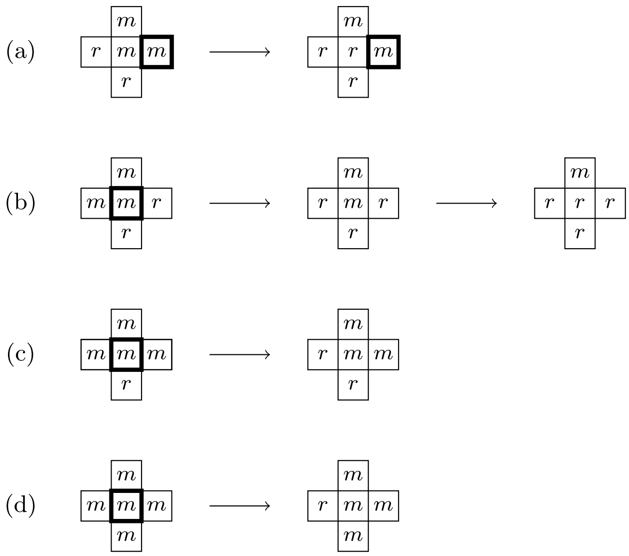


Fig. 10. Pictorial illustration of the final part of the proof of 4.5.

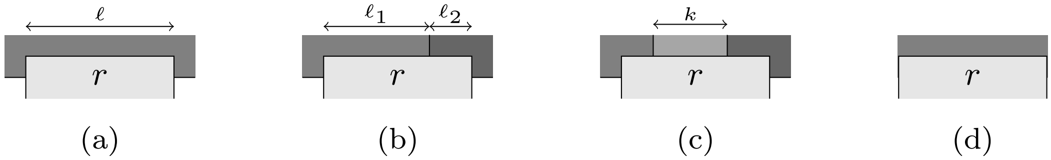


Fig. 11. Pictorial illustration of the four possible external boundaries of a side of the r-rectangle $\bar{R}$ studied in the final part of Case 3 of the proof of Proposition 4.5. Notice that to avoid considering redundant cases, in (b) and (c), we assume $\ell_1, \ell_2 < \ell$ and $k < \ell$, respectively.

as the one depicted in Fig. 4(a), then in the new configuration it is the center of a stable tile as depicted in Fig. 4(f), see Fig. 10(d). Iterating this procedure, since the volume of Λ is finite, we end up either decreasing the energy by 2 or removing all the stable v-tiles as depicted in Fig. 4(f). In the former case, we conclude the proof. In the latter case, we obtain a configuration where each r-cluster does not have angles of $\frac{3}{2}\pi$. In other words, each r-cluster is either a rectangle or a strip. Notice that the same proof holds when considering the vertex above the center of the tile depicted in Fig. 10(a). Let us now focus on one of these r-rectangles, say $\bar{R}$. We argue similarly as in Case 2 by considering the possible external boundaries of $\bar{R}$ constructed by the stable tiles as in Fig. 4(a), (c), (f), and (h). We will often refer to Fig. 11.

Case 3.1 Assume that $\bar{R}$ has at least a side, say of length ℓ, which is fully surrounded by the same spin value $m \in S$, $m \neq 1, r$, see Fig. 11(a). In this setting, starting from one corner and flipping from r to m the spins r on the vertices defining that side, the first $\ell - 1$ spin-updates preserve the energy, while after the last one it decreases by 2 and we conclude.

Case 3.2 Assume now that each side of $\bar{R}$ is surrounded by clusters of two different spins, see Fig. 11(b). Notice that with 'surrounded' we mean that the stable tiles on the corner of that side of $\bar{R}$ are of the type represented in Fig. 4(f). We further refer to Fig. 12 to aid the understanding of the proof. Flipping from r to m all the spins on those vertices which are the center of stable tiles as in Fig. 4(f) leads us to a configuration, say η, in which the $\bar{R}$ is now as in Fig. 12(b) and whose energy is the same as the initial configuration. Consider now η. Flipping from r to z all the spins on the first $\ell - 1$ vertices adjacent to the z-cluster, which are the center of stable tiles of the type depicted in Fig. 4(f), leads to a configuration $\bar{\eta}$ in which the r-cluster is now as in Fig. 12(c) and such that $H_{\text{pos}}(\eta) = H_{\text{pos}}(\bar{\eta})$. Finally, flipping from r to z the spin on the last vertex, see Fig. 12(d), decreases the energy by 1 and it is enough to conclude.

Case 3.3 Consider now the case depicted in Fig. 11(c). If there exists at least a side of $\bar{R}$ for which there exists an s-cluster, for some $s \in S$, $s \neq 1, r$, with a side, say of length k, fully adjacent to it, then we proceed as follows. When we have one of the situations (or of a their possible generalization) depicted in Fig. 13 (c_1) and (c_2), then starting from one corner of the side of the s-cluster adjacent to $\bar{R}$ and flipping from s to r all the following k spins leads to increase the energy by 1 after the first spin-update, to preserve the energy after the following $k-2$ spin-updates and to decrease it by 1 after the last spin-update. Iterating this procedure, so flipping from s to r the k or less spins s on the row immediately above, at a certain point the opposite horizontal side is reached and the second spin s which is updated to r on this last row leads to decrease the energy by at least 1. Notice that the opposite side could be adjacent to the opposite horizontal side of $\bar{R}$ in view of the periodic boundary conditions and also that the same arguments hold also for a generic geometric shape of the s-cluster, the only fundamental request is the existence of at least one of its sides fully adjacent to a side of $\bar{R}$. Otherwise, if we are in one of the cases depicted in Fig. 13 (c_3) and (c_4), where the s-cluster enlarges, we

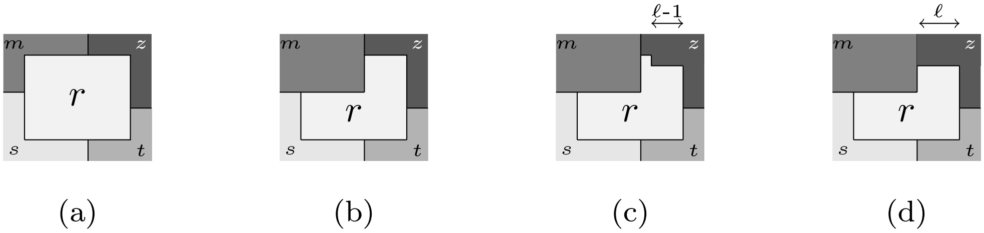


Fig. 12. Pictorial illustration of the Case 3.2 of the proof of Proposition 4.5.

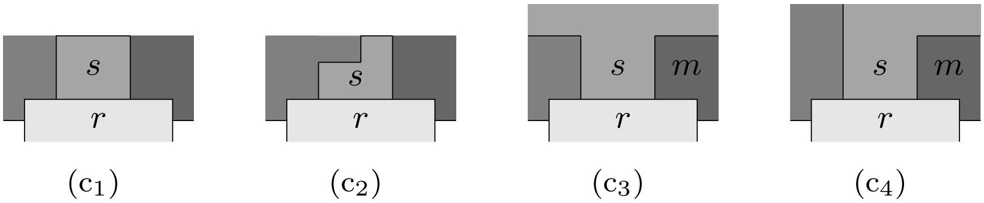


Fig. 13. Pictorial illustration of the cases considered in Case 3.3 of the proof of Proposition 4.5.

argue as follows. First, we proceed as above flipping from s to r those spins s in the portion of the s-cluster of width k (or lower). Along this path, the maximum energy increase is 1 and the energy difference between the final and the initial configuration is 0. Second, we flip from m to r those spins m on the vertical side adjacent to the new protuberance of $\bar{R}$. More precisely, first we flip from m to r the spin m on the lowest corner, second we flip the spin m on the vertex on the row immediately above and so on until we still find stable tiles as in Fig. 4(f). Finally, we find a spin m which has two nearest neighbors with spin r (the one on the left and the one below if we consider the cases depicted in Fig. 13 (c_3) and (c_4)), one nearest neighbor with spin s and one nearest neighbor with spin m. Flipping from m to r that spin leads to decrease the energy by 1 and we conclude.

Case 3.4 Let us now assume that $\bar{R}$ has at least a side completely adjacent to an s-cluster, for some $s \in S$, $s \neq 1, r$. In this case we apply a similar procedure to the one described above: proceeding from a corner to another corner and row by row and flipping from r to s all the spins inside $\bar{R}$, when the opposite side of length ℓ of $\bar{R}$ is reached the energy decreases by at least 1. In general, note that following this procedure and flipping from r to s all the spins r inside $\bar{R}$, leads to construct a path whose height is 1 and such that the energy difference between the final and the initial configuration is ℓ, i.e., the overall number of disagreeing edges which are removed once the two opposite sides of $\bar{R}$ end to coincide. □

5.3. Energy landscape and asymptotic behavior: proof of the main results

We are now able to prove Corollary 4.4 and Theorem 4.8.

Proof of Corollary 4.4. By [63, Lemma 3.6] we have that $\widetilde{\Gamma}_{\text{pos}}(\mathcal{X}\backslash\mathcal{X}^s_{\text{pos}})$ is the maximum energy that the process started in $\eta \in \mathcal{X}\backslash\mathcal{X}^s_{\text{pos}}$ has to overcome in order to arrive in $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$, i.e.,

$$\widetilde{\Gamma}_{\text{pos}}(\mathcal{X}\backslash\mathcal{X}^s_{\text{pos}}) = \max_{\eta\in\mathcal{X}\backslash\mathcal{X}^s_{\text{pos}}} \Gamma_{\text{pos}}(\eta, \mathcal{X}^s_{\text{pos}}). \tag{5.46}$$

Hence, let us proceed to estimate $\Gamma_{\text{pos}}(\eta, \mathcal{X}^s_{\text{pos}})$ for any $\eta \in \mathcal{X}\backslash\mathcal{X}^s_{\text{pos}}$. Let $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$. Note that for any $\eta \in \mathcal{X}\backslash(\mathcal{X}^s_{\text{pos}} \cup \mathcal{X}^m_{\text{pos}})$ there are not initial cycles $C^{\eta}_{\mathcal{X}^s_{\text{pos}}}(\Gamma_{\text{pos}}(\eta, \mathcal{X}^s_{\text{pos}}))$ deeper that $C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$. While for any $\mathbf{z} \in \mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$, the initial cycles $C^{\mathbf{z}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$ are as deep as $C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$. By this fact, that holds since we are in the metastability scenario as in [63, Subsection 3.5, Example 1], we get that for any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$

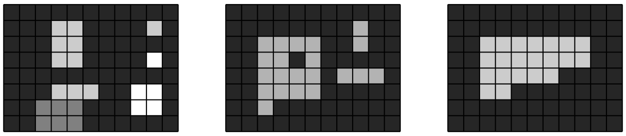
$$\Gamma_{\text{pos}}(\eta, \mathcal{X}^s_{\text{pos}}) = \Phi_{\text{pos}}(\eta, \mathcal{X}^s_{\text{pos}}) - H_{\text{pos}}(\eta) \le \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) - H_{\text{pos}}(\mathbf{m}) = \Gamma_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}), \tag{5.47}$$

where the last equality follows by Theorem 4.3. Thus, we conclude that (4.3) is verified since, using (4.1), we have $\Gamma_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) = \Gamma^m_{\text{pos}}$. □

Proof of Theorem 4.8. The theorem follows by [50, Theorem 2.3]. In order to apply this result it is enough to show that the pair $(\mathbf{m}, G)$ verifies the assumption

$$\sup_{\eta\in\mathcal{X}} \mathbb{P}\left(\tau^{\eta}_{\{\mathbf{m},G\}} > R\right) \le \delta, \tag{5.48}$$

with $R < \mathbb{E}(\tau^{\mathbf{m}}_G)$ and δ sufficiently small. To prove (5.48), let us distinguish two cases.



(a) (b) (c)

Fig. 14. Illustration of three examples of $\sigma \in \mathscr{D}^m_{\text{pos}}$ when $\ell^* = 5$. We color black the vertices with spin m. In (a) the $\ell^*(\ell^*-1)+1 = 21$ spins different from m have not all the same spin value and they belong to more clusters. In (b) these spins different from m have the same spin value and they belong to three different clusters. In (c) the spins different from m have the same spin value and they belong to a single cluster.

Case 1. Let $\eta \in C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$. By Equation (2.20) of [59, Theorem 2.17] applied to the cycle $C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$ we get that almost surely the process visits $\mathbf{m}$ before exiting from the cycle $C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$. More precisely, we have that there exists $k_1 > 0$ such that for any β sufficiently large

$$\mathbb{P}(\tau^\eta_{\mathbf{m}} > \tau^\eta_{\partial C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})}) \le e^{-k_1\beta}. \tag{5.49}$$

Since $\tau^\eta_G > \tau^\eta_{\partial C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})}$, it follows that the process almost surely visits $\mathbf{m}$ before hitting G. Furthermore, since $\{\mathbf{1},\dots,\mathbf{q}\}\backslash\{\mathbf{m}\} \subset G$, we obtain that almost surely the process starting from η visits $\mathbf{m}$ before hitting $\{\mathbf{1},\dots,\mathbf{q}\}\backslash\{\mathbf{m}\}$, i.e., $\tau^\eta_{\mathbf{m}} = \tau^\eta_{\{\mathbf{1},\dots,\mathbf{q}\}}$. Using the recurrence property given in Theorem 4.6, we conclude that (5.48) is satisfied by choosing $R = e^{\beta(2+\epsilon)}$ with $2+\epsilon < \Gamma^m_{\text{pos}}$ and $\delta = e^{-e^{k_2\beta}}$ with $k_2 > 0$.

Case 2. Let $\eta \in \mathcal{X}\backslash C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$. In this case (5.48) is trivially verified for any R and δ sufficiently small since η belongs to the target. □

6. Minimal gates and tube of typical trajectories

In this section we investigate on the minimal gates and the tube of typical paths for the transition from any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$ to $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$. We further identify the union of all minimal gates also for the transition from a metastable state to the other metastable states.

6.1. Identification of critical configurations for the transition from a metastable to the stable state

The goal of this subsection is to investigate the set of critical configurations for the transition from any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$ to $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$. The idea of the proof of the following lemmas and proposition generalizes the proof of similar results given in [33, Section 6] for the Blume Capel model.

First we need to give some further definitions. For any $m \in S\backslash\{1\}$ we define $\mathscr{D}^m_{\text{pos}} \subset \mathcal{X}$ as the set of those configurations with $|\Lambda| - (\ell^*(\ell^*-1)+1)$ spins equal to m

$$\mathscr{D}^m_{\text{pos}} := \{\sigma \in \mathcal{X} : N_m(\sigma) = |\Lambda| - (\ell^*(\ell^*-1)+1)\}. \tag{6.1}$$

Furthermore, we define

$$\mathscr{D}^{m,+}_{\text{pos}} := \{\sigma \in \mathcal{X} : N_m(\sigma) > |\Lambda| - (\ell^*(\ell^*-1)+1)\}, \tag{6.2}$$

note that $\mathbf{m} \in \mathscr{D}^{m,+}_{\text{pos}}$, and

$$\mathscr{D}^{m,-}_{\text{pos}} := \{\sigma \in \mathcal{X} : N_m(\sigma) < |\Lambda| - (\ell^*(\ell^*-1)+1)\}. \tag{6.3}$$

For any $\sigma \in \mathscr{D}^m_{\text{pos}}$, we remark that σ has $\ell^*(\ell^*-1)+1$ spins different from m and they may have not the same spin value and may belong to one or more clusters, see Fig. 14. A *two dimensional polyomino* on $\mathbb{Z}^2$ is a finite union of unit squares. The area of a polyomino is the number of its unit squares, while its perimeter is the cardinality of its boundary, namely, the number of unit edges of the dual lattice which intersect only one of the unit squares of the polyomino itself. Thus, the perimeter is the number of interfaces on $\mathbb{Z}^2$ between the sites inside the polyomino and those outside. The polyominoes with minimal perimeter among those with the same area are said to be *minimal polyominoes*.

Lemma 6.1. *For any $m \in \{2,\dots,q\}$ the minimum of the energy in $\mathscr{D}^m_{pos}$ is achieved by those configurations in which the $\ell^*(\ell^*-1)+1$ spins different from m are 1 and they belong to a unique cluster of perimeter $4\ell^*$. More precisely,*

$$\mathscr{F}(\mathscr{D}^m_{pos}) = \{\sigma \in \mathscr{D}^m_{pos} : \sigma \text{ has all spins } m \text{ except those in a unique cluster } C^1(\sigma) \text{ of spins } 1 \text{ of perimeter } 4\ell^*\}. \tag{6.4}$$

Moreover,

$$H_{pos}(\mathscr{F}(\mathscr{D}^m_{pos})) = H_{pos}(\mathbf{m}) + \Gamma_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}) = \Phi_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}). \tag{6.5}$$

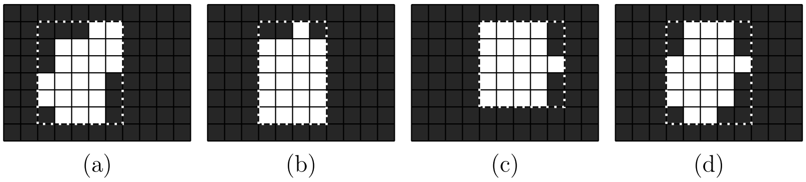


Fig. 15. Examples of $\sigma \in \tilde{\mathscr{D}}^m_{\rm pos}$ (a) and of $\sigma \in \hat{\mathscr{D}}^m_{\rm pos}$ (b) and (c) when $\ell^* = 5$. We associate the color black to the spin m, the color white to the spin 1. The dotted rectangle represents $R(C^1(\sigma))$. Figure (d) is an example of configuration that does not belong to $\hat{\mathscr{D}}^m_{\rm pos}$.

Proof. Let $m \in \{2, \dots, q\}$. From the definition of the Hamiltonian $H_{\rm pos}$, (2.2), we get that the presence of disagreeing edges increases the energy, thus in order to identify the bottom of $\mathscr{D}^m_{\rm pos}$ we have to consider those configurations $\sigma \in \mathscr{D}^m_{\rm pos}$ in which the $\ell^*(\ell^*-1)+1$ spins different from m belong to a single cluster. Moreover, given the number of the disagreeing edges, the presence of each spin 1 decreases the energy by h compared of the presence with other spins. Hence, the single cluster is full of spins 1, say $C^1(\sigma)$, and it is inside a homogeneous sea of spins m. Arguing like in the second part of the proof of Proposition 5.8, we have that $4\ell^*$ is the minimal perimeter of a polyomino of area $\ell^*(\ell^*-1)+1$. Thus, for any $\sigma \in \mathscr{F}(\mathscr{D}^m_{\rm pos})$, $C^1(\sigma)$ must have perimeter $4\ell^*$. Hence, all the characteristics given in (6.4) are verified. Let us now prove (6.5). By (4.11) we get that $\mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}) \subset \mathscr{F}(\mathscr{D}^m_{\rm pos})$, that is $H_{\rm pos}(\mathscr{F}(\mathscr{D}^m_{\rm pos})) = H_{\rm pos}(\mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}))$. Thus, (6.5) is satisfied since for any $\eta \in \mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos})$,

$$H_{\rm pos}(\eta) - H_{\rm pos}(\mathbf{m}) = 4\ell^* - h(\ell^*(\ell^*-1)+1) = \Gamma_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}). \quad \square \tag{6.6}$$

In the next corollary we show that every optimal path from $\mathbf{m} \in \mathscr{X}^m_{\rm pos}$ to $\mathscr{X}^s_{\rm pos} = \{\mathbf{1}\}$ visits at least once $\mathscr{F}(\mathscr{D}^m_{\rm pos})$, i.e., we prove that $\mathscr{F}(\mathscr{D}^m_{\rm pos})$ is a gate for the transition from $\mathbf{m}$ to $\mathscr{X}^s_{\rm pos}$.

Corollary 6.2. *Let $\mathbf{m} \in \mathscr{X}^m_{pos}$. For any $\omega \in \Omega^{opt}_{\mathbf{m},\mathscr{X}^s_{pos}}$ we have $\omega \cap \mathscr{F}(\mathscr{D}^m_{pos}) \neq \emptyset$. Hence, $\mathscr{F}(\mathscr{D}^m_{pos})$ is a gate for the transition $\mathbf{m} \to \mathscr{X}^s_{pos}$.*

Proof. Every path from $\mathbf{m} \in \mathscr{X}^m_{\rm pos}$ to the stable configuration $\mathbf{1}$ has to pass through the set $\mathscr{V}^m_k := \{\sigma \in \mathscr{X} : N_m(\sigma) = k\}$ for any $k = |V|, \dots, 0$. In particular, given $k^* := \ell^*(\ell^*-1)+1$, any $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{m},\mathscr{X}^s_{\rm pos}}$ visits at least once the set $\mathscr{V}^m_{|\Lambda|-k^*} \equiv \mathscr{D}^m_{\rm pos}$. Hence, there exists $i \in \{0, \dots n\}$ such that $\omega_i \in \mathscr{D}^m_{\rm pos}$. Since from (6.5) we have that the energy value of any configuration belonging to $\mathscr{F}(\mathscr{D}^m_{\rm pos})$ is equal to the min–max reached by any optimal path from $\mathbf{m}$ to $\mathscr{X}^s_{\rm pos}$, we conclude that $\omega_i \in \mathscr{F}(\mathscr{D}^m_{\rm pos})$. $\square$

In the last result of this subsection, we prove that, for any $\mathbf{m} \in \mathscr{X}^m_{\rm pos}$, every optimal path $\omega \in \Omega^{opt}_{\mathbf{m},\mathscr{X}^s_{\rm pos}}$ is such that $\omega \cap \mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}) \neq \emptyset$. Hence, we show that $\mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos})$ is a gate for the transition $\mathbf{m} \to \mathscr{X}^s_{\rm pos}$.

Proposition 6.3. *Let $\mathbf{m} \in \mathscr{X}^m_{pos}$. Then, any $\omega \in \Omega^{opt}_{\mathbf{m},\mathscr{X}^s_{pos}}$ visits $\mathscr{W}_{pos}(\mathbf{m}, \mathscr{X}^s_{pos})$. Hence, $\mathscr{W}_{pos}(\mathbf{m}, \mathscr{X}^s_{pos})$ is a gate for the transition $\mathbf{m} \to \mathscr{X}^s_{pos}$.*

Proof. For any $m \in S$, $m \neq 1$, let $\tilde{\mathscr{D}}^m_{\rm pos}$ and $\hat{\mathscr{D}}^m_{\rm pos}$ be the subsets of $\mathscr{F}(\mathscr{D}^m_{\rm pos})$ defined as follows. $\tilde{\mathscr{D}}^m_{\rm pos}$ is the set of those configurations of $\mathscr{F}(\mathscr{D}^m_{\rm pos})$ in which the boundary of the polyomino $C^1(\sigma)$ intersects each side of the boundary of its smallest surrounding rectangle $R(C^1(\sigma))$ on a set of the dual lattice $\mathbb{Z}^2 + (1/2, 1/2)$ made by at least two consecutive unit segments, see Fig. 15(a). On the other hand, $\hat{\mathscr{D}}^m_{\rm pos}$ is the set of those configurations of $\mathscr{F}(\mathscr{D}^m_{\rm pos})$ in which the boundary of the polyomino $C^1(\sigma)$ intersects at least one side of the boundary of $R(C^1(\sigma))$ in a single unit segment, see Fig. 15(b) and (c).

In particular note that $\mathscr{F}(\mathscr{D}^m_{\rm pos}) = \tilde{\mathscr{D}}^m_{\rm pos} \cup \hat{\mathscr{D}}^m_{\rm pos}$. The proof proceeds in five steps.

Step 1. Our first aim is to prove that

$$\hat{\mathscr{D}}^m_{\rm pos} = \mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}) \cup \mathscr{W}'_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}). \tag{6.7}$$

From (4.11) we have $\mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}) \cup \mathscr{W}'_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}) \subseteq \hat{\mathscr{D}}^m_{\rm pos}$. Thus we reduce our proof to show that $\sigma \in \hat{\mathscr{D}}^m_{\rm pos}$ implies $\sigma \in \mathscr{W}_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos}) \cup \mathscr{W}'_{\rm pos}(\mathbf{m}, \mathscr{X}^s_{\rm pos})$. Note that this implication is not straightforward, since given $\sigma \in \hat{\mathscr{D}}^m_{\rm pos}$, the boundary of the polyomino $C^1(\sigma)$ could intersect the other three sides of the boundary of its smallest surrounding rectangle $R(C^1(\sigma))$ in a proper subsets of the sides itself, see Fig. 15(d) for an illustration of this hypothetical case. Hence, consider $\sigma \in \hat{\mathscr{D}}^m_{\rm pos}$ and let $R(C^1(\sigma)) = R_{(\ell^*+a)\times(\ell^*+b)}$ with $a, b \in \mathbb{Z}$. In view of the proof of Lemma 6.1 we have that $C^1(\sigma)$ is a minimal polyomino and by [33, Lemma 6.16] it is also convex and monotone, i.e., its perimeter of value $4\ell^*$ is equal to the one of $R(C^1(\sigma))$. Hence, the following equality holds

$$4\ell^* = 4\ell^* + 2(a+b). \tag{6.8}$$

In particular, (6.8) is satisfied only by $a = -b$. Now, let $\tilde{R}$ be the smallest rectangle surrounding the polyomino, say $\tilde{C}^1(\sigma)$, obtained by removing the unit protuberance from $C^1(\sigma)$. If $C^1(\sigma)$ has the unit protuberance adjacent to a side of length $\ell^* + a$, then $\tilde{R}$ is a rectangle $(\ell^*+a)\times(\ell^*-a-1)$. Note that $\tilde{R}$ must have an area larger than or equal to the number of spins 1 of the polyomino $\tilde{C}^1(\sigma)$, that is $\ell^*(\ell^*-1)$. Thus, we have

$$\text{Area}(\tilde{R}) = (\ell^*+a)(\ell^*-a-1) = \ell^*(\ell^*-1) - a^2 - a \geq \ell^*(\ell^*-1) \iff -a^2 - a \geq 0. \tag{6.9}$$

Since $a \in \mathbb{Z}$, $-a^2 - a \geq 0$ is satisfied only if either $a = 0$ or $a = -1$. Otherwise, if $C^1(\sigma)$ has the unit protuberance adjacent to a side of length $\ell^* - a$, then $\tilde{R}$ is a rectangle $(\ell^* + a - 1) \times (\ell^* - a)$ and

$$\mathrm{Area}(\tilde{R}) = (\ell^* + a - 1)(\ell^* - a) = \ell^*(\ell^* - 1) - a^2 + a \geq \ell^*(\ell^* - 1) \iff -a^2 + a \geq 0. \tag{6.10}$$

Since $a \in \mathbb{Z}$, $-a^2 + a \geq 0$ is satisfied only if either $a = 0$ or $a = 1$. In both cases we get that $\tilde{R}$ is a rectangle of side lengths ℓ^* and $\ell^* - 1$. Thus, if the protuberance is attached to one of the longest sides of $\tilde{R}$, then $\sigma \in \mathcal{W}_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}})$, otherwise $\sigma \in \mathcal{W}'_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}})$. In any case we conclude that (6.7) is satisfied.

Step 2. For any $\mathbf{m} \in \mathcal{X}^m_{\mathrm{pos}}$ and for any path $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\mathrm{pos}}}$, let

$$g_m(\omega) := \{i \in \mathbb{N} : \omega_i \in \mathcal{F}(\mathcal{D}^m_{\mathrm{pos}}),\ N_1(\omega_{i-1}) = \ell^*(\ell^* - 1),\ N_m(\omega_{i-1}) = |\Lambda| - \ell^*(\ell^* - 1)\}. \tag{6.11}$$

We claim that $g_m(\omega) \neq \emptyset$. Let $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\mathrm{pos}}}$ and let $j^* \leq n$ be the smallest integer such that after j^* the path leaves $\mathcal{D}^{m,+}_{\mathrm{pos}}$, i.e., $(\omega_{j^*}, \dots, \omega_n) \cap \mathcal{D}^{m,+}_{\mathrm{pos}} = \emptyset$. Since ω_{j^*-1} is the last configuration in $\mathcal{D}^{m,+}_{\mathrm{pos}}$, it follows that $\omega_{j^*} \in \mathcal{D}^m_{\mathrm{pos}}$ and, by the proof of Corollary 6.2, we have that $\omega_{j^*} \in \mathcal{F}(\mathcal{D}^m_{\mathrm{pos}})$. Moreover, since ω_{j^*-1} is the last configuration in $\mathcal{D}^{m,+}_{\mathrm{pos}}$, we have that $N_m(\omega_{j^*-1}) = |\Lambda| - \ell^*(\ell^* - 1)$ and ω_{j^*} is obtained by ω_{j^*-1} by flipping a spin m from m to $s \neq m$. Note that $N_m(\omega_{j^*-1}) = |\Lambda| - \ell^*(\ell^* - 1)$ implies $N_s(\omega_{j^*-1}) \leq \ell^*(\ell^* - 1)$ for any $s \in S \backslash \{m\}$. By Lemma 6.1, $\omega_{j^*} \in \mathcal{F}(\mathcal{D}^m_{\mathrm{pos}})$ implies $N_1(\omega_{j^*}) = \ell^*(\ell^* - 1) + 1$, thus $N_1(\omega_{j^*-1}) < \ell^*(\ell^* - 1)$ is not feasible since ω_{j^*} and ω_{j^*-1} differ by a single spin update which increases the number of spins 1 of at most one. Then, $j^* \in g_m(\omega)$ and the claim is proved.

Step 3. We claim that for any path $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\mathrm{pos}}}$ one has $\omega_i \in \hat{\mathcal{D}}^m_{\mathrm{pos}}$ for any $i \in g_m(\omega)$. We argue by contradiction. Assume that there exists $i \in g_m(\omega)$ such that $\omega_i \notin \hat{\mathcal{D}}^m_{\mathrm{pos}}$ and $\omega_i \in \tilde{\mathcal{D}}^m_{\mathrm{pos}}$. Since ω_{i-1} is obtained from ω_i by flipping a spin 1 to m and since any configuration belonging to $\tilde{\mathcal{D}}^m_{\mathrm{pos}}$ has all the spins 1 with at least two nearest neighbors with spin 1, using (2.8) we have

$$H_{\mathrm{pos}}(\omega_{i-1}) - H_{\mathrm{pos}}(\omega_i) \geq (2 - 2) + h = h > 0. \tag{6.12}$$

In particular, from (6.12) we get a contradiction. Indeed,

$$\Phi^{\mathrm{pos}}_\omega \geq H_{\mathrm{pos}}(\omega_{i-1}) > H_{\mathrm{pos}}(\omega_i) = H_{\mathrm{pos}}(\mathbf{m}) + \Gamma_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}) = \Phi_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}), \tag{6.13}$$

where the first equality follows by (6.5). Thus by (6.13) ω is not an optimal path, which is a contradiction, the claim is proved and we conclude the proof of Step 3.

Step 4. Now we claim that for any $\mathbf{m} \in \mathcal{X}^m_{\mathrm{pos}}$ and for any path $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\mathrm{pos}}}$,

$$\omega_i \in \mathcal{F}(\mathcal{D}^m_{\mathrm{pos}}) \implies \omega_{i-1}, \omega_{i+1} \notin \mathcal{D}^m_{\mathrm{pos}}. \tag{6.14}$$

Using Corollary 6.2, for any $\mathbf{m} \in \mathcal{X}^m_{\mathrm{pos}}$ and any path $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\mathrm{pos}}}$ there exists an integer i such that $\omega_i \in \mathcal{F}(\mathcal{D}^m_{\mathrm{pos}})$. Assume by contradiction that $\omega_{i+1} \in \mathcal{D}^m_{\mathrm{pos}}$. In particular, since ω_i and ω_{i+1} have the same number of spins m, note that ω_{i+1} is obtained by flipping a spin 1 from 1 to $t \neq 1$. Since $\omega_i(v) \neq t$ for every $v \in V$, the above flip increases the energy, i.e., $H_{\mathrm{pos}}(\omega_{i+1}) > H_{\mathrm{pos}}(\omega_i)$. Hence, using this inequality and (6.5), we have

$$\Phi^{\mathrm{pos}}_\omega \geq H_{\mathrm{pos}}(\omega_{i+1}) > H_{\mathrm{pos}}(\omega_i) = H_{\mathrm{pos}}(\mathbf{m}) + \Gamma_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}) = \Phi_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}), \tag{6.15}$$

which implies the contradiction because ω is not optimal. Thus $\omega_{i+1} \notin \mathcal{D}^m_{\mathrm{pos}}$ and similarly we show that also $\omega_{i-1} \notin \mathcal{D}^m_{\mathrm{pos}}$.

Step 5. In this last step of the proof we claim that for any $\mathbf{m} \in \mathcal{X}^m_{\mathrm{pos}}$ and for any path $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\mathrm{pos}}}$ there exists a positive integer i such that $\omega_i \in \mathcal{W}_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}})$. Arguing by contradiction, assume that there exists $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\mathrm{pos}}}$ such that $\omega \cap \mathcal{W}_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}) = \emptyset$. Thanks to Corollary 6.2, we know that ω visits $\mathcal{F}(\mathcal{D}^m_{\mathrm{pos}})$ and thanks to Step 4 we have that the configurations along ω belonging to $\mathcal{F}(\mathcal{D}^m_{\mathrm{pos}})$ are not consecutive. More precisely, they are linked by a sub-path that belongs either to $\mathcal{D}^{m,+}_{\mathrm{pos}}$ or $\mathcal{D}^{m,-}_{\mathrm{pos}}$. If n is the length of ω, then let $j \leq n$ be the smallest integer such that $\omega_j \in \mathcal{F}(\mathcal{D}^m_{\mathrm{pos}})$ and such that $(\omega_j, \dots, \omega_n) \cap \mathcal{D}^{m,+}_{\mathrm{pos}} = \emptyset$, thus, $j \in g_m(\omega)$ since j plays the same role of j^* in the proof of Step 2. Using (6.7), Step 3 and the assumption $\omega \cap \mathcal{W}_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}) = \emptyset$, it follows that $\omega_j \in \mathcal{W}'_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}})$. Moreover, starting from $\omega_j \in \mathcal{F}(\mathcal{D}^m_{\mathrm{pos}})$ the energy along the path decreases only by either

(i) flipping the spin in the unit protuberance from 1 to m, or
(ii) flipping a spin, with two nearest neighbors with spin 1, from m to 1.

Since by the definition of j we have that ω_{j-1} is the last that visits $\mathcal{D}^{m,+}_{\mathrm{pos}}$, $\omega_{j+1} \notin \mathcal{D}^{m,+}_{\mathrm{pos}}$, (i) is not feasible. Considering (ii), we have $H_{\mathrm{pos}}(\omega_{j+1}) = H_{\mathrm{pos}}(\mathbf{m}) + \Gamma_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}) - h$. Starting from ω_{j+1} we consider only moves which imply either a decrease of energy or an increase by at most h. Since $C^1(\omega_{j+1})$ is a polyomino $\ell^* \times (\ell^* - 1)$ with a bar made of two adjacent unit squares on a shortest side, the only feasible moves are

(iii) flipping a spin, with two nearest neighbors with spin m, from m to 1,
(iv) flipping a spin, with two nearest neighbors with spin 1, from 1 to m.

By means of the moves (iii) and (iv), the process reaches a configuration σ in which all the spins are equal to m except those, that are 1, in a connected polyomino $C^1(\sigma)$ that is convex and such that $R(C^1(\sigma)) = R_{(\ell^*+1)\times(\ell^*-1)}$. We cannot repeat the move

(iv) otherwise we get a configuration that does not belong to $\mathscr{D}^m_{\rm pos}$. While applying one time (iv) and iteratively (iii), until we fill the rectangle $R_{(\ell^*+1)\times(\ell^*-1)}$ with spins 1, we get a set of configurations in which the one with the smallest energy is σ such that $C^1(\sigma) \equiv R(C^1(\sigma))$. Moreover, from any configuration in this set, a possible move is reached by flipping from m to 1 a spin m with three nearest neighbors with spin m that implies to enlarge the circumscribed rectangle. This spin-flip increases the energy by $2-h$. Thus, we obtain

$$\begin{aligned}\Phi^{\rm pos}_\omega &\geq 4\ell^* - h(\ell^*+1)(\ell^*-1) + 2 - h + H_{\rm pos}(\mathbf{m}) \\ &= 4\ell^* - h(\ell^*)^2 + 2 + H_{\rm pos}(\mathbf{m}) \\ &> \Gamma_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}) + H_{\rm pos}(\mathbf{m}) = \Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}),\end{aligned} \tag{6.16}$$

which is a contradiction by the definition of an optimal path. Note that the last inequality follows by $2 > h(\ell^*-1)$ since $0 < h < \frac{1}{2}$, see Assumption 4.1. It follows that it is not possible to have $\omega \cap \mathcal{W}_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}) = \emptyset$ for any $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\rm pos}}$, namely $\mathcal{W}_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos})$ is a gate for this type of transition. $\square$

6.2. Minimal gates for the transition from a metastable state to the other metastable states

This subsection is devoted to the study of the transition from a metastable state to the set of the other metastable configurations. In Propositions 6.7 and 6.10 we identify geometrically two gates for this type of transition and in Theorem 4.11 we show that the union of these sets gives the union of all the minimal gates for the same transition. Furthermore, in this subsection we also give some more details for the transition from any metastable state to the stable configuration **1**. More precisely, in Proposition 6.9 we prove that for any $\mathbf{m} \in \mathcal{X}^m_{\rm pos}$ almost surely any optimal path $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^s_{\rm pos}}$ does not visit any metastable state different from the initial one during the transition. Let us begin by proving the following useful lemma.

Lemma 6.4. *For any $m \in \{2,\dots,q\}$, let $\eta \in \bar{B}^1_{\ell^*-1,\ell^*}(m,1)$ and let $\bar\eta \in \mathcal{X}$ a configuration which communicates with η by one step of the dynamics. Then, either $H_{pos}(\eta) < H_{pos}(\bar\eta)$ or $H_{pos}(\eta) > H_{pos}(\bar\eta)$.*

Proof. Since η and $\bar\eta$ differ by a single-spin update, let us define $\bar\eta := \eta^{v,t}$ for some $v \in V$ and $t \in S$, $t \neq \eta(v)$. Note that $\eta \in \bar{B}^1_{\ell^*-1,\ell^*}(m,1)$ implies that η is characterized by all spins m except those that are 1 in a quasi-square $(\ell^*-1)\times\ell^*$ with a unit protuberance on one of the longest sides. In particular, for any $v \in V$, either $\eta(v) = m$ or $\eta(v) = 1$. If $\eta(v) = m$, then for any $t \in S\backslash\{m\}$, depending on the distance between the vertex v and the 1-cluster, we have

$$H_{\rm pos}(\bar\eta) - H_{\rm pos}(\eta) = \begin{cases} 4 - h\mathbb{1}_{\{t=1\}}, & \text{if } n_m(v) = 4\ ; \\ 3 - \mathbb{1}_{\{t=1\}} - h\mathbb{1}_{\{t=1\}}, & \text{if } n_m(v) = 3,\ n_1(v) = 1; \\ 2 - 2\mathbb{1}_{\{t=1\}} - h\mathbb{1}_{\{t=1\}}, & \text{if } n_m(v) = 2,\ n_1(v) = 2. \end{cases} \tag{6.17}$$

Otherwise, if $\eta(v) = 1$, for any $t \in S\backslash\{1\}$, depending on the distance between the vertex v and the boundary of the 1-cluster, we get

$$H_{\rm pos}(\bar\eta) - H_{\rm pos}(\eta) = \begin{cases} 4 + h, & \text{if } n_1(v) = 4; \\ 3 - \mathbb{1}_{\{t=m\}} + h, & \text{if } n_m(v) = 1,\ n_1(v) = 3; \\ 2 - 2\mathbb{1}_{\{t=m\}} + h, & \text{if } n_m(v) = 2,\ n_1(v) = 2; \\ 1 - 3\mathbb{1}_{\{t=m\}} + h, & \text{if } n_m(v) = 3,\ n_1(v) = 1. \end{cases} \tag{6.18}$$

We conclude that $H_{\rm pos}(\eta) \neq H_{\rm pos}(\bar\eta)$. $\square$

In the next proposition we prove that the communication energy between metastable states is equal to the one between a metastable state and the stable state $\mathcal{X}^s_{\rm pos} = \{\mathbf{1}\}$.

Proposition 6.5. *For any $\mathbf{m} \in \mathcal{X}^m_{pos}$,*

$$\Phi_{pos}(\mathbf{m}, \mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}) = 4\ell^* - h(\ell^*(\ell^*-1)+1) + H_{pos}(\mathbf{m}) = \Phi_{pos}(\mathbf{m}, \mathcal{X}^s_{pos}). \tag{6.19}$$

Proof. Let us divide the proof in two steps. First we compute an upper bound of $\Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^m_{\rm pos}\backslash\{\mathbf{m}\})$, second a lower bound.

Upper bound. For any $\mathbf{z} \in \mathcal{X}^m_{\rm pos}\backslash\{\mathbf{m}\}$, we use Definition 5.4 to construct two reference paths $\tilde\omega^{(1)} : \mathbf{m} \to \mathbf{1}$ and $\tilde\omega^{(2)} : \mathbf{z} \to \mathbf{1}$. Thus, we define the reference path $\omega^* : \mathbf{m} \to \mathbf{z}$ as the concatenation of the reference path $\tilde\omega^{(1)}$ and the time reversal $(\tilde\omega^{(2)})^T : \mathbf{1} \to \mathbf{z}$. Thus, $\omega^* = (\tilde\omega^{(1)}, (\tilde\omega^{(2)})^T)$. By this definition of ω^*, we have $\max_{\xi\in\omega^*} H_{\rm pos}(\xi) = \max\{\max_{\xi\in\tilde\omega^{(1)}} H_{\rm pos}(\xi), \max_{\xi\in(\tilde\omega^{(2)})^T} H_{\rm pos}(\xi)\}$. In the proof of Lemma 5.6, using Eqs. (A.1)–(A.3) and (A.7), we get that

$$\max_{\xi\in\omega^*} H_{\rm pos}(\xi) = 4\ell^* - h(\ell^*(\ell^*-1)+1) + H_{\rm pos}(\mathbf{m}). \tag{6.20}$$

Thus, applying the definition of the communication energy, we conclude that

$$\Phi_{\rm pos}(\mathbf{m}, \mathbf{z}) = \min_{\omega:\mathbf{m}\to\mathbf{z}} \max_{\xi\in\omega} H_{\rm pos}(\xi) \leq 4\ell^* - h(\ell^*(\ell^*-1)+1) + H_{\rm pos}(\mathbf{m}). \tag{6.21}$$

Lower bound. During the transition from $\mathbf{m}$ to any $\mathbf{z} \in \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}$ the process has to intersect at least once the set $\mathcal{V}^m_k := \{\sigma \in \mathcal{X} : N_m(\sigma) = k\}$ for any $k = KL, \dots, 0$. In particular, given $k^* := \ell^*(\ell^*-1)+1$, the process has to visit at least once the set $\mathcal{V}^m_{|\Lambda|-k^*}$. Since $\mathcal{V}^m_{|\Lambda|-k^*} \equiv \mathcal{D}^m_{\rm pos}$, from Lemma 6.1, we have

$$H_{\rm pos}(\mathcal{F}(\mathcal{V}^m_{|\Lambda|-k^*})) = 4\ell^* - h(\ell^*(\ell^*-1)+1) + H_{\rm pos}(\mathbf{m}). \tag{6.22}$$

It follows that the process visits at least once the set $\mathcal{V}^m_{|\Lambda|-k^*}$ in a configuration with energy larger than or equal to the r.h.s. of (6.22), i.e., we obtain the following lower bound for the communication height between metastable states

$$\Phi_{\rm pos}(\mathbf{m}, \mathbf{z}) \geq \ 4\ell^* - h(\ell^*(\ell^*-1)+1) + H_{\rm pos}(\mathbf{m}). \tag{6.23}$$

Thanks to (6.21) and (6.23), we conclude that (6.19) is satisfied. □

Exploiting the equality $\Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}) = \Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\})$ for any $\mathbf{m} \in \mathcal{X}^m_{\rm pos}$, we are now able to state the following corollary and proposition.

Corollary 6.6. *Let $\mathbf{m} \in \mathcal{X}^m_{pos}$ and let $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{pos} \backslash \{\mathbf{m}\}}$. Then, $\omega \cap \mathcal{F}(\mathcal{D}^m_{pos}) \neq \emptyset$. Hence, $\mathcal{F}(\mathcal{D}^m_{pos})$ is a gate for the transition $\mathbf{m} \to \mathcal{X}^m_{pos} \backslash \{\mathbf{m}\}$.*

Proof. Thanks to (6.19) the proof is analogous to the one of Corollary 6.2. We refer to Appendix A.2.1 for the explicit proof. □

Proposition 6.7. *For any $\mathbf{m} \in \mathcal{X}^m_{pos}$, $\mathcal{W}_{pos}(\mathbf{m}, \mathcal{X}^s_{pos})$ is a gate for the transition $\mathbf{m} \to \mathcal{X}^m_{pos} \backslash \{\mathbf{m}\}$.*

Proof. Thanks to (6.19) the proof is analogous to the one of Proposition 6.3. See Appendix A.2.2 for the detailed proof. □

Given $\mathbf{m} \in \mathcal{X}^m_{\rm pos}$, the reader may be surprised that $\mathcal{W}_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos})$ is a minimal gate for both the transitions $\mathbf{m} \to \mathcal{X}^s_{\rm pos}$ and $\mathbf{m} \to \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}$. Intuitively, the set $\Omega_{\mathbf{m}, \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}}$ is partitioned in two non-empty subsets, i.e., the set containing those paths $\omega \in \Omega_{\mathbf{m}, \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}}$ such that $\omega \cap C^{\mathbf{1}}_{\mathcal{X}^m_{\rm pos}}(\Gamma^m_{\rm pos}(\mathbf{1}, \mathcal{X}^m_{\rm pos})) \neq \emptyset$ and the set containing those paths that do not enter this cycle. Corollary 4.12 points out that the $\Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}}$ is a subset of the first set, i.e.,

$$\Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}} \subseteq \{\omega \in \Omega_{\mathbf{m}, \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}} : \omega \cap C^{\mathbf{1}}_{\mathcal{X}^m_{\rm pos}}(\Gamma^m_{\rm pos}(\mathbf{1}, \mathcal{X}^m_{\rm pos})) \neq \emptyset\}. \tag{6.24}$$

More precisely, in Proposition 6.9 we show that almost surely the process started in $\mathbf{m} \in \mathcal{X}^m_{\rm pos}$ does not visit any other metastable states before hitting the stable configuration $\mathcal{X}^s_{\rm pos} = \{\mathbf{1}\}$. In order to prove this result, first we need to introduce the following habitat and to show that almost surely during the transition from a metastable to the stable state the process does not exit from it. For any $\mathbf{m} \in \mathcal{X}^m_{\rm pos}$, let

$$\mathcal{A}_{\rm pos} := \{\sigma \in \mathcal{X} : \ H_{\rm pos}(\sigma) < H_{\rm pos}(\mathbf{m}) + \Gamma_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}) + \hat{\delta}/2\}, \tag{6.25}$$

where $\hat{\delta}$ is the minimum energy gap between an optimal and a non-optimal path from $\mathbf{m}$ to $\mathcal{X}^s_{\rm pos}$. Note that $\mathcal{A}_{\rm pos}$ is a cycle and that the choice to give some results on the dynamics from a metastable to the stable state inside $\mathcal{A}_{\rm pos}$ is justified by the following result.

Proposition 6.8. *Let $\mathcal{A}_{pos}$ be the habitat defined in (6.25). Then, $\mathcal{F}(\mathcal{A}_{pos}) = \mathcal{X}^s_{pos}$ and $V(\mathcal{A}_{pos}) = \Gamma^m_{pos}$. Moreover, for any $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^s_{pos}}$, during the transition from any $\mathbf{m} \in \mathcal{X}^m_{pos}$ to $\mathcal{X}^s_{pos}$ the process does not exit almost surely from $\mathcal{A}_{pos}$, i.e.,*

$$\lim_{\beta\to\infty} \mathbb{P}(\tau^{\mathbf{m}}_{\mathcal{X}^s_{pos}} < \tau^{\mathbf{m}}_{\partial\mathcal{A}_{pos}}) = 1. \tag{6.26}$$

Proof. By Proposition 4.2 we have $\mathcal{X}^s_{\rm pos} = \{\mathbf{1}\}$ and by the definition (6.25) we have $\mathbf{1} \in \mathcal{A}_{\rm pos}$. Hence, $\mathcal{F}(\mathcal{A}_{\rm pos}) = \mathcal{X}^s_{\rm pos} = \{\mathbf{1}\}$. Furthermore, by (6.25) we also get that $\mathcal{X}^m_{\rm pos} \subset \mathcal{A}_{\rm pos}$. Hence, using Theorem 4.3 we have that $V(\mathcal{A}_{\rm pos}) = \Gamma^m_{\rm pos}$. Finally, (6.26) is verified thanks to Equation (2.20) of [59, Theorem 2.17] applied to the cycle $\mathcal{A}_{\rm pos}$. □

We are now able to prove the following result.

Proposition 6.9 (*Study of the Transition from any $\mathbf{m} \in \mathcal{X}^m_{pos}$ to $\mathcal{X}^s_{pos}$*). *For any $\mathbf{m} \in \mathcal{X}^m_{pos}$ every optimal path from $\mathbf{m}$ to $\mathcal{X}^s_{pos}$ almost surely does not intersect other metastable states. More precisely,*

$$\lim_{\beta\to\infty} \mathbb{P}(\tau^{\mathbf{m}}_{\mathcal{X}^m_{pos} \backslash \{\mathbf{m}\}} > \tau^{\mathbf{m}}_{\mathcal{X}^s_{pos}}) = 1. \tag{6.27}$$

Proof. Let $\omega = (\omega_0 = \mathbf{m}, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^s_{\rm pos}}$ and, for some $j < n$, let $\omega_j \in \mathcal{W}_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos})$. By Corollary 4.10 and by Corollary 4.12 we get that almost surely the process started in $\mathbf{m} \in \mathcal{X}^m_{\rm pos}$ visits $\mathcal{W}_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos})$ before hitting $\mathcal{X}^s_{\rm pos} \cup \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}$. Hence, almost surely we have

$$(\omega_0, \dots, \omega_j) \cap (\mathcal{X}^s_{\rm pos} \cup \mathcal{X}^m_{\rm pos} \backslash \{\mathbf{m}\}) = \emptyset. \tag{6.28}$$

Thus, our claim is to show that starting from ω_j, the process arrives in $\mathcal{X}^s_{\text{pos}}$ before visiting $\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$. By Lemma 6.4, we have that ω_{j+1} does not have the same energy value of ω_j. Thus, starting from ω_j, the path passes to a configuration with energy strictly lower or strictly higher than $H_{\text{pos}}(\omega_j)$. More precisely, for some $v \in V$ and some $t \in S$, let $\omega_{j+1} := \omega_j^{v,t}$. We have to consider the following possibilities:

(a) v is the vertex in the unit protuberance in the 1-cluster in ω_j and $t = m$;
(b) v is a vertex with spin m with two nearest-neighbors with spin m and two nearest-neighbors with spin 1 in ω_j and $t = 1$;
(c) v has spin 1 (respectively m), $t = m$ (respectively $t = 1$) and v is not a vertex that follows in case (a) (respectively case (b));
(d) $t \in S\backslash\{1, m\}$.

If v and t are as in case (a), then $\omega_{j+1} \in C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$ and, starting from this configuration, almost surely the process comes back to $\mathbf{m}$. Indeed, by Equation (2.20) of [59, Theorem 2.17] applied the cycle $C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})$ we have that there exists $k_1 > 0$ such that for every β sufficiently large

$$\mathbb{P}(\tau^{\omega_{j+1}}_{\mathbf{m}} > \tau^{\omega_{j+1}}_{\partial C^{\mathbf{m}}_{\mathcal{X}^s_{\text{pos}}}(\Gamma^m_{\text{pos}})}) \leq e^{-k_1\beta}. \tag{6.29}$$

Thus, we repeat the same arguments to reduce the proof again to the cases (a)–(d) above. If v and t are as in case (b), then $\omega_{j+1} \in C^{\mathbf{1}}_{\mathcal{X}^m_{\text{pos}}}(\Gamma_{\text{pos}}(\mathbf{1}, \mathcal{X}^m_{\text{pos}}))$ and almost surely the process visits $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$ before exiting from this cycle. Indeed, by Equation (2.20) of [59, Theorem 2.17] applied to the cycle $C^{\mathbf{1}}_{\mathcal{X}^m_{\text{pos}}}(\Gamma_{\text{pos}}(\mathbf{1}, \mathcal{X}^m_{\text{pos}}))$ we have that there exists $k_2 > 0$ such that for every β sufficiently large

$$\mathbb{P}(\tau^{\omega_{j+1}}_{\mathcal{X}^s_{\text{pos}}} > \tau^{\omega_{j+1}}_{\partial C^{\mathbf{1}}_{\mathcal{X}^m_{\text{pos}}}(\Gamma_{\text{pos}}(\mathbf{1},\mathcal{X}^m_{\text{pos}}))}) \leq e^{-k_2\beta}. \tag{6.30}$$

Since almost surely (6.28) holds, we conclude that (6.27) is verified.

Finally, we consider v and t as in case (c) and (d) and our claim is to prove that almost surely ω_{j+1} as in these two cases does not belong to any optimal path from $\mathbf{m}$ to $\mathcal{X}^s_{\text{pos}}$. Indeed, $H_{\text{pos}}(\omega_{j+1}) > H_{\text{pos}}(\omega_j)$ and since the minimum increase of energy is h, it follows that

$$H_{\text{pos}}(\omega_{j+1}) \geq H_{\text{pos}}(\omega_j) + h = \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) + h, \tag{6.31}$$

where the last equality follows by $\omega_j \in \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. Hence, by (6.31) and by the definition of the habitat $\mathcal{A}_{\text{pos}}$, we get that $\omega_{j+1} \notin \mathcal{A}_{\text{pos}}$. However, by Proposition 6.8 we have that almost surely the process started in $\mathbf{m}$ does not exit from $\mathcal{A}_{\text{pos}}$ before hitting its bottom, and thus the cases (c) and (d) do not belong to any optimal path from $\mathbf{m}$ to $\mathcal{X}^s_{\text{pos}}$. □

Exploiting Proposition 6.9, we are now able to identify another gate for the transition from a metastable state to the set of the other metastable states.

Proposition 6.10. *$\bigcup_{\mathbf{z}\in\mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}} \mathcal{W}_{pos}(\mathbf{z}, \mathcal{X}^s_{pos})$ is a gate for the transition $\mathbf{m} \to \mathcal{X}^m_{pos}\backslash\{\mathbf{m}\}$.*

Proof. Using Proposition 6.9 we get that the process started in $\mathbf{m}$ almost surely visits $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$ earlier than $\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$. It follows that almost surely $\mathbf{1} \in \omega$ for any $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}}$, and since it is necessary to complete the transition to $\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$, ω almost surely has a subpath which goes from $\mathbf{1}$ to some $\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$. Thus, exploiting the reversibility of the dynamics, Proposition 6.3 and since any optimal path $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}}$ hits $\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$ in any metastable state different from $\mathbf{m}$ with the same probability, i.e., $\frac{1}{q-2}$, we get that $\bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z}, \mathcal{X}^s_{\text{pos}})$ is a gate for the transition $\mathbf{1} \to \mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$. □

6.3. Minimal gates: proof of the main results

We are now able to prove Theorems 4.9 and 4.11.

Proof of Theorem 4.9. For any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$, by Proposition 6.3 we get that $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ is a gate for the transition from $\mathbf{m}$ to $\mathcal{X}^s_{\text{pos}} = \{\mathbf{1}\}$. In order to prove that $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ is a minimal gate, we exploit [59, Theorem 5.1] and we show that any $\eta \in \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ is an essential saddle. To this end, in view of the definition of an essential saddle given in Section 3.1, for any $\eta \in \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ we define an optimal path from $\mathbf{m}$ to $\mathcal{X}^s_{\text{pos}}$ that passes through η and such that it reaches its maximum energy only in this configuration. In particular, the optimal path is defined by modifying the reference path $\tilde{\omega}$ of Definition 5.4 in a such a way that $\tilde{\omega}_{\ell^*(\ell^*-1)+1} = \eta$ in which $C^1(\eta)$ is a quasi-square $\ell^* \times (\ell^* - 1)$ with a unit protuberance. This is possible by choosing the initial vertex (i, j) such that during the construction the cluster $C^1(\tilde{\omega}_{\ell^*(\ell^*-1)})$ coincides with the quasi-square in η and in the next step the unit protuberance is added in the site as in η. It follows that $\tilde{\omega} \cap \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) = \{\eta\}$ and by the proof of Lemma 5.6 we get $\arg\max_{\tilde{\omega}} H_{\text{pos}} = \{\eta\}$. To conclude, we prove (4.12), i.e., that $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ is the unique minimal gate for the transition $\mathbf{m} \to \mathcal{X}^s_{\text{pos}}$. Note that the above reference paths $\tilde{\omega}$ reach the energy $\Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$ only in $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. Thus, we get that for any $\eta_1 \in \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$, the set $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})\backslash\{\eta_1\}$ is not a gate for the transition $\mathbf{m} \to \mathcal{X}^s_{\text{pos}}$ since, in view of the above construction, we have that there exists an optimal path $\tilde{\omega}$ such that $\tilde{\omega} \cap \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})\backslash\{\eta_1\} = \emptyset$. Note that the uniqueness of the minimal gate follows by the condition $\frac{2}{h} \notin \mathbb{N}$, see Assumption 4.1(ii). □

Remark 6.11. A saddle $\eta \in \mathcal{S}(\sigma,\sigma')$ is unessential if for any $\omega \in \Omega^{opt}_{\sigma,\sigma'}$ such that $\omega \cap \eta \neq \emptyset$ the following conditions are both satisfied:

(i) $\{\text{argmax}_{\omega} H\}\backslash\{\eta\} \neq \emptyset$,
(ii) there exists $\omega' \in \Omega^{opt}_{\sigma,\sigma'}$ such that $\{\text{argmax}_{\omega'} H\} \subseteq \{\text{argmax}_{\omega} H\}\backslash\{\eta\}$.

Proof of Theorem 4.11. By Proposition 6.7 we have that the set given in (a) is a gate for the transition $\mathbf{m} \to \mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$. Hence, our aim is to prove that $\mathcal{W}_{\text{pos}}(\mathbf{m},\mathcal{X}^s_{\text{pos}})$ is a minimal gate for the same transition. In order to show that this set satisfies the definition of minimal gate given in Section 3.1, we show that for any $\eta \in \mathcal{W}_{\text{pos}}(\mathbf{m},\mathcal{X}^s_{\text{pos}})$ there exists an optimal path $\omega' \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}}$ such that $\omega' \cap (\mathcal{W}_{\text{pos}}(\mathbf{m},\mathcal{X}^s_{\text{pos}})\backslash\{\eta\}) = \emptyset$. We construct this optimal path ω' as the reference path ω^* defined in the proof of the upper bound of Proposition 6.5 in such a way that at the step k^*-1 the rectangular $\ell^* \times (\ell^*-1)$ s-cluster is as in η without the protuberance. For $k^* \le k \le k^*+\ell^*-1$, we proceed as follows. At step k^* the unit protuberance is added in the same position as in η, and in the following steps the same side is filled flipping consecutively to s spins 1 that have two nearest neighbors with spin s. Thus, $\omega' \cap \mathcal{W}_{\text{pos}}(\mathbf{m},\mathcal{X}^s_{\text{pos}}) = \{\eta\}$ and the condition of minimality is satisfied. By Proposition 6.10 the set depicted in (b) is a gate for the transition $\mathbf{m} \to \mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$. Thus, our aim is to prove that $\bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}})$ is a minimal gate for the same transition. Similarly to the previous case we show that for any $\eta \in \bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}})$ there exists an optimal path $\omega' \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}}$ such that $\omega' \cap (\bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}})\backslash\{\eta\}) = \emptyset$. We define this optimal path ω' as the reference path ω^* constructed in the proof of the upper bound of Proposition 6.5 in such a way that at the step k^*-1 the rectangular $\ell^* \times (\ell^*-1)$ s-cluster is as in η without the protuberance. For $k^* \le k \le k^*+\ell^*-1$, we proceed as follows. At step k^* the unit protuberance is added in the same position as in η, and in the following steps the same side is filled flipping consecutively to s spins 1 that have two nearest neighbors with spin s. Thus, $\omega' \cap \bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}}) = \{\eta\}$ and the condition of minimality is verified. Thus $\bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}}) \subseteq \mathcal{G}_{\text{pos}}(\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\})$, and we conclude exploiting [59, Theorem 5.1] and showing that any

$$\eta \in \mathcal{S}_{\text{pos}}(\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\})\backslash \bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}}) \tag{6.32}$$

is an unessential saddle for the transition $\mathbf{m} \to \mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}$. To this end we prove that any η as in (6.32) satisfies conditions Remark 6.11(i) and (ii). Indeed, let $\omega \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}}$ such that $\omega \cap \{\eta\} \neq \emptyset$. Note that condition (i) in Remark 6.11 is satisfied since ω intersects at least once both $\mathcal{W}_{\text{pos}}(\mathbf{m},\mathcal{X}^s_{\text{pos}})$ and $\bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}})$. Next we define an optimal path $\omega' \in \Omega^{opt}_{\mathbf{m},\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}}$ in order to prove that also condition (ii) in Remark 6.11 is satisfied. From Propositions 6.7 and 6.10, there exist $\eta_1^* \in \omega \cap \mathcal{W}_{\text{pos}}(\mathbf{m},\mathcal{X}^s_{\text{pos}})$ and $\eta_2^* \in \omega \cap \bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}})$. Thus, we construct ω' as the reference path defined in the proof of the upper bound of Proposition 6.5 in such a way that $\omega' \cap \mathcal{W}_{\text{pos}}(\mathbf{m},\mathcal{X}^s_{\text{pos}}) = \{\eta_1^*\}$, $\omega' \cap \bigcup_{\mathbf{z}\in\mathcal{X}^m_{\text{pos}}\backslash\{\mathbf{m}\}} \mathcal{W}_{\text{pos}}(\mathbf{z},\mathcal{X}^s_{\text{pos}}) = \{\eta_2^*\}$ and $\{\text{argmax}_{\omega'} H\} = \{\eta_1^*,\eta_2^*\}$. □

6.4. Tube of typical paths: proof of the main results

In order to give the proof of Theorem 4.13, first we prove the following lemmas.

Lemma 6.12. *For any $m \in S\backslash\{1\}$, consider the local minimum $\eta \in \bar{R}_{\ell,\ell-1}(m,1)$ with $\ell \le \ell^*$ and $\zeta \in \bar{R}_{\ell,\ell}(m,1)$ with $\ell \le \ell^*-1$. Let $C(\eta)$ and $C(\zeta)$ be the non-trivial cycles whose bottom are η and ζ, respectively. Thus,*

$$\mathcal{B}(C(\eta)) = \bar{B}^1_{\ell-1,\ell-1}(m,1); \tag{6.33}$$

$$\mathcal{B}(C(\zeta)) = \bar{B}^1_{\ell-1,\ell}(m,1). \tag{6.34}$$

Proof. For any $m \in S\backslash\{1\}$, let $\eta_1 \in \bar{R}_{\ell,\ell-1}(m,1)$ with $\ell \le \ell^*$. Using (4.17), our aim is to prove the following

$$\bar{B}^1_{\ell-1,\ell-1}(m,1) = \mathcal{F}(\partial C(\eta_1)). \tag{6.35}$$

In η_1, for any $v \in V$ the corresponding v-tile (see before Lemma 5.10 for the definition) is one among those depicted in Fig. 4(a), (b), (d), (e) and (n) with $r = m$. Starting from η_1, the spin-flip to m (resp. 1) the spin 1 (resp. m) on a vertex whose tile is one among those depicted in Fig. 4(b), (e) (resp. (a), (d)), the process visits a configuration σ_1 such that

$$H_{\text{pos}}(\sigma_1) - H_{\text{pos}}(\eta_1) \ge 2 - h. \tag{6.36}$$

Thus, the smallest energy increase is given by h by flipping to m a spin 1 on a vertex v_1 centered in a tile as in Fig. 4(n) with $r = m$. Let $\eta_2 := \eta_1^{v_1,m} \in \bar{B}^{\ell-2}_{\ell-1,\ell-1}(m,1)$. In η_2, for any $v \in V$ the corresponding v-tile is one among those depicted in Fig. 4(a), (b), (d), (e), (n) and (l) with $r = 1$. Since $H_{\text{pos}}(\eta_2) = H_{\text{pos}}(\eta_1) + h$, the spin-flips on a vertex whose tile is one among those depicted in Fig. 4(a), (b), (d), (e) lead to $H_{\text{pos}}(\sigma_2) - H_{\text{pos}}(\eta_1) \ge 2$. Thus, as in the previous case, the smallest energy increase is given by flipping to m a spin 1 on a vertex v_2 centered in a tile as Fig. 4(n). Note that starting from η_2 the only spin-flip which decreases the energy leads to the bottom of $C(\eta_1)$, namely in η_1. Iterating the strategy, the same arguments hold as long as the uphill path towards $\mathcal{F}(\partial C(\eta_1))$ visits $\eta_{\ell-1} \in \bar{B}^1_{\ell-1,\ell-1}(m,1)$. Indeed, in this type of configuration for any $v \in V$ the corresponding v-tile is one among those depicted in Fig. 4(a), (b), (d), (e), (n) and unstable tile (s) with $t = r = s = m$, and it is possible to decrease the energy by passing to a configuration that does not belong to $C(\eta_1)$. More precisely, there exists a vertex w such that its tile is as the one in Fig. 4(s) with

$t = r = s = m$. By flipping to m the spin 1 on w the energy decreases by $2 - h$, and the process enters a new cycle visiting its bottom, i.e., a local minimum belonging to $\bar{R}_{\ell-1,\ell-1}(m, 1)$. Let us now note that

$$H_{\text{pos}}(\eta_{\ell-1}) - H_{\text{pos}}(\eta_1) = h(\ell - 2). \tag{6.37}$$

Since $\ell \le \ell^*$, comparing (6.36) with (6.37), we get that $\eta_{\ell-1} \in \mathscr{F}(\partial C(\eta_1))$, and (6.35) is verified.

Let us now consider for any $m \in S\backslash\{1\}$ the local minimum $\zeta_1 \in \bar{R}_{\ell,\ell}(m, 1) \subset \mathscr{M}_{\text{pos}}$ with $\ell \le \ell^* - 1$. Arguing similarly to the previous case, we verify (6.34) by proving that

$$\bar{B}^1_{\ell-1,\ell}(m, 1) = \mathscr{F}(\partial C(\zeta_1)). \quad \square \tag{6.38}$$

Lemma 6.13. *For any $m \in S\backslash\{1\}$, consider the local minimum $\eta \in \bar{R}_{\ell_1,\ell_2}(m, 1)$ with $\min\{\ell_1, \ell_2\} \ge \ell^*$. Let $C(\eta)$ be the non-trivial cycle whose bottom is η. Thus,*

$$\mathcal{B}(C(\eta)) = \bar{B}^1_{\ell_1,\ell_2}(m, 1) \cup \bar{B}^1_{\ell_2,\ell_1}(m, 1). \tag{6.39}$$

Proof. For any $m \in S\backslash\{1\}$, let $\eta_1 \in \bar{R}_{\ell_1,\ell_2}(m, 1)$ with $\ell^* \le \ell_1 \le \ell_2$. Using (4.17), our aim is to prove the following

$$\bar{B}^1_{\ell_1,\ell_2}(m, 1) \cup \bar{B}^1_{\ell_2,\ell_1}(m, 1) = \mathscr{F}(\partial C(\eta_1)). \tag{6.40}$$

In η_1, for any $v \in V$ the corresponding v-tile is one among those depicted in Fig. 4(a), (b), (d), (e) and (n) with $r = m$. Let $v_1 \in V$ such that the v_1-tile is as the one depicted in Fig. 4(d), and let $\eta_2 := \eta_1^{v_1,1}$. Note that if v_1 is adjacent to a side of length ℓ_2, then $\eta_2 \in \bar{B}^1_{\ell_1,\ell_2}(m, 1)$, otherwise $\eta_2 \in \bar{B}^1_{\ell_2,\ell_1}(m, 1)$. Without loss of generality, let us assume that $\eta_2 \in \bar{B}^1_{\ell_1,\ell_2}(m, 1)$. By simple algebraic calculation we obtain that $H_{\text{pos}}(\eta_2) - H_{\text{pos}}(\eta_1) = 2 - h$. In η_2 for any $v \in V$ the corresponding v-tile is one among those depicted in Fig. 4(a), (b), (d), (e), (n) and (l) with $r = 1$. By flipping to 1 a spin m on a vertex w whose tile is as the one depicted in Fig. 4(l) with $r = 1$ the energy decreases by h and the process enters a cycle different from the previous one that is either the cycle $\bar{C}$ whose bottom is a local minimum belonging to $\bar{R}_{\ell_1+1,\ell_2}(m, 1)$, or a trivial cycle for which iterating this procedure the process enters $\bar{C}$. Thus, $\bar{B}^1_{\ell_1,\ell_2}(m, 1) \subseteq \partial C(\eta_1)$. Similarly we prove that $\bar{B}^1_{\ell_2,\ell_1}(m, 1) \subseteq \partial C(\eta_1)$.

Let us now note that starting from η_1 the smallest energy increase is h, and it is given by flipping to m a spin 1 on a vertex whose tile is as the one depicted in Fig. 4(n) with $r = m$. Let us consider the uphill path ω started in η_1 and constructed by flipping to m all the spins 1 along a side of the rectangular $\ell_1 \times \ell_2$ 1-cluster, say one of length ℓ_1. Using the discussion given in the proof of Lemma 6.12 and the construction of ω, we get that the process intersects $\partial C(\eta)$ in a configuration σ belonging to $\bar{B}^1_{\ell_2-1,\ell_1}(m, 1)$. By simple algebraic computations, we obtain the following

$$H_{\text{pos}}(\sigma) - H_{\text{pos}}(\eta_1) = h(\ell_2 - 1). \tag{6.41}$$

Since $\ell_2 \ge \ell^*$, it follows that $H_{\text{pos}}(\sigma) > H_{\text{pos}}(\eta_2)$. Since by flipping to m (resp. 1) the vertex centered in a tile as depicted in Fig. 4(b), (e) (resp. (a)), the energy increase is largest than or equal to $2 + h$, it follows that (6.40) is satisfied. $\square$

We are now able to prove Theorem 4.13.

Proof of Theorem 4.13. Following the same approach as [68, Section 6.7], we geometrically characterize the tube of typical trajectories for the transition using the so-called "standard cascades". See [68, Figure 6.3] for an example of these objects. We describe the standard cascades in terms of the paths that are started in $\mathbf{m}$ and are vtj-connected to $\mathcal{X}^s_{\text{pos}}$. See (4.20) for the formal definition and see [63, Lemma 3.12] for an equivalent characterization of these paths. We remark that any typical path from $\mathbf{m}$ to $\mathcal{X}^s_{\text{pos}}$ is also an optimal path for the same transition. In order to describe these typical paths we proceed similarly to [68, Section 7.4], where the authors apply the model-independent results given in [68, Section 6.7] to identify the tube of typical paths in the context of the Ising model. Thus, we define a vtj-connected cycle-path that is the concatenation of both trivial and non-trivial cycles that satisfy (4.19). In Theorem 4.9 we give the geometric characterization of all the minimal gates for the transition $\mathbf{m} \to \mathcal{X}^s_{\text{pos}}$. Let η_1 be a configuration belonging to one of these minimal gates. We begin by studying the first descent from η_1 both to $\mathbf{m}$ and to $\mathcal{X}^s_{\text{pos}}$. Then, we complete the description of $\mathfrak{T}_{\mathcal{X}^s_{\text{pos}}}(\mathbf{m})$ by joining the time reversal of the first descent from η_1 to $\mathbf{m}$ with the first descent from η_1 to $\mathcal{X}^s_{\text{pos}}$.

Let us begin by studying the standard cascades from η_1 to $\mathbf{m}$. Since a spin-flip from 1 to $t \notin \{1, m\}$ implies an increase of the energy value equal to the increase of the number of the disagreeing edges, we consider only the spin-flips from 1 to m on those vertices belonging to the 1-cluster. Thus, starting from η_1 and given v_1 a vertex such that $\eta_1(v_1) = 1$, since $H_{\text{pos}}(\eta_1) = \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$, we get

$$H_{\text{pos}}(\eta_1^{v_1,m}) = \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) + n_1(v_1) - n_m(v_1) + h. \tag{6.42}$$

It follows that the only possibility in which the assumed optimality of the path is not contradicted is the one where $n_1(v_1) = 1$ and $n_m(v_1) = 3$. Thus, along the first descent from η_1 to $\mathbf{m}$ the process visits η_2 in which all the vertices have spin m except those, which are 1, in a rectangular cluster $\ell^* \times (\ell^* - 1)$, i.e., $\eta_2 \in \bar{R}_{\ell^*-1,\ell^*}(m, 1)$. According to (4.19) we have to describe the non-trivial cycle whose bottom is η_2 and its principal boundary. Starting from η_2, the next configuration along a typical path is defined by flipping to m a spin 1 on a vertex v_2 on one of the four corners of the rectangular 1-cluster. Indeed, since $H_{\rm pos}(\eta_2) = \Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}) - 2 + h$, we have

$$H_{\rm pos}(\eta_2^{v_2,1}) = H_{\rm pos}(\eta_2) + n_1(v_2) - n_m(v_2) + h = \Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}) - 2 + 2h + n_1(v_2) - n_m(v_2),$$

and the only possibility in which the assumed optimality of the path is not contradicted is $n_1(v_2) = 2$ and $n_m(v_2) = 2$. Then, a typical path towards $\mathbf{m}$ proceeds by eroding the $\ell^* - 2$ unit squares with spin 1 belonging to a side of length $\ell^* - 1$ that are corners of the 1-cluster and that belong to the same side of v_2. Each of the first $\ell^* - 3$ spin-flips increases the energy by h, i.e., the smallest energy increase for any single step of the dynamics, and these uphill steps are necessary in order to exit from the cycle whose bottom is the local minimum η_2. After these $\ell^* - 3$ steps, the process hits the bottom of the boundary of this cycle in a configuration $\eta_{\ell^*} \in \bar{B}^1_{\ell^*-1,\ell^*-1}(m, 1)$, see Lemma 6.12. The last spin-update, that flips from 1 to m the spin 1 on the unit protuberance of the 1-cluster, decreases the energy by $2 - h$. Thus, the typical path arrives in a local minimum $\eta_{\ell^*+1} \in \bar{R}_{\ell^*-1,\ell^*-1}(m, 1)$, i.e., it enters a new cycle whose bottom is a configuration in which all the vertices have spin 1, except those, which are 1, in a square $(\ell^*-1)\times(\ell^*-1)$ 1-cluster. Summarizing the construction above, we have the following sequence of vtj-connected cycles

$$\{\eta_1\}, C^{\eta_2}_{\mathbf{m}}(h(\ell^* - 2)), \{\eta_{\ell^*}\}, C^{\eta_{\ell^*+1}}_{\mathbf{m}}(h(\ell^* - 2)). \tag{6.43}$$

Iterating this argument, we obtain that the first descent from $\eta_1 \in \mathcal{W}_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos})$ to $\mathbf{m}$ is characterized by the concatenation of those vtj-connected cycle-subpaths between the cycles whose bottom is a local minimum in which all the vertices have spin equal to m, except those, which are 1, in either a quasi-square $(\ell - 1) \times \ell$ or a square $(\ell - 1) \times (\ell - 1)$ for any $\ell = \ell^*, \dots, 1$, and whose depth is given by $h(\ell - 2)$. More precisely, from a quasi-square to a square, a typical path proceeds by flipping to m those spins 1 on one of the shortest sides of the 1-cluster. On the other hand, from a square to a quasi-square, it proceeds by flipping to m those spins 1 belonging to one of the four sides of the square. Thus, a standard cascade from η_1 to $\mathbf{m}$ is characterized by the sequence of those configurations that belong to

$$\bigcup_{\ell=1}^{\ell^*}\Bigg[\bigcup_{l=1}^{\ell-1} \bar{B}^l_{\ell-1,\ell}(m, 1) \cup \bar{R}_{\ell-1,\ell}(m, 1) \cup \bigcup_{l=1}^{\ell-2} \bar{B}^l_{\ell-1,\ell-1}(m, 1) \cup \bar{R}_{\ell-1,\ell-1}(m, 1)\Bigg]. \tag{6.44}$$

Let us now consider the first descent from $\eta_1 \in \bar{B}^1_{\ell^*-1,\ell^*}(m, 1)$ to $\mathcal{X}^s_{\rm pos} = \{\mathbf{1}\}$. In order to not contradict the definition of an optimal path, we have only to consider those steps which flip to 1 a spin m. Indeed, adding a spin different from m and 1 leads to a configuration with energy value strictly larger than $\Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos})$. Thus, let w_1 be a vertex such that $\eta_1(w_1) = m$. Flipping the spin m on the vertex w_1, we get

$$H_{\rm pos}(\eta_1^{w_1,1}) = \Phi_{\rm pos}(\mathbf{m}, \mathcal{X}^s_{\rm pos}) + n_m(w_1) - n_1(w_1) - h, \tag{6.45}$$

and the only feasible choice is $n_m(w_1) = 2$ and $n_1(w_1) = 2$ in η_1. Thus, $\eta_1^{w_1,1} \in \bar{B}^2_{\ell^*-1,\ell^*}(m, 1)$, namely the bar is now of length two. Arguing similarly, we get that along the descent to $\mathbf{1}$ a typical path proceeds by flipping from m to 1 the spins m with two nearest-neighbors with spin 1 and two nearest-neighbors with spin m belonging to the incomplete side of the 1-cluster. More precisely, it proceeds downhill visiting $\bar{\eta}_i \in \bar{B}^i_{\ell^*-1,\ell^*}(m, 1)$ for any $i = 2, \dots, \ell^* - 1$ and $\bar{\eta}_{\ell^*} \in \bar{R}_{\ell^*,\ell^*}(m, 1)$. In order to exit from the cycle whose bottom is $\bar{\eta}_{\ell^*}$, the process crosses the bottom of its boundary by creating a unit protuberance of spin 1 adjacent to one of the four edges of the 1-square, i.e., visits $\{\bar{\eta}_{\ell^*+1}\}$ where $\bar{\eta}_{\ell^*+1} \in \bar{B}^1_{\ell^*,\ell^*}(m, 1)$. Indeed, starting from $\bar{\eta}_{\ell^*} \in \bar{R}_{\ell^*,\ell^*}(m, 1)$ the energy minimum increase is obtained by flipping a spin m with three nearest-neighbors with spin 1 and one nearest-neighbor with spin m. Starting from $\{\bar{\eta}_{\ell^*+1}\}$, a typical path towards $\mathbf{1}$ proceeds by enlarging the protuberance to a bar of length two to $\ell^* - 1$, thus it visits $\bar{\eta}_{\ell^*+i} \in \bar{B}^i_{\ell^*,\ell^*}(m, 1)$ for any $i = 2, \dots, \ell^* - 1$. Each of these steps decreases the energy by h, and after them the descent arrives in the bottom of the cycle, i.e., in the local minimum $\bar{\eta}_{2\ell^*} \in \bar{R}_{\ell^*,\ell^*+1}(m, 1)$. Then, the process exits from this cycle through the bottom of its boundary, i.e., by adding a unit protuberance of spin 1 on *any* one of the four edges of the rectangular $\ell^* \times (\ell^* + 1)$ 1-cluster in $\bar{\eta}_{2\ell^*}$, see Lemma 6.13. Thus, it visits the trivial cycle $\{\bar{\eta}_{2\ell^*+1}\}$, where $\bar{\eta}_{2\ell^*+1} \in \bar{B}^1_{\ell^*,\ell^*+1}(m, 1) \cup \bar{B}^1_{\ell^*+1,\ell^*}(m, 1)$. Note that the resulting standard cascade is different from the one towards $\mathbf{m}$. Thus, summarizing the construction above, we have defined the following sequence of vtj-connected cycles

$$\{\eta_1\}, C^{\bar{\eta}_{\ell^*}}_{\mathbf{1}}(h(\ell^* - 1)), \{\bar{\eta}_{\ell^*+1}\}, C^{\bar{\eta}_{2\ell^*}}_{\mathbf{1}}(h(\ell^* - 1)), \{\bar{\eta}_{2\ell^*+1}\}. \tag{6.46}$$

Note that if $\bar{\eta}_{2\ell^*} \in \bar{B}^1_{\ell^*,\ell^*+1}(m, 1)$, then the process enters the cycle whose bottom is a configuration belonging to $\bar{R}_{\ell^*+1,\ell^*+1}(m, 1)$. On the other hand, if $\bar{\eta}_{2\ell^*} \in \bar{B}^1_{\ell^*+1,\ell^*}(m, 1)$, then the standard cascade enters the cycle whose bottom is a configuration belonging to $\bar{R}_{\ell^*,\ell^*+2}(m, 1)$. In the first case the cycle has depth $h\ell^*$, in the second case the cycle has depth $h(\ell^* - 1)$. Iterating this argument, we get that the first descent from η_1 to $\mathcal{X}^s_{\rm pos} = \{\mathbf{1}\}$ is characterized by vtj-connected cycle-subpaths from $\bar{R}_{\ell_1,\ell_2}(m, 1)$ to $\bar{R}_{\ell_1,\ell_2+1}(m, 1)$ defined as the sequence of those configurations belonging to $\bar{B}^l_{\ell_1,\ell_2}(m, 1)$ for any $l = 1, \dots, \ell_2 - 1$. Enlarging the 1-cluster, at a certain point, the process arrives in a configuration in which this cluster is either a vertical or a horizontal strip, i.e., it intersects one of the two sets defined in (4.25)–(4.26). If the descent arrives in $\mathcal{S}^v_{\rm pos}(m, 1)$, then it proceeds by enlarging the vertical strip column by column. Otherwise, if it arrives in $\mathcal{S}^h_{\rm pos}(m, 1)$, then it enlarges the horizontal strip row by row. In both cases, starting from a configuration with an 1-strip, the path exits from its cycle by adding a unit protuberance with a spin 1 adjacent to one of the two vertical (resp. horizontal) edges and increasing the energy by $2-h$. Starting from the trivial cycle given by this configuration with an 1-strip with a unit protuberance, the standard cascade enters a new cycle and it proceeds downhill by filling the column (resp. row) with spins 1. More precisely, the standard cascade visits $K - 1$ (resp. $L - 1$) configurations such that each of them is defined by the previous one flipping from m to 1 a spin m with two nearest-neighbors with spin m and two nearest-neighbors with spin 1. Each of these spin-updates decreases the energy by h. The process arrives in this way to the bottom of the cycle, i.e., in a configuration in which the thickness of the 1-strip has been enlarged by a column (resp. row). Starting from this state with the new 1-strip, we repeat the same arguments above until the standard cascade arrives in the trivial cycle of a configuration σ with an 1-strip of thickness $L-2$ (resp. $K-2$) and with a unit protuberance. Starting from $\{\sigma\}$, the process enters the cycle whose bottom is $\mathbf{1}$ and it proceeds downhill either by flipping from m to 1 those spins m with two nearest-neighbors with spin m and two nearest-neighbors with spin 1, or by flipping to 1 all the spins m with three nearest-neighbors with spin 1 and one nearest-neighbor with spin m. The last step flips from m to 1 the last spin m with four nearest-neighbors with spin 1. Note that if the vtj-connected cycle path $(\mathcal{C}_1,\dots,\mathcal{C}_n)$ is such that $(\mathcal{C}_1,\dots,\mathcal{C}_n)\cap\mathcal{S}^v_{\text{pos}}(m,1)\neq\varnothing$ (resp. $(\mathcal{C}_1,\dots,\mathcal{C}_n)\cap\mathcal{S}^h_{\text{pos}}(m,1)\neq\varnothing$), then $(\mathcal{C}_1,\dots,\mathcal{C}_n)\cap\mathcal{S}^h_{\text{pos}}(m,1)=\varnothing$ (resp. $(\mathcal{C}_1,\dots,\mathcal{C}_n)\cap\mathcal{S}^v_{\text{pos}}(m,1)=\varnothing$). Thus, the first descent from η_1 to $\mathcal{X}^s_{\text{pos}}$ is characterized by the sequence of those configurations that belong to

$$\bigcup_{\ell_1=\ell^*}^{K-1}\bigcup_{\ell_2=\ell^*}^{K-1}\bar{R}_{\ell_1,\ell_2}(m,1)\cup\bigcup_{\ell_1=\ell^*}^{K-1}\bigcup_{\ell_2=\ell^*}^{K-1}\bigcup_{l=1}^{\ell_2-1}\bar{B}^l_{\ell_1,\ell_2}(m,1)\cup\bigcup_{\ell_1=\ell^*}^{L-1}\bigcup_{\ell_2=\ell^*}^{L-1}\bar{R}_{\ell_1,\ell_2}(m,1)$$
$$\cup\bigcup_{\ell_1=\ell^*}^{L-1}\bigcup_{\ell_2=\ell^*}^{L-1}\bigcup_{l=1}^{\ell_2-1}\bar{B}^l_{\ell_1,\ell_2}(m,1)\cup\mathcal{S}^v_{\text{pos}}(m,1)\cup\mathcal{S}^h_{\text{pos}}(m,1). \tag{6.47}$$

Finally, the standard cascade from $\mathbf{m}$ to $\mathcal{X}^s_{\text{pos}}$ is given by (6.44)–(6.47). Finally, (4.28) follows by [63, Lemma 3.13]. □

Appendix

A.1. Additional material for Section 5.2

A.1.1. Definition 5.4

For any $\mathbf{m}\in\mathcal{X}^m_{\text{pos}}$, we define a *reference path* $\tilde{\omega}:\mathbf{m}\to\mathbf{1}$, $\tilde{\omega}=(\omega^*_0,\dots,\omega^*_{KL})$ as the concatenation of the two paths $\tilde{\omega}^{(1)}:=(\mathbf{1}=\tilde{\omega}_0,\dots,\tilde{\omega}_{(K-1)^2})$ and $\tilde{\omega}^{(2)}:=(\tilde{\omega}_{(K-1)^2},\dots,\mathbf{m}=\tilde{\omega}_{KL})$. The path $\tilde{\omega}^{(1)}$ is defined as follows. We set $\tilde{\omega}_0:=\mathbf{m}$. Then, we define $\tilde{\omega}_1:=\tilde{\omega}_0^{(i,j),1}$, where (i,j) denotes the vertex which belongs to the row r_i and to the column c_j of Λ, for some $i=0,\dots,K-1$ and $j=0,\dots,L-1$. Sequentially, we flip clockwise from m to 1 all the vertices that surround the vertex (i,j) in order to depict a 3×3 square of spins 1. We iterate this construction until we get $\tilde{\omega}_{(K-1)^2}\in\bar{R}_{K-1,K-1}(m,1)$. See [19, Figure 6(a)] for an illustration of this procedure. This time the white squares denote those vertices with spin m, the black ones denote the vertices with spin 1. Note that by considering the periodic boundary conditions the definition of $\tilde{\omega}$ is general for any i and j.

The path $\tilde{\omega}^{(2)}$ is defined as follows. Without loss of generality, assume that $\tilde{\omega}_{(K-1)^2}\in\bar{R}_{K-1,K-1}(m,1)$ has the cluster of spin 1 in the first $c_0,\dots,c_{K-2}$ columns, see [19, Figure 6(b)]. Starting from this last configuration $\tilde{\omega}_{(K-1)^2}$ of $\tilde{\omega}^{(1)}$, we define $\tilde{\omega}_{(K-1)^2+1},\dots,\tilde{\omega}_{(K-1)^2+K-1}$ as a sequence of configurations in which the cluster of spin 1 grows gradually by flipping the spins m on the vertices $(K-1,j)$, for $j=0,\dots,K-2$. Thus, $\tilde{\omega}_{(K-1)^2+K-1}\in\bar{R}_{K-1,K}(m,1)$, as depicted in [19, Figure 6(c)]. Finally, we define $\tilde{\omega}_{(K-1)^2+K},\dots,\tilde{\omega}_{KL}$ as a sequence of configurations in which the cluster of spin s grows gradually column by column. More precisely, starting from $\tilde{\omega}_{(K-1)^2+K-1}\in\bar{R}_{K-1,K}(m,1)$, $\tilde{\omega}^{(2)}$ passes through configurations in which the spins m on columns $c_K,\dots,c_{L-1}$ become 1. The procedure ends with $\tilde{\omega}_{KL}=\mathbf{m}$.

A.1.2. Proof of Lemma 5.5

Consider the reference path of Definition 5.4 and note that for any $i=0,\dots,KL$, $N_1(\tilde{\omega}_i)=i$. The reference path may be constructed in such a way that $\tilde{\omega}_{\ell^*(\ell^*-1)}:=\sigma$. Let $\gamma:=(\tilde{\omega}_{\ell^*(\ell^*-1)}=\sigma,\tilde{\omega}_{\ell^*(\ell^*-1)-1},\dots,\tilde{\omega}_0=\mathbf{m})$ be the time reversal of the subpath $(\tilde{\omega}_0,\dots,\tilde{\omega}_{\ell^*(\ell^*-1)})$ of $\tilde{\omega}$. We claim that $\max_{\xi\in\gamma}H_{\text{pos}}(\xi)<4\ell^*-h(\ell^*(\ell^*-1)+1)+H_{\text{pos}}(\mathbf{m})$. Indeed, $\tilde{\omega}_{\ell^*(\ell^*-1)}=\sigma,\tilde{\omega}_{\ell^*(\ell^*-1)-1},\dots,\tilde{\omega}_1$ is a sequence of configurations in which all the spins are equal to m except those, which are 1, in either a quasi-square $\ell\times(\ell-1)$ or a square $(\ell-1)\times(\ell-1)$ possibly with one of the longest sides not completely filled. For any $\ell=\ell^*,\dots,2$, the path γ moves from $\bar{R}_{\ell,\ell-1}(m,1)$ to $\bar{R}_{\ell-1,\ell-1}(m,1)$ by flipping to m the $\ell-1$ spins 1 on one of the shortest sides of the 1-cluster. In particular, $\tilde{\omega}_{\ell(\ell-1)-1}$ is obtained by $\tilde{\omega}_{\ell(\ell-1)}\in\bar{R}_{\ell,\ell-1}(m,1)$ by flipping the spin on a corner of the quasi-square from 1 to m and this increases the energy by h. The next $\ell-3$ steps are defined by flipping the spins on the incomplete shortest side from 1 to m, thus each step increases the energy by h. Finally, $\tilde{\omega}_{(\ell-1)^2}\in\bar{R}_{\ell-1,\ell-1}(m,1)$ is defined by flipping the last spin 1 to m and this decreases the energy by $2-h$. For any $\ell=\ell^*,\dots,2$, $h(\ell-2)<2-h$. Indeed, from (3.14) and from Assumption 4.1, we have $2-h>h(\ell^*-2)\geq h(\ell-2)$. Hence, $\max_{\xi\in\gamma}H_{\text{pos}}(\xi)=H_{\text{pos}}(\sigma)=4\ell^*-h(\ell^*(\ell^*-1)+1)-(2-h)+H_{\text{pos}}(\mathbf{m})$ and the claim is verified. □

A.1.3. Proof of Lemma 5.6

Let $\mathbf{m}\in\mathcal{X}^m_{\text{pos}}$ and let $\sigma\in\bar{B}^2_{\ell^*-1,\ell^*}(m,1)$. Consider the reference path of Definition 5.4 and assume that it is constructed in such a way that $\tilde{\omega}_{\ell^*(\ell^*-1)+2}:=\sigma$. Let $\gamma:=(\tilde{\omega}_{\ell^*(\ell^*-1)+2}=\sigma,\tilde{\omega}_{\ell^*(\ell^*-1)+3},\dots,\tilde{\omega}_{KL-1},\mathbf{1})$. Our aim is to prove that $\max_{\xi\in\gamma}H_{\text{pos}}(\xi)<4\ell^*-h(\ell^*(\ell^*-1)+1)+H_{\text{pos}}(\mathbf{m})$. In particular, we prove this claim by showing that $\max_{\xi\in\tilde{\omega}}H_{\text{pos}}(\xi)=4\ell^*-h(\ell^*(\ell^*-1)+1)+H_{\text{pos}}(\mathbf{m})$ and that γ does not visit the unique configuration in which this maximum is reached.

Consider $\ell\leq K-2$. We recall that $\tilde{\omega}^{(1)}$ is defined as a sequence of configurations in which all the spins are equal to m except those, which are 1, in either a square $\ell\times\ell$ or a quasi-square $\ell\times(\ell-1)$ possibly with one of the longest sides not completely filled. For some $\ell\leq K-2$, let $\tilde{\omega}_{\ell(\ell-1)}\in\bar{R}_{\ell-1,\ell}(m,1)$ and $\tilde{\omega}_{\ell^2}\in\bar{R}_{\ell,\ell}(m,1)$, then

$$\max_{\sigma\in\{\tilde{\omega}_{\ell(\ell-1)},\tilde{\omega}_{\ell(\ell-1)+1},\dots,\tilde{\omega}_{\ell^2}\}}H_{\text{pos}}(\sigma)=H_{\text{pos}}(\tilde{\omega}_{\ell(\ell-1)+1})=4\ell-h\ell^2+h\ell-h+H_{\text{pos}}(\mathbf{m}). \tag{A.1}$$

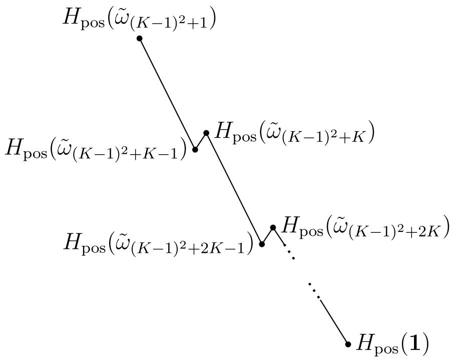


Fig. A.16. Qualitative illustration of the energy of the configurations belonging to $\tilde{\omega}^{(2)}$.

Otherwise, if $\tilde{\omega}_{\ell^2} \in \bar{R}_{\ell,\ell}(m,1)$ and $\tilde{\omega}_{\ell(\ell+1)} \in \bar{R}_{\ell,\ell+1}(m,1)$, then

$$\max_{\sigma \in \{\tilde{\omega}_{\ell^2}, \tilde{\omega}_{\ell^2+1}, \ldots, \tilde{\omega}_{\ell(\ell+1)}\}} H_{\mathrm{pos}}(\sigma) = H_{\mathrm{pos}}(\tilde{\omega}_{\ell^2+1}) = 4\ell - h\ell^2 + 2 - h + H_{\mathrm{pos}}(\mathbf{m}). \tag{A.2}$$

Let $k^* := \ell^*(\ell^*-1)+1$. By recalling the condition $\frac{2}{h} \notin \mathbb{N}$ of Assumption 4.1 and by studying the maxima of H_{pos} as a function of ℓ, we have

$$\arg\max_{\tilde{\omega}^{(1)}} H_{\mathrm{pos}} = \{\tilde{\omega}_{k^*}\}. \tag{A.3}$$

Note that if $\frac{2}{h}$ belonged to $\mathbb{N}$, then $\tilde{\omega}_{k^*}$ and $\tilde{\omega}_{(\ell^*)^2+1}$ would have the same energy value.

Let us now study the maximum energy value reached along $\tilde{\omega}^{(2)}$. This path is constructed as a sequence of configurations whose clusters of spins 1 wrap around Λ. Moreover, the maximum of the energy is reached at the first configuration of $\tilde{\omega}^{(2)}$, see Fig. A.16 for a qualitative representation of the energy of the configurations in $\tilde{\omega}^{(2)}$. Indeed,

$$\begin{aligned}
&H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+j}) - H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+j-1}) = -2-h,\ \ j = 2, \ldots, K-1,\\
&H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+K}) - H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+K-1}) = 2-h,\\
&H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+j}) - H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+j-1}) = -h,\ \ j = K+1, \ldots, 2K-1,\\
&H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+2K}) - H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+2K-1}) = 2-h.
\end{aligned}$$

Note that

$$\begin{aligned}
H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+1}) - H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+K}) &= 4K-4-h(K-1)^2 - (2K - h((K-1)^2+K))\\
&= 2K-4+h(K-1) > 0,
\end{aligned} \tag{A.4}$$

where the last inequality follows by $K \geq 3\ell^*$. Moreover,

$$\begin{aligned}
&H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+K}) - H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+2K})\\
&\qquad = 2K+2-h(K-1)^2+2K-2K+2-h((K-1)^2+K) = K > 0.
\end{aligned} \tag{A.5}$$

By iterating the analysis of the energy gap between two consecutive configurations along $\tilde{\omega}^{(2)}$, we conclude that

$$\arg\max_{\tilde{\omega}^{(2)}} H_{\mathrm{pos}} = \{\tilde{\omega}_{(K-1)^2+1}\}. \tag{A.6}$$

In particular,

$$H_{\mathrm{pos}}(\tilde{\omega}_{(K-1)^2+1}) < H_{\mathrm{pos}}(\tilde{\omega}_{k^*}). \tag{A.7}$$

This inequality is proved in [19, Appendix A.1]. Hence, $\arg\max_{\tilde{\omega}} H_{\mathrm{pos}} = \{\tilde{\omega}_{k^*}\}$. Since γ is constructed as the subpath of $\tilde{\omega}$ which goes from $\tilde{\omega}_{\ell^*(\ell^*-1)+2} = \sigma$ to $\mathbf{1}$, γ does not visit the configuration $\tilde{\omega}_{k^*}$. Hence, the claim is verified. □

A.1.4. Proof of Proposition 5.8

For any $k = 1, \ldots, |V|$, let $\mathcal{V}_k^1 := \{\sigma \in \mathcal{X} : N_1(\sigma) = k\}$. Every path ω from any $\mathbf{m} \in \{\mathbf{2}, \ldots, \mathbf{q}\}$ to the stable configuration $\mathbf{1}$ has to intersect the set $\mathcal{V}_k^1$ for every $k = 1, \ldots, |V|$. In particular, it has to visit the set $\mathcal{V}_{k^*}^1$ at least once, where $k^* = \ell^*(\ell^*-1)+1$. We prove the lower bound given in (5.10) by showing that $H_{\mathrm{pos}}(\mathscr{F}(\mathcal{V}_{k^*}^1)) = 4\ell^* - h(\ell^*(\ell^*-1)+1) + H_{\mathrm{pos}}(\mathbf{m})$. Note that from (5.5), we get that the presence of disagreeing edges increases the energy. Thus, in order to describe the bottom $\mathscr{F}(\mathcal{V}_{k^*}^1)$ we have to consider those configurations in which the $\ell^*(\ell^*-1)+1$ spins 1 belong to a unique cluster inside a homogeneous sea of spin $m \in S\backslash\{1\}$. Hence, consider $\tilde{\omega}$ be the reference path of Definition 5.4 whose configurations satisfy this characterization. Note that $\tilde{\omega} \cap \mathcal{V}^1_{k^*} = \{\tilde{\omega}_{k^*}\}$ with $\tilde{\omega}_{k^*} \in \bar{B}^1_{\ell^*-1,\ell^*}(m,1)$. In particular,

$$H_{\text{pos}}(\tilde{\omega}_{k^*}) - H_{\text{pos}}(\mathbf{m}) = 4\ell^* - h(\ell^*(\ell^*-1)+1), \tag{A.8}$$

where $4\ell^*$ is the perimeter of the cluster of spins 1 in $\tilde{\omega}_{k^*}$. We want to show that it is not possible to have a configuration with k^* spins 1 in a cluster of perimeter smaller than $4\ell^*$. Since the perimeter is an even integer, we suppose that there exists a configuration belonging to $\mathcal{V}^1_{k^*}$ such that the 1-cluster has perimeter $4\ell^*-2$. Since $4\ell^*-2 < 4\sqrt{k^*}$, where $\sqrt{k^*}$ is the side-length of the square $\sqrt{k^*} \times \sqrt{k^*}$ of minimal perimeter among those in $\mathbb{R}^2$ of area k^*, using that the square is the figure that minimizes the perimeter for a given area, we conclude that there is no configuration with k^* spins 1 in a cluster with perimeter strictly smaller than $4\ell^*$. Hence, $\tilde{\omega}_{k^*} \in \mathcal{F}(\mathcal{V}^1_{k^*})$ and (5.10) is satisfied thanks to (A.8). □

A.1.5. Proof of Lemma 5.9

Let $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$. In the proof of Proposition 5.8 we noted that any path $\omega : \mathbf{m} \to \mathbf{1}$ has to visit $\mathcal{V}^1_k$ at least once for every $k = 0, \dots, |V|$. Consider $\mathcal{V}^1_{\ell^*(\ell^*-1)}$. In [2, Theorem 2.6] the authors show that the unique configuration of minimal energy in $\mathcal{V}^1_{\ell^*(\ell^*-1)}$ is the one in which all spins are m except those that are 1 in a quasi-square $\ell^* \times (\ell^*-1)$. In particular, this configuration has energy $\Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) - (2-h) = 4\ell^* - 2 - h\ell^*(\ell^*-1) + H_{\text{pos}}(\mathbf{m})$. Note that $4\ell^*-2$ is the perimeter of its 1-cluster. Since the perimeter is an even integer, we have that the other configurations belonging to $\mathcal{V}^1_{\ell^*(\ell^*-1)}$ have energy that is larger than or equal to $4\ell^* - h\ell^*(\ell^*-1) + H_{\text{pos}}(\mathbf{m})$. Thus, they are not visited by any optimal path. Indeed, $4\ell^* - h\ell^*(\ell^*-1) + H_{\text{pos}}(\mathbf{m}) > \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. Thus, we conclude that any optimal path intersects $\mathcal{V}^1_{\ell^*(\ell^*-1)}$ in a configuration belonging to $\bar{R}_{\ell^*-1,\ell^*}(m,1)$. □

A.2. Additional material for Section 6.2

A.2.1. Proof of Corollary 6.6

Every path from $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$ to the other metastable configurations in $\mathcal{X}^m_{\text{pos}} \backslash \{\mathbf{m}\}$ has to pass through the set $\mathcal{V}^m_k := \{\sigma \in \mathcal{X} : N_m(\sigma) = k\}$ for any $k = |V|, \dots, 0$. In particular, given $k^* := \ell^*(\ell^*-1)+1$, any $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \backslash \{\mathbf{m}\}}$ visits at least once the set $\mathcal{V}^m_{|\Lambda|-k^*} \equiv \mathcal{D}^m_{\text{pos}}$. Hence, there exists $i \in \{0, \dots n\}$ such that $\omega_i \in \mathcal{D}^m_{\text{pos}}$. Thanks to (6.5) and to (6.19) we have that the energy value of any configuration belonging to $\mathcal{F}(\mathcal{D}^m_{\text{pos}})$ is equal to the min–max reached by any optimal path from $\mathbf{m}$ to $\mathcal{X}^m_{\text{pos}} \backslash \{\mathbf{m}\}$. Thus, we conclude that $\omega_i \in \mathcal{F}(\mathcal{D}^m_{\text{pos}})$. □

A.2.2. Proof of Proposition 6.7

For any $m \in S$, $m \neq 1$, let $\tilde{\mathcal{D}}^m_{\text{pos}}$ and $\hat{\mathcal{D}}^m_{\text{pos}}$ be the subsets of $\mathcal{F}(\mathcal{D}^m_{\text{pos}})$ defined as follows. $\tilde{\mathcal{D}}^m_{\text{pos}}$ is the set of those configurations of $\mathcal{F}(\mathcal{D}^m_{\text{pos}})$ in which the boundary of the polyomino $C^1(\sigma)$ intersects each side of the boundary of its smallest surrounding rectangle $R(C^1(\sigma))$ on a set of the dual lattice $\mathbb{Z}^2 + (1/2, 1/2)$ made by at least two consecutive unit segments, see Fig. 15(a). On the other hand, $\hat{\mathcal{D}}^m_{\text{pos}}$ is the set of those configurations of $\mathcal{F}(\mathcal{D}^m_{\text{pos}})$ in which the boundary of the polyomino $C^1(\sigma)$ intersects at least one side of the boundary of $R(C^1(\sigma))$ in a single unit segment, see Fig. 15(b) and (c). In particular note that $\mathcal{F}(\mathcal{D}^m_{\text{pos}}) = \tilde{\mathcal{D}}^m_{\text{pos}} \cup \hat{\mathcal{D}}^m_{\text{pos}}$. The proof proceeds in five steps.

Step 1. Our first aim is to prove that

$$\hat{\mathcal{D}}^m_{\text{pos}} = \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) \cup \mathcal{W}'_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}). \tag{A.9}$$

From (4.11) we have $\mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) \cup \mathcal{W}'_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) \subsetneq \hat{\mathcal{D}}^m_{\text{pos}}$. Thus we reduce our proof to show that $\sigma \in \hat{\mathcal{D}}^m_{\text{pos}}$ implies $\sigma \in \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) \cup \mathcal{W}'_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. Note that this implication is not straightforward, since given $\sigma \in \hat{\mathcal{D}}^m_{\text{pos}}$, the boundary of the polyomino $C^1(\sigma)$ could intersect the other three sides of the boundary of its smallest surrounding rectangle $R(C^1(\sigma))$ in a proper subsets of the sides itself, see Fig. 15(d) for an illustration of this hypothetical case. Hence, consider $\sigma \in \hat{\mathcal{D}}^m_{\text{pos}}$ and let $R(C^1(\sigma)) = R_{(\ell^*+a)\times(\ell^*+b)}$ with $a, b \in \mathbb{Z}$. In view of the proof of Lemma 6.1 we have that $C^1(\sigma)$ is a minimal polyomino and by [33, Lemma 6.16] it is also convex and monotone, i.e., its perimeter of value $4\ell^*$ is equal to the one of $R(C^1(\sigma))$. Hence, the following equality holds

$$4\ell^* = 4\ell^* + 2(a+b). \tag{A.10}$$

In particular, (A.10) is satisfied only by $a = -b$. Now, let $\tilde{R}$ be the smallest rectangle surrounding the polyomino, say $\tilde{C}^1(\sigma)$, obtained by removing the unit protuberance from $C^1(\sigma)$. If $C^1(\sigma)$ has the unit protuberance adjacent to a side of length ℓ^*+a, then $\tilde{R}$ is a rectangle $(\ell^*+a)\times(\ell^*-a-1)$. Note that $\tilde{R}$ must have an area larger than or equal to the number of spins 1 of the polyomino $\tilde{C}^1(\sigma)$, that is $\ell^*(\ell^*-1)$. Thus, we have

$$\text{Area}(\tilde{R}) = (\ell^*+a)(\ell^*-a-1) = \ell^*(\ell^*-1) - a^2 - a \geq \ell^*(\ell^*-1) \iff -a^2 - a \geq 0. \tag{A.11}$$

Since $a \in \mathbb{Z}$, $-a^2 - a \geq 0$ is satisfied only if either $a = 0$ or $a = -1$. Otherwise, if $C^1(\sigma)$ has the unit protuberance adjacent to a side of length $\ell^* - a$, then $\tilde{R}$ is a rectangle $(\ell^*+a-1)\times(\ell^*-a)$ and

$$\text{Area}(\tilde{R}) = (\ell^*+a-1)(\ell^*-a) = \ell^*(\ell^*-1) - a^2 + a \geq \ell^*(\ell^*-1) \iff -a^2 + a \geq 0. \tag{A.12}$$

Since $a \in \mathbb{Z}$, $-a^2 + a \geq 0$ is satisfied only if either $a = 0$ or $a = 1$. In both cases we get that $\tilde{R}$ is a rectangle of side lengths ℓ^* and ℓ^*-1. Thus, if the protuberance is attached to one of the longest sides of $\tilde{R}$, then $\sigma \in \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$, otherwise $\sigma \in \mathcal{W}'_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. In any case we conclude that (A.9) is satisfied.

Step 2. For any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$ and for any path $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}}$, let

$$g_m(\omega) := \{i \in \mathbb{N} : \omega_i \in \mathcal{F}(\mathcal{D}^m_{\text{pos}}),\ N_1(\omega_{i-1}) = \ell^*(\ell^* - 1),\ N_m(\omega_{i-1}) = |\Lambda| - \ell^*(\ell^* - 1)\}. \tag{A.13}$$

We claim that $g_m(\omega) \neq \emptyset$. Let $\omega = (\omega_0, \dots, \omega_n) \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}}$ and let $j^* \le n$ be the smallest integer such that after j^* the path leaves $\mathcal{D}^{m,+}_{\text{pos}}$, i.e., $(\omega_{j^*}, \dots, \omega_n) \cap \mathcal{D}^{m,+}_{\text{pos}} = \emptyset$. Since ω_{j^*-1} is the last configuration in $\mathcal{D}^{m,+}_{\text{pos}}$, it follows that $\omega_{j^*} \in \mathcal{D}^m_{\text{pos}}$ and, by the proof of Corollary 6.6, we have that $\omega_{j^*} \in \mathcal{F}(\mathcal{D}^m_{\text{pos}})$. Moreover, since ω_{j^*-1} is the last configuration in $\mathcal{D}^{m,+}_{\text{pos}}$, we have that $N_m(\omega_{j^*-1}) = |\Lambda| - \ell^*(\ell^* - 1)$ and ω_{j^*} is obtained by ω_{j^*-1} by flipping a spin m from m to $s \neq m$. Note that $N_m(\omega_{j^*-1}) = |\Lambda| - \ell^*(\ell^* - 1)$ implies $N_s(\omega_{j^*-1}) \le \ell^*(\ell^* - 1)$ for any $s \in S \setminus \{m\}$. By Lemma 6.1, $\omega_{j^*} \in \mathcal{F}(\mathcal{D}^m_{\text{pos}})$ implies $N_1(\omega_{j^*}) = \ell^*(\ell^* - 1) + 1$, thus $N_1(\omega_{j^*-1}) < \ell^*(\ell^* - 1)$ is not feasible since ω_{j^*} and ω_{j^*-1} differ by a single spin update which increases the number of spins 1 of at most one. Then, $j^* \in g_m(\omega)$ and the claim is proved.

Step 3. We claim that for any path $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}}$ one has $\omega_i \in \hat{\mathcal{D}}^m_{\text{pos}}$ for any $i \in g_m(\omega)$. We argue by contradiction. Assume that there exists $i \in g_m(\omega)$ such that $\omega_i \notin \hat{\mathcal{D}}^m_{\text{pos}}$ and $\omega_i \in \tilde{\mathcal{D}}^m_{\text{pos}}$. Since ω_{i-1} is obtained from ω_i by flipping a spin 1 to m and since any configuration belonging to $\tilde{\mathcal{D}}^m_{\text{pos}}$ has all the spins 1 with at least two nearest neighbors with spin 1, using (2.8) we have

$$H_{\text{pos}}(\omega_{i-1}) - H_{\text{pos}}(\omega_i) \ge (2 - 2) + h = h > 0. \tag{A.14}$$

In particular, from (A.14) we get a contradiction. Indeed,

$$\Phi^{\text{pos}}_\omega \ge H_{\text{pos}}(\omega_{i-1}) > H_{\text{pos}}(\omega_i) = H_{\text{pos}}(\mathbf{m}) + \Gamma_{\text{pos}}(\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}) = \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}), \tag{A.15}$$

where the equality follows by (6.5). Thus by (A.15) ω is not an optimal path, which is a contradiction, the claim is proved and we conclude the proof of Step 3.

Step 4. Now we claim that for any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}$ and for any path $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}}$,

$$\omega_i \in \mathcal{F}(\mathcal{D}^m_{\text{pos}}) \implies \omega_{i-1}, \omega_{i+1} \notin \mathcal{D}^m_{\text{pos}}. \tag{A.16}$$

Using Corollary 6.6, for any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$ and any path $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}}$ there exists an integer i such that $\omega_i \in \mathcal{F}(\mathcal{D}^m_{\text{pos}})$. Assume by contradiction that $\omega_{i+1} \in \mathcal{D}^m_{\text{pos}}$. In particular, since ω_i and ω_{i+1} have the same number of spins m, note that ω_{i+1} is obtained by flipping a spin 1 from 1 to $t \neq 1$. Since $\omega_i(v) \neq t$ for every $v \in V$, the above flip increases the energy, i.e., $H_{\text{pos}}(\omega_{i+1}) > H_{\text{pos}}(\omega_i)$. Hence, using this inequality and (6.5), we have

$$\Phi^{\text{pos}}_\omega \ge H_{\text{pos}}(\omega_{i+1}) > H_{\text{pos}}(\omega_i) = H_{\text{pos}}(\mathbf{m}) + \Gamma_{\text{pos}}(\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}) = \Phi_{\text{pos}}(\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}), \tag{A.17}$$

which implies the contradiction because ω is not optimal. Thus $\omega_{i+1} \notin \mathcal{D}^m_{\text{pos}}$ and similarly we show that also $\omega_{i-1} \notin \mathcal{D}^m_{\text{pos}}$.

Step 5. In this last step of the proof we claim that for any $\mathbf{m} \in \mathcal{X}^m_{\text{pos}}$ and for any path $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}}$ there exists a positive integer i such that $\omega_i \in \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. Arguing by contradiction, assume that there exists $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}}$ such that $\omega \cap \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) = \emptyset$. Thanks to Corollary 6.6, we know that ω visits $\mathcal{F}(\mathcal{D}^m_{\text{pos}})$ and thanks to Step 4 we have that the configurations along ω belonging to $\mathcal{F}(\mathcal{D}^m_{\text{pos}})$ are not consecutive. More precisely, they are linked by a sub-path that belongs either to $\mathcal{D}^{m,+}_{\text{pos}}$ or $\mathcal{D}^{m,-}_{\text{pos}}$. If n is the length of ω, then let $j \le n$ be the smallest integer such that $\omega_j \in \mathcal{F}(\mathcal{D}^m_{\text{pos}})$ and such that $(\omega_j, \dots, \omega_n) \cap \mathcal{D}^{m,+}_{\text{pos}} = \emptyset$, thus, $j \in g_m(\omega)$ since j plays the same role of j^* in the proof of Step 2. Using (A.9), Step 3 and the assumption $\omega \cap \mathcal{W}_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}}) = \emptyset$, it follows that $\omega_j \in \mathcal{W}'_{\text{pos}}(\mathbf{m}, \mathcal{X}^s_{\text{pos}})$. Moreover, starting from $\omega_j \in \mathcal{F}(\mathcal{D}^m_{\text{pos}})$ the energy along the path decreases only by either

(i) flipping the spin in the unit protuberance from 1 to m, or
(ii) flipping a spin, with two nearest neighbors with spin 1, from m to 1.

Since by the definition of j we have that ω_{j-1} is the last that visits $\mathcal{D}^{m,+}_{\text{pos}}$, $\omega_{j+1} \notin \mathcal{D}^{m,+}_{\text{pos}}$, (i) is not feasible. Considering (ii), we have $H_{\text{pos}}(\omega_{j+1}) = H_{\text{pos}}(\mathbf{m}) + \Gamma_{\text{pos}}(\mathbf{m}, \mathcal{X}^m_{\text{pos}} \setminus \{\mathbf{m}\}) - h$. Starting from ω_{j+1} we consider only moves which imply either a decrease of energy or an increase by at most h. Since $C^1(\omega_{j+1})$ is a polyomino $\ell^* \times (\ell^* - 1)$ with a bar made of two adjacent unit squares on a shortest side, the only feasible moves are

(iii) flipping a spin, with two nearest neighbors with spin m, from m to 1,
(iv) flipping a spin, with two nearest neighbors with spin 1, from 1 to m.

By means of the moves (iii) and (iv), the process reaches a configuration σ in which all the spins are equal to m except those, that are 1, in a connected polyomino $C^1(\sigma)$ that is convex and such that $R(C^1(\sigma)) = R_{(\ell^*+1)\times(\ell^*-1)}$. We cannot repeat the move (iv) otherwise we get a configuration that does not belong to $\mathcal{D}^m_{\text{pos}}$. While applying one time (iv) and iteratively (iii), until we fill the rectangle $R_{(\ell^*+1)\times(\ell^*-1)}$ with spins 1, we get a set of configurations in which the one with the smallest energy is σ such that $C^1(\sigma) \equiv R(C^1(\sigma))$. Moreover, from any configuration in this set, a possible move is reached by flipping from m to 1 a spin m with three nearest neighbors with spin m that implies to enlarge the circumscribed rectangle. This spin-flip increases the energy by $2 - h$.

Thus, we obtain

$$\begin{aligned} \Phi_\omega^{\mathrm{pos}} &\geq 4\ell^* - h(\ell^*+1)(\ell^*-1) + 2 - h + H_{\mathrm{pos}}(\mathbf{m}) \\ &> \Gamma_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^m_{\mathrm{pos}} \backslash \{\mathbf{m}\}) + H_{\mathrm{pos}}(\mathbf{m}) = \Phi_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^m_{\mathrm{pos}} \backslash \{\mathbf{m}\}), \end{aligned} \tag{A.18}$$

which is a contradiction by the definition of an optimal path. Note that the last inequality follows by $2 > h(\ell^*-1)$ since $0 < h < \frac{1}{2}$, see Assumption 4.1. It follows that it is not possible to have $\omega \cap \mathcal{W}_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}}) = \emptyset$ for any $\omega \in \Omega^{opt}_{\mathbf{m}, \mathcal{X}^m_{\mathrm{pos}} \backslash \{\mathbf{m}\}}$, namely $\mathcal{W}_{\mathrm{pos}}(\mathbf{m}, \mathcal{X}^s_{\mathrm{pos}})$ is a gate for this type of transition. □

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