Economic MPC without terminal constraints: Gradient-correcting end penalties enforce asymptotic stability☆

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Abstract

In recent years, Economic MPC (EMPC) has gained popularity due to the promise of increasing performance by directly optimizing the performance index rather than tracking a given steady state. Moreover, EMPC formulations without terminal cost nor constraints are appealing for the simplicity of implementation. However, the stability and convergence analysis for such formulations is rather involved and so far only practical stability (in discrete time), respectively, practical convergence (in sampled-data continuous time) has been proven; i.e., convergence to a horizon-dependent neighborhood of the optimal steady state. In this paper, we prove that, whenever the cost has a non-zero gradient at the optimal steady-state and the MPC formulation satisfies a regularity assumption, nominal stability to the economic optimum cannot be achieved. Consequently, the average performance of EMPC is bound to be worse than that of tracking MPC. We propose to solve this problem by introducing a linear terminal penalty correcting the gradient at steady state. We prove that this simple correction enforces uniform exponential stability of the economically optimal steady state. We illustrate our findings in simulations using three examples.

Keywords: Economic MPC, strict dissipativity, cost rotation, optimal control

1. Introduction

In recent years, there has been a growing interest in generalized formulations of Nonlinear Model Predictive Control (NMPC) beyond the classical control tasks of setpoint stabilization and tracking. This includes schemes with purely *economic* objectives [31, 40, 10, 4, 50] and dual formulations [34, 5], wherein a combination of tracking and economic objectives is considered. The former approach is termed *Economic MPC* (EMPC) in [40]. Instead of designing cost functions in order to solve a given control or stabilization problem, in EMPC user-provided economic objectives are considered and optimized in a receding horizon fashion. In this setting, one aims at directly designing an economic control scheme, thus avoiding the often cumbersome translation of economic objectives into corresponding control tasks, cf. [37, 31].

Recent progress on EMPC includes Lyapunov-based stability results [12], dissipativity-based approaches using terminal constraints [10, 4], and dissipativity-based approaches without end penalties and terminal constraints [23, 26, 14, 13]. In dissipativity-based EMPC approaches without end penalties, one relies on the observation that, under mild reachability assumptions, dissipativity of the Optimal Control Problem (OCP) implies the existence of a turnpike in the open-loop predictions, whereby the turnpike happens to be the optimal steady state [24, 23, 18, 17]. Furthermore, we remark that, again under mild assumptions, the existence of a turnpike implies recursive feasibility of the OCP [14, 15]. We refer to [12, 16] for recent overviews. The main difference between the dissipativity-based approaches with and without terminal constraints is that, in the former, one can establish Lyapunov stability of the optimal steady state (provided that the terminal constraint and penalty are chosen appropriately); while, in the latter, one proves convergence to a neighborhood of the optimal steady state without requiring a priori knowledge of this target.[1]

The present paper tries to close this evident gap between dissipativity-based EMPC with and without terminal constraints in terms of convergence and stability properties. That is, we investigate under which conditions EMPC with terminal penalty and no terminal constraint enforces asymptotic stability of the optimal steady state. It is well known that in the context of tracking MPC this can be achieved by using (global) control Lyapunov functions as end penalties [29]. Here, instead of using control Lyapunov functions, we pursue a different route by designing the terminal penalty so as to correct the gradient of the underlying cost function. This, in turn, corrects the gradient of the underlying steady-state optimization problem. We prove that the required gradient correction can be achieved by a linear end penalty. Further, we prove that this correction

☆Both authors are supported by the Deutsche Forschungsgemeinschaft, Grants WO 2056/1 and WO 2056/4-1. Furthermore, Timm Faulwasser acknowledges further support from the Baden-Württemberg Stiftung under the Elite Programme for Postdocs.

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[1]We remark that the majority of discrete-time results [10, 4, 23] establishes (practical) asymptotic stability properties, while sampled-data continuous-time counterparts typically establish asymptotic convergence due to the use of Barbalat's Lemma [18].

is equivalent to a linear rotation of the stage cost and a zero terminal cost. Our approach combines linearly approximated storage functions, which have been introduced for terminally constrained EMPC in [10], with recent results on EMPC without terminal constraints [14] and recent insights on indefinite LQR control and approximate EMPC [47, 48, 49].

The contributions of this paper are as follows. We begin with a formal investigation the effect of cost rotations on primal-dual solutions of OCPs. We prove that EMPC without terminal constraints nor end penalties can never stabilize the optimal steady state whenever the cost has a non-zero gradient at the optimal steady state and the OCP satisfies a regularity assumption, as per Definition 1. En passant, this shows that non-singular, i.e. regular, OCPs do not exhibit exact turnpikes. Furthermore, we establish a crucial relation between the local geometry of any storage function and the dual variables of the underlying steady-state optimization problem. Finally, we establish sufficient conditions sampled-data continuous-time EMPC with a linear gradient-correcting end penalty uniformly exponentially stabilizes the optimal steady-state.

The remainder of the paper is structured as follows. Section 2 introduces the problem setting and recalls an EMPC convergence result. Section 3 presents the main results and their proofs. Section 4 first illustrates our findings using two simple examples and then applies the developed theory to a practical example from the literature on chemical processes. The paper concludes with a discussion in Section 5.

Notation

The inner product of $x_1, x_2 \in \mathbb{R}^{n_x}$ is written as $\langle x_1, x_2\rangle$. We denote the state and control of a system, respectively, as $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$. We denote partial derivatives of functions by a subscript: e.g. for function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ we use $F_x(v,w) = \frac{\partial F}{\partial x}\Big|_{\substack{x=v\\u=w}}$ and $F_{xx}(v,w) = \frac{\partial^2 F}{\partial x^2}\Big|_{\substack{x=v\\u=w}}$ Moreover, we define $z := (x,u) \in \mathbb{R}^{n_z} = \mathbb{R}^{n_x+n_u}$, and use the shorthand notation $F(z) = F(x,u)$. Finally, we omit the dependence of the function on its variables whenever it is clear from context, especially in the case $F_z(z)$. The notation $x(\cdot, x_0, u(\cdot, x_0))$ refers to a trajectory of $\dot{x} = f(x,u)$ originating at x_0 and driven by $u(\cdot, x_0)$.

2. Problem Statement and Preliminaries

In this paper, we consider sampled-data continuous-time NMPC formulations with economic cost functions. However, the results obtained are expected to essentially hold also in the discrete-time framework, modulo proper adaptations. In the following, we introduce the problem, recall important results and definitions from the literature and establish an equivalence between different problem formulations.

2.1. Nonlinear Model Predictive Control

Consider the dynamic system given by

$$\dot{x} = f(x,u), \quad x(0) = x_0, \tag{1}$$

	$L^i(x,u)$	$M^i(x(0),x(T))$
OCP^1	$F(x,u)$	$E(x(T))$
OCP^2	$F(x,u) - \langle S_x, f(x,u)\rangle$	$E(x(T)) + S(x(T))$
OCP^3	$F(x,u)$	$E(x(T)) + S(x(0))$

Table 1: Considered cost functionals.

and subject to the mixed state-input constraints $z := (x,u) \in \mathcal{Z} \subset \mathbb{R}^{n_x+n_u}$,

$$\mathcal{Z} = \{z \in \mathbb{R}^{n_x+n_u} \,|\, g_j(z) \leq 0, j \in \mathcal{G}\}, \tag{2}$$

where $\mathcal{G} = \{1, \ldots, n_g\}$ is the index set of the mixed state-input constraints. Occasionally, we use the shorthand notation $g(x,u) = [g_1(x,u), \ldots, g_{n_g}(x,u)]^\top$. We assume w.l.o.g. that $f(0,0) = 0$.

NMPC is based on repeatedly solving a given OCP according to the following strategy:

1. Get the current state x_0 at time t_0;
2. Solve OCP (3);
3. Apply the optimal control law $u^*(\cdot)$ over the time interval $[t_0, t_0+\delta)$, set $t_0 \leftarrow t_0 + \delta$ and go to Step 1.

In this paper, we compare NMPC formulations based on members of the following family of OCPs

$$V^i(x_0) := \min_{x(\cdot),u(\cdot)} \int_0^T L^i(x(t),u(t))\,\mathrm{d}t + M^i(x(0),x(T)) \tag{3a}$$

$$\text{s.t. } x(0) = x_0, \tag{3b}$$

$$\dot{x}(t) = f(x(t),u(t)), \quad t \in [0,T], \tag{3c}$$

$$0 \geq g(x(t),u(t)), \quad t \in [0,T]. \tag{3d}$$

In many engineering applications, the input $u(\cdot)$ is required to be piecewise continuous. Hence, we restrict ourselves to this function space. To avoid cumbersome technicalities, we assume that the problem data of (3) is sufficiently smooth, i.e. at least twice differentiable, and that the minimum exists.[2]

We consider three variants of OCP (3), denoted as OCP^i, $i \in \{1,2,3\} := \mathcal{I}$ differing in the considered cost functionals as listed in Table 1, where $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ denotes the *running cost*, $E : \mathbb{R}^{n_x} \to \mathbb{R}$ denotes what is usually called *terminal cost*. The function $S : \mathbb{R}^{n_x} \to \mathbb{R}$, is typically a *storage function*, which is used to establish closed-loop stability and whose properties will be defined later. Unless explicitly stated otherwise, S should be considered as any differentiable function. Consistently with the literature on EMPC [9, 1], we will call $\hat{F}(x,u) := F(x,u) - \langle S_x, f(x,u)\rangle = L^2(x,u)$ the *rotated cost*. If it exists, we denote the optimal pair of OCP^i, $i \in \mathcal{I}$ as $z^i(\cdot) := (x^i(\cdot), u^i(\cdot))$. Moreover, we denote the corresponding optimal adjoint as $\lambda^i(\cdot)$.

[2] For a detailed investigation of conditions ensuring the existence of minima in OCP^i we refer the interested reader to [32, 45].

Henceforth, OCP^1 is referred to as *original OCP*, OCP^2 as *rotated OCP* [10] and OCP^3 as *OCP with initial penalty* [14]. Typically, OCP^1 is the formulation of a problem in its "natural" form, i.e. as it makes the most sense from a modelling or engineering point of view. OCP^2 and OCP^3 instead, are used in the economic MPC literature order to analyse the stability properties of the closed-loop system. Note that, these OCP formulations are very similar, e.g. the cost of OCP^1 and OCP^3 only differs by a constant term, therefore they must deliver the same primal solution. However, the three formulations also present some important differences which we will highlight, together with the similarities, in Theorem 2. In particular, we will prove that while the primal solutions $z^i(\cdot)$ coincide, the adjoint solutions $\lambda^i(\cdot)$ differ. Particular attention should be dedicated to OCP^3, as the formal definition of OCP^3 does not require any differentiability or continuity properties of S, while OCP^2 does. Moreover, we will prove in Theorem 2 that while the dual solution satisfies $\lambda^3(t) = \lambda^1(t)$, $t \in (0,T]$, it also holds that $\lambda^3(0) \neq \lambda^1(0)$ and $V^3(x_0) = V^2(x_0) \neq V^1(x_0)$. In particular this last fact has been used to prove convergence of EMPC.

2.2. Necessary Optimality Conditions

In our later developments, we rely on the Necessary Conditions of Optimality (NCO) of OCP^i. The set of NCOs considered hereafter require the following technical assumption.

Assumption 1 (Linear independence of $g(x,u)$).
Along any optimal pair $z^i(\cdot)$, $i \in \mathcal{I}$ and for all $t \in [0,T]$, the following constraint qualification holds

$$\text{rank}\left[\text{diag}\left(g(z^i(t))\right) \quad g_u(z^i(t))\right] = n_g.$$

Using this assumption, the NCO of OCP^i can be expressed via the Hamiltonian

$$H^i(z,\lambda,\mu) = L^i(z) + \langle \lambda, f(z) \rangle + \langle \mu, g(z) \rangle. \tag{4}$$

We restrict the consideration to OCPs [33, 44] without terminal constraints; i.e. we consider *normal* OCPs and thus drop the constant adjoint for the Lagrange function L^i. The NCOs can then be stated as follows

$$\dot{x}^i = H^i_\lambda(z^i,\lambda^i,\mu^i), \qquad x^i(0) = x_0, \tag{5a}$$

$$\dot{\lambda}^i = -H^i_x(z^i,\lambda^i,\mu^i), \qquad \lambda^i(T) = \left.\frac{\partial M^i}{\partial x(T)}\right|_{(x_0,x^i(T))}, \tag{5b}$$

$$0 = H^i_u(z^i(t),\lambda^i(t),\mu^i(t)), \tag{5c}$$

$$0 = \langle \mu^i(t), g(z^i(t)) \rangle, \qquad \mu^i_j(t) \geq 0 \tag{5d}$$

where $\lambda^i(\cdot)$ is the adjoint/costate and $\mu^i(\cdot)$, $\mu^i(t) \geq 0$ for all $t \in [0,T]$, is the multiplier function associated with the constraints g.

Here, we are not interested in discussing the most general form of the NCO; we rather aim at keeping the exposition at an accessible level. Hence, we restrict ourselves to cases where the optimal solutions $u^i(\cdot), \lambda^i(\cdot), \mu^i(\cdot)$ are piecewise continuous, respectively, absolutely continuous in case of $x^i(\cdot)$. We remark that most of our results can be extended to more general formulations of OCPs such as problems with pure state constraints. This however requires working with technically cumbersome versions of the Pontryagin Maximum Principle (PMP), cf. [27] for an overview of PMPs for state-constrained OCPs.

We define three steady-state optimization problems SOP^i, $i \in \mathcal{I}$ corresponding to OCP^i, $i \in \mathcal{I}$ as

$$\min_{x,u} \; L^i(x,u) \qquad \text{s.t. } 0 = f(x,u), \qquad g(x,u) \leq 0, \tag{6}$$

where the cost functions are listed in Table 1. If it exists, we denote the optimal pair of SOP^i, $i \in \mathcal{I}$ as $\bar{z}^i = (\bar{x}^i, \bar{u}^i)$. Note that the differentiability assumption on the problem data of OCP^i, $i \in \mathcal{I}$ implies similar smoothness in SOP^i, $i \in \mathcal{I}$. Furthermore, we require regularity in the following sense:

Assumption 2 (Regularity of SOP^i).
Whenever SOP^i*, $i \in \mathcal{I}$ has an optimal solution $\bar{z}^i \in \text{int}\,\mathcal{Z}$, the corresponding dual variables $\bar{\lambda}^i, i \in \mathcal{I}$ are unique; i.e., we assume that the linear independence constraint qualification (LICQ) holds.*

We note that due to $\bar{z}^i \in \text{int}\,\mathcal{Z}$ Assumption 2 refers to LICQ of the steady-state equality constraints, while Assumption 1 requires linear independence of the mixed state-input constraints. In Theorem 2 we will prove that the primal solutions of SOP^i, $i \in \mathcal{I}$ coincide; i.e., $\bar{z}^i = \bar{z}$, while the dual solutions differ. Finally, without loss of generality, we assume that $\bar{z} = 0$. Note that, in case $\bar{z} \neq 0$, one can use the transformed state and control space given by $z - \bar{z}$, such that this assumption is not restrictive.

In the considered setting, the NCO of SOP^i, $i \in \mathcal{I}$ can be stated in terms of the Hamiltonian (4) as

$$0 = H^i_z(\bar{z}^i, \bar{\lambda}^i, \bar{\mu}^i), \tag{7a}$$

$$0 = H^i_\lambda(\bar{z}^i, \bar{\lambda}^i, \bar{\mu}^i), \tag{7b}$$

$$0 = H^i_\mu(\bar{z}^i, \bar{\lambda}^i, \bar{\mu}^i), \tag{7c}$$

$$0 = \bar{\mu}^i_j g_j(\bar{z}^i), \qquad \bar{\mu}^i_j \geq 0, \qquad j = 1,\ldots,n_g. \tag{7d}$$

Henceforth, we denote the optimal dual of the equality constraint $0 = f(x,u)$ in SOP^i, $i \in \mathcal{I}$ as $\bar{\lambda}^i$.

2.3. Useful Notions

For the purpose of this paper, we define regularity of an OCP as follows:

Definition 1 (OCP regularity at the optimal steady-state).
OCP^i*, $i \in \mathcal{I}$ is said to be regular at the steady-state $\bar{z} = (\bar{x},\bar{u}) \in \text{int}\,\mathcal{Z}$ if its Hamiltonian H^i is twice differentiable at $\bar{z}$ and* $\det H^i_{uu}(\bar{z}) \neq 0$*. If additionally* $\det H^i_{uu}(\bar{z}) \succ 0$*, the OCP is said to be regular positive at $\bar{z}$.*[3]

[3] We remark that the regular positivity is needed to enforce satisfaction of sufficient second-order conditions for OCPs, cf. [35, Thm. 2.2]. If one employs other types of sufficient conditions, one could drop this assumption.

In this paper, we will occasionally approximate OCP^1 locally around optimal solutions $\bar{z}^1 = 0 \in \operatorname{int}\mathcal{Z}$ to SOP^1 by the linear-quadratic problem:

$$\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T \frac{1}{2} z(t)^\top W z(t) + w^\top z(t)\,\mathrm{d}t \\
& \qquad + \frac{1}{2} x(T)^\top P_T x(T) + x^\top p_T(T) && \text{(8a)}\\
\text{s.t.} \quad & x(0) = x_0, && \text{(8b)}\\
& \dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0,T], && \text{(8c)}\\
& Cx(t) + Du(t) - g(\bar{z}^1) \le 0, \quad t \in [0,T]. && \text{(8d)}
\end{aligned}$$

where the linear dynamics and path constraints are defined via the Jacobians

$$A = f_x, \qquad B = f_u, \qquad C = g_x, \qquad D = g_u, \tag{8e}$$

and the quadratic objective is given by $W = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$, $w = \begin{bmatrix} q \\ r \end{bmatrix}$, with

$$\begin{aligned}
& Q = H^1_{xx}, \quad S = H^1_{xu}, \quad R = H^1_{uu}, \quad q = L^1_x, \quad r = L^1_u, && \text{(8f)}\\
& P_T = E_{xx}, \quad p_T = E_x, && \text{(8g)}
\end{aligned}$$

all evaluated at $\bar{z}^1, \bar{\lambda}^1$ and $\bar{\mu}^1 = 0$. Whenever necessary, in order to clearly distinguish OCP^1 from the LQ OCP (8), we will denote the latter as $\mathrm{OCP}^1_{\mathrm{LQ}}$. The same notation—i.e., the addition of the subscript LQ—is used for the linear-quadratic approximation of SOP (6).

The NCOs of $\mathrm{OCP}^1_{\mathrm{LQ}}$ (8) coincide with a form of linearisation of the NCOs (5) of OCP^1 evaluated at the optimal steady state, where in (5d) function g is linearized but the inner product is not, to obtain $0 = \langle \mu^i(t), Cx(t) + Du(t) - g(\bar{z}^1)\rangle$. This is similar to the discrete-time case studied in [48]. Note that, by assumption $g(\bar{z}^1) < 0$, and no path constraint is active at the optimal steady state. It is also worth to be noted that this time-invariant approximation is similar but not identical to the classical second variation, which is usually stated as a time-varying LQ approximation around optimal trajectories, cf. [6]. Moreover, in second variations one usually does not have the linear term in the cost (8a), see cf. [6, 35] for the time-varying case. In this paper, similar to [44], we will use the LQ Problem (8) to characterize the situation in which the solution of the original OCP (3) remains at steady-state when $x(0) = \bar{x}$. Then, Problem (8) yields an LQ approximation around the optimal stationary trajectory $z(t) \equiv \bar{z}$.

Consider the LQ approximation (8) and denote the value of time-varying variables, computed for a prediction horizon T, evaluated at time t by $\xi(t,T)$ with $\xi \in \{u,p,P,K\}$.

Lemma 1 (Indefinite LQR solution).
Consider the unconstrained version of $\mathrm{OCP}^1_{\mathrm{LQ}}$ (8). *Suppose that* (A,B) *is stabilizable and let the end penalty in* (8a) *be such that* $P_T = P_T^\top$. *Then the optimal input is given by*

$$u^1_{LQ}(t,T) = -K(t,T)x^1_{LQ}(t) - R^{-1}(B^\top p(t,T) + r), \tag{9a}$$

where

$$\begin{aligned}
K(t,T) &= R^{-1}(S + B^\top P(t,T)), && \text{(9b)}\\
-\dot{P}(t,T) &= A^\top P(t,T) + P(t,T)A + Q - (P(t,T)B + S)K(t,T), && \text{(9c)}\\
-\dot{p}(t,T) &= \left(A^\top - K(t,T)^\top B^\top\right) p(t,T) + q - K(t,T)^\top r, && \text{(9d)}\\
P(T,T) &= P_T, \qquad p(T,T) = p_T; && \text{(9e)}
\end{aligned}$$

and the optimal value function is given by $V(x_0) = \frac{1}{2} x_0^\top P(0,T) x_0 + \langle p(0,T), x_0\rangle$.

PROOF. The proof of this result is given in Appendix A. □

The next lemma recalls conditions under which $\mathrm{OCP}^1_{\mathrm{LQ}}$ (8) yields a first-order approximation of the solution of OCP^1, cf. [35, Theorem 3.1].

Lemma 2 (Local LQ approximation of OCP^1).
Let OCP^1 *be regular positive at* $\bar{z}$ *in the sense of Definition 1 and let* $z(t) \equiv \bar{z}$ *be the optimal solution to* OCP^1 *for* $x(0) = \bar{x}$. *Assume that* F *and* f *are at least* C^2 *in a neighborhood of* $\bar{z}$.

Then, $\mathrm{OCP}^1_{\mathrm{LQ}}$ (8) *with initial constraint* $x(0) = x_0$ *(and* x_0 *in a neighborhood of* $\bar{x}$*) yields a first-order approximation of the solution of* OCP^1*, i.e.*

$$a^1(t) = a^1_{\mathrm{LQ}}(t) + O(\|x_0 - \bar{x}\|^2), \quad \text{with } a \in \{x,u,\lambda\}. \tag{10}$$

Occasionally, we will require the following properties of functions $G : \mathbb{R}^{n_x} \to \mathbb{R}^m$ evaluated along optimal trajectories of OCP^i.

Definition 2 (Absolute continuity along optimal pairs).
Consider OCP^i*,* $i \in \mathcal{I}$ *for some horizon* $T > 0$. *A function* $G : \mathbb{R}^{n_x} \to \mathbb{R}^m$ *is said to be* absolutely continuous along optimal pairs $z^i(\cdot)$ of OCP^i, $i \in \mathcal{I}$, *if* $\gamma : [0,T] \to \mathbb{R}^m$, $\gamma(t) := G(x^i(t))$ *is such that*

(i) the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = G_x f(x^i(t), u^i(t))$ *exist almost everywhere on* $[0,T]$,

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$ *is componentwise Lebesgue integrable and*

$$\gamma(t_2) = \gamma(t_1) + \int_{t_1}^{t_2} G_x(x^i(t)) f(x^i(t), u^i(t))\,\mathrm{d}t$$

holds, for any interval $[t_1,t_2] \subseteq [0,T]$.

Essentially, the above definition is satisfied for differentiable functions G and absolutely continuous optimal state trajectories $x^i(\cdot)$. In case that G is not continuously differentiable the definition requires that the set of time points for which optimal pairs $z^i(\cdot)$ are at any point of non-differentiability of G is of measure zero. One can regard this as an extension of absolute continuity of trajectories to absolute continuity of state-dependent functions evaluated along trajectories.

2.4. Economic and Tracking NMPC

We introduce and discuss next the definition of *economic* and *tracking* MPC that we will use throughout the paper.

Definition 3 (Tracking and economic MPC). *Let $\bar{z} \in \operatorname{int}\mathcal{Z}$ be the optimal steady-state from SOP* (6)*. We say a predictive control scheme is a* tracking MPC *(TMPC) if*

$$L^i(\bar{z}) < L^i(z), \quad \forall z \in \mathcal{Z} \setminus \bar{z}. \tag{11}$$

We label as economic MPC *(EMPC) those MPC schemes for which* (11) *does not hold, i.e. the cost function does not have a strict (global) minimum at the optimal steady state.*[4]

Note that (11) implies that $F(z)$ is a positive-definite function, and thus it is lower bounded by $\alpha^i \in \mathcal{K} : \alpha^i(\|z-\bar{z}\|) \leq L^i(z)$; moreover, $\bar{z} = \operatorname{argmin}_{z\in\mathcal{Z}} L^i(z)$. Finally, Equation (11) implies $\bar{\lambda}^1 = 0$. Therefore, the condition $\bar{\lambda}^1 \neq 0$ implies that removing the constraint $0 = f(x,u)$ from SOP^i (6) would yield a lower cost. This, in turn, implies that the MPC scheme is economic, though the converse is in general not true. In the following, whenever it will be necessary to clearly distinguish tracking and economic MPC formulations, we will denote the tracking stage cost as F^{t}, while F will refer to economic stage costs.

In the sense of Definition 3, tracking and economic MPC differ in how stability can be proven. In classical tracking MPC approaches one can enforce stability in different ways. One typical option is to choose E as a local control Lyapunov function and enforce additional terminal constraints, see e.g. [19, 36, 7, 25, 41]. If no terminal constraints are imposed, the terminal cost E can still be chosen as global control Lyapunov function guaranteeing closed-loop stability [30, 29]. However, as control Lyapunov functions are often difficult to compute, in practice a common choice is $E(x) = 0$ and no terminal constraint is enforced. In this case, asymptotic stability can still be proven for TMPC provided that the prediction horizon is long enough and certain controllability assumptions hold [25, 42].

In contrast to TMPC, stability results for EMPC often require that there exists a function $S : \mathbb{R}^{n_x} \to \mathbb{R}$ satisfying the following strict dissipation inequality

$$S(x(T)) - S(x_0) \leq \int_0^T -\alpha(\|z-\bar{z}\|) + F(z(t)) - F(\bar{z})\ \mathrm{d}t, \tag{12}$$

where α is of class $\mathcal{K}$, see [3, 10, 1, 23, 14].[5] Hence S is called *storage function*. It has been observed in [23, 17, 8] that dissipativity combined with reachability of the best steady state implies the existence of a turnpike property in the OCP. Furthermore, recent works [24, 18] show that, under certain assumptions, dissipativity of the OCP is very close to being equivalent to the existence of a turnpike in the OCP.[6]

Assuming w.l.o.g. that $F(\bar{z}) = 0$, an important consequence of (12), enabling stability proofs for EMPC, is that $\int_0^T L^2(x,u)\ \mathrm{d}t \geq \int_0^T \alpha(\|z-\bar{z}\|)\ \mathrm{d}t$. Note that, F and f being continuous, this implies that $\nabla_z L^2|_{z=\bar{z}} = 0$, so that the storage function must have a specific slope. We will provide a formal proof in Theorem 4.

By relying on S being a storage function, using $E(x) = 0$, and considering no terminal constraint, it can be shown that the closed-loop system converges to a neighborhood of the optimal steady-state $\bar{x}$, cf. [14, 23, 26]. In other words, one establishes practical convergence in this case as summarized next:[7]

Theorem 1 (Practical convergence of EMPC [14]). *Consider an EMPC controller based on* OCP^3 *with $T < \infty$, and $E(x) = 0$. Suppose that*

- *A1 for all $x_0 \in \mathcal{X}_0$, the strict dissipation inequality* (12) *holds along all optimal pairs $z(\cdot, x_0)$;*
- *A2 the optimal steady steady is admissibly reachable in finite time from $x_0 \in \mathcal{X}_0$;*
- *A3 the Jacobian linearization of* (1) *at $\bar{z} \in \operatorname{int}\mathcal{Z}$, (A,B), is controllable.*

Then, there exists a finite horizon $T < \infty$, a sampling period $\delta > 0$, and a constant $\rho(T,\delta) > 0$ such that OCP^3 *is recursively feasible and*

$$\lim_{t\to\infty} \left(\max\{\|x(t,x_0,u_{EMPC}(\cdot)) - \bar{x}\|, \rho(T,\delta)\} \right) = \rho(T,\delta), \tag{13}$$

holds for all $x_0 \in \mathcal{X}_0$, and furthermore $\lim_{T,\frac{1}{\delta}\to\infty} \rho(T,\delta) = 0$.

The proof of this result is given in [14] and thus omitted.[8] It relies on the existence of a turnpike at $\bar{z}$, which is implied by the dissipativity and reachability assumptions. The turnpike allows one to conclude that, for a sufficiently long horizon T, the open-loop predictions will stay close to $\bar{z}$ during large parts of the horizon. However, they may leave the neighborhood of $\bar{z}$ towards the end of the horizon. Furthermore, it is important to note that the size of the neighborhood to which the closed-loop EMPC solutions eventually converge, $\rho(T,\delta)$, shrinks with increasing horizon length and decreasing sampling period δ. Naturally, it is fair to ask, if this result for OCP^3 carries over to

[4] In principle, this notion of EMPC is as general as the term *nonlinear system*. Yet, with this definition we aim at characterizing schemes where some generic stage cost L, which happens to be not a tracking cost, *is given* instead of *being designed*.

[5] It is worth to be remarked that while the early papers [3, 10, 1, 23] require dissipativity along all admissible trajectories, the recent paper [14] shows that satisfaction of the dissipation inequality (12) is only required along optimal trajectories. Furthermore, we remark that [3, 10, 1, 23] require strict dissipativity with respect to $\alpha(\|x-\bar{x}\|)$, while [14] requires $\alpha(\|z-\bar{z}\|)$. For more details on the implications of this subtle difference, we refer to [24, 18].

[6] In order establish converse {turnpike, dissipativity} results [24, 18] as well as [39] require non-negativity of S. However, it should be noted that the EMPC stability proofs typically do not require this. Moreover, on compact sets the non-negativity does not pose any restriction as one can always add a constant to shift the storage S.

[7] We remark that [26] establishes practical stability, while the sampled-data result [14] shows practical convergence.

[8] The proof given in [14] relies on a regularity property of the underlying turnpike. It is, however, straightfoward to show that the respective assumption holds by considering the rotated problem in a neighborhood of $\bar{x}$. Moreover, [18, Thm. 2] implies that finite-time reachability in A2 can be relaxed to exponential reachability.

OCP^i, $i \in \{1,2\}$. This question will be answered by Theorem 2 in Section 3.

In the context of this paper, it is important to note that the existence of a turnpike in the open-loop predictions is not affected by end penalties, cf. the proof of [18, Thm. 2]. Yet, we remark that the existence of a leaving arc will be affected by end penalties. The next result is a direct consequence of [18, Thm. 2] and Theorem 1 showing that, as long the horizon T is sufficiently large, adding an end penalty does not jeopardize the convergence to a neighborhood of $\bar{z}$.[9]

Corollary 1 (Terminal penalties preserve stability). *Let Conditions A1-A3 of Theorem 1 hold and consider a twice-differentiable end penalty $E(x) \neq 0$ in OCP^3. Then, there exists a finite horizon $T > \infty$, a sampling period $\delta > 0$, and a constant $\rho(T, \delta) > 0$ such that* (13) *holds for the closed EMPC loop.*

3. Main Results

The main question we answer in this paper is:

> *Under which conditions does the linear end penalty, $E(x) = x^\top p_T$, in $\mathrm{OCP}^i, i \in \mathfrak{I}$ enforce stability of the optimal steady state $\bar{x}$?*

Throughout our derivations, it will become clear that the addition of the linear end penalty $E(x) = x^\top p_T$ is equivalent to the linear rotation of the stage cost given by $S(x) = -x^\top p_T$ with a zero terminal cost; i.e., eventually we consider an OCP with Lagrange term $\hat{F}(x,u) = F(x,u) - \langle p_T, f(x,u)\rangle$ and no Mayer term ($\hat{E}(x) = 0$).

To establish the result, we first show that general rotations of the cost of an OCP do not affect its primal solutions. Furthermore, we verify that, provided that regularity of the OCP holds at the optimal steady state pair $\bar{z}$, without any terminal constraint or terminal penalty, the optimal steady state $\bar{z}$ is not an equilibrium of the closed-loop system arising from EMPC. Thereafter, we turn towards the relation between the geometry of the storage functions and the dual variables of the SOP. Finally, we prove that linear end penalties allow the recovery of closed-loop stability of the optimal steady state, provided that p_T is chosen appropriately.

3.1. Invariance of Rotated OCPs

Theorem 2 (Rotation invariance of primal solutions). *For any function $S : \mathbb{R}^{n_x} \to \mathbb{R}$, which is absolutely continuous along optimal pairs, the families of OCPs* (3) *and SOPs* (6) *have the following properties:*

(i) If $(x^i(\cdot), u^i(\cdot))$ is optimal for OCP^i, $i \in \mathfrak{I}$, then it is optimal for OCP^j, $j \in \mathfrak{I}$.

(ii) The value functions $V^i(x_0)$ of OCP^i, $i \in \mathfrak{I}$ satisfy

$$V^1(x_0) + S(x_0) = V^2(x_0) = V^3(x_0). \tag{14}$$

(iii) Additionally, suppose that Assumption 2 holds, let S be differentiable at $\bar{x}^2$, and let $(\bar{x}^i, \bar{u}^i) \in \operatorname{int} \mathcal{Z}$ and $\bar{\lambda}^i$ be the optimal primal solution, respectively, the dual solution of SOP^i, $i \in \mathfrak{I}$. Then $(\bar{x}^i, \bar{u}^i)$ is also optimal for SOP^j, $j \in \mathfrak{I}$. Moreover, $\bar{\lambda}^1 = \bar{\lambda}^2 - S_x(\bar{x}^2) = \bar{\lambda}^3$.

(iv) Let $\lambda^i(\cdot)$ be the optimal (piecewise continuous) adjoint for OCP^i, $i \in \mathfrak{I}$, let Assumption 1 hold, and let S be continuously differentiable and S_x is absolutely continuous along optimal pairs, then

$$\lambda^1(t) = \lambda^2(t) - S_x(x^2(t)) = \lambda^3(t) \tag{15}$$

holds almost everywhere on $(0, T]$.

Proof. Claim (i) and Claim (ii): Because of the constraint $x(0) = x_0$, adding the term $S(x(0))$ to the cost functional (3a) only shifts it by a constant value. Therefore, $z^1(\cdot) = z^3(\cdot)$ and $V^1(x_0) + S(x_0) = V^3(x_0)$.

Moreover, $\int_0^T -\langle S_x, f(x,u)\rangle \,\mathrm{d}t = S(x(0)) - S(x(T))$ by Definition 2, so that

$$\int_0^T F(x,u) - \langle S_x, f(x,u)\rangle \,\mathrm{d}t + S(x(T)) = \int_0^T F(x,u)\,\mathrm{d}t + S(x(0)).$$

Therefore, $z^2(\cdot) = z^3(\cdot)$ and $V^2(x_0) = V^3(x_0)$.

Claim (iii): The first part of Claim (iii) is an immediate consequence of the steady-state constraint of Problem (6), i.e. $f(x,u) = 0$. This entails that, for all feasible (x,u), the cost is given by $L^i(x,u) = F(x,u)$, $i \in \mathfrak{I}$, such that the primal solutions of the three problems coincide.

The second part of the claim is obtained using Assumption 2 and $\bar{z}^i \in \operatorname{int} \mathcal{Z}$ by writing the NCOs for the three SOPs. Because (7b)-(7d) coincide for the three formulations, we only detail (7a):

$$\begin{aligned}
\mathrm{SOP}^1: &\quad & F_x + \langle f_x, \bar{\lambda}^1\rangle &= 0,\\
&& F_u + \langle f_u, \bar{\lambda}^1\rangle &= 0.\\
\mathrm{SOP}^2: && F_x - \langle S_{xx}, f(x,u)\rangle + \langle f_x, \bar{\lambda}^2 - S_x\rangle &= 0,\\
&& F_u + \langle f_u, \bar{\lambda}^2 - S_x\rangle &= 0.\\
\mathrm{SOP}^3: && F_x + \langle f_x, \bar{\lambda}^3\rangle &= 0,\\
&& F_u + \langle f_u, \bar{\lambda}^3\rangle &= 0.
\end{aligned}$$

Since $0 = f(x,u)$ it follows immediately that $\bar{\lambda}^1 = \bar{\lambda}^2 - S_x(\bar{x}) = \bar{\lambda}^3$.

Claim (iv): NCOs (5a) and (5d) coincide for the three OCPs. Consider the adjoint equations of $\mathrm{OCP}^i, i \in \mathfrak{I}$; i.e., consider NCO (5b)

$$\begin{aligned}
\dot{\lambda}^i &= -F_x - \langle f_x, \lambda^i\rangle - \langle g_x, \mu^i\rangle, \quad i = 1,3\\
\dot{\lambda}^2 &= -F_x + \langle S_{xx}, f\rangle + \langle f_x, S_x - \lambda^2\rangle - \langle g_x, \mu^2\rangle,
\end{aligned}$$

with the respective terminal conditions

$$\begin{aligned}
\lambda^i(T) &= E_x(x^i(T)), \quad i = 1,3,\\
\lambda^2(T) &= E_x(x^2(T)) + S_x(x^2(T)).
\end{aligned}$$

[9] One can also easily show that the size of the neighborhood is not affected by the end penalty. Due to space limitations, we do not investigate this in detail.

Using Definition 2, we have that $\frac{\mathrm{d}}{\mathrm{d}t}[S_x] = \langle S_{xx}, f\rangle$ holds almost everywhere. Hence, we rewrite the adjoint dynamics of OCP^2

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda^2 - S_x\right] = -F_x - \langle f_x, \lambda^2 - S_x\rangle + \langle g_x, \mu^2\rangle,$$
$$\lambda^2(T) - S_x(x^2(T)) = E_x(x^2(T)).$$

Now, recall that the already proven Claim (i) states $z^1(\cdot) = z^2(\cdot) = z^3(\cdot)$. This implies that (a) NCO (5c) must coincide for the three OCPs and (b) $\mu^1(\cdot) = \mu^2(\cdot) = \mu^3(\cdot)$. Therefore, we conclude that, almost everywhere on $t \in (0,T]$, we have $\lambda^1(t) = \lambda^2(t) - S_x(x^2(t)) = \lambda^3(t)$. $\square$

It is worth to stress that, due to the penalty on the initial condition in OCP^3, $S(x(0))$, the adjoint variable $\lambda^3(\cdot)$ is discontinuous at $t = 0$. More precisely, by differentiation of (14) one obtains $\lambda^1(0) + S_x(x_0) = V_x^1(x_0) + S_x(x_0) = V_x^3(x_0) = \lambda^3(0)$. Hence, the equivalence (15) does not hold at $t = 0$.

We remark that the proof of Theorem 2 can be easily extended to OCPs that do not satisfy Assumption 1. However, this implies working with more technical versions of the PMP.

Theorem 2 is particularly important for the developments of this paper as it states that SOP^i and OCP^i yield the same primal solution when formulated using any of the stage costs L^i, $i \in \mathcal{I}$. In case of the rotated OCP, i.e. when the stage cost L^2 is considered, in order to obtain the exact same primal solution, the addition of the term $S(x(T))$ to the terminal cost is necessary, cf. Table 1. We remark that the result also holds in case S does not satisfy the dissipation inequality (12). In other words, Theorem 2 shows that rotating the objective by means of *any function S, which is absolutely continuous along optimal pairs,* does not change the primal solutions.

3.2. Closed-loop Stability of EMPC at Optimal Steady States

Theorem 3 (EMPC is not stabilizing the system to $\bar{z}$). *Consider an EMPC controller based on* OCP^i, $i \in \mathcal{I}$ *with* $T < \infty$, $E(x) = 0$ *and no terminal constraint. Let Assumption 2 hold. Furthermore, suppose that*

1. OCP^i, $i \in \mathcal{I}$ *is regular positive at* $\bar{z} \in \operatorname{int}\mathcal{Z}$*;*
2. $\bar{\lambda}^1 \neq 0$, *;i.e., the cost has a non-zero gradient at the optimal steady state* $\bar{z}$*, which implies that the scheme is not of tracking type;*
3. *the Jacobian linearization of* (1) *at* $\bar{z} \in \operatorname{int}\mathcal{Z}$, (A,B)*, is controllable.*

Then, the EMPC controller cannot stabilize the system to $\bar{z}$*.*

Before proving Theorem 3, we turn to the easier linear-quadratic case (8), with generic data

$$\dot{x} = Ax + Bu, \qquad F(z) = \frac{1}{2}z^\top W z + w^\top z, \tag{16a}$$
$$W = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}, \qquad w = \begin{bmatrix} q \\ r \end{bmatrix}. \tag{16b}$$

Lemma 3 (Linear EMPC is not stabilizing the system to $\bar{z}$). *Consider* $\mathrm{OCP}^i_{\mathrm{LQ}}, i \in \mathcal{I}$ *with the problem data from* (16)*. Suppose that Assumption 2 holds and that*

1. *the optimal steady state is* $\bar{z} = 0$ *and all* $\mathrm{OCP}^i_{\mathrm{LQ}}, i \in \mathcal{I}$ *are regular at* $\bar{z} \in \operatorname{int}\mathcal{Z}$*;*
2. (A,B) *is controllable.*

Then, whenever $\bar{\lambda}^1_{\mathrm{LQ}} \neq 0$*, any EMPC scheme based on* $\mathrm{OCP}^i_{\mathrm{LQ}}, i \in \mathcal{I}$ *with* $E(x) = 0$ *and without additional terminal constraints does not stabilize the system at the optimal steady state* $\bar{z}$*.*

Proof. Since $\mathrm{OCP}^i_{\mathrm{LQ}}, i \in \{2,3\}$ and $\mathrm{OCP}^1_{\mathrm{LQ}}$ have identical primal solutions, cf. Theorem 2, it suffices to consider $\mathrm{OCP}^1_{\mathrm{LQ}}$. In order to prove the Lemma, we show that, when starting at the initial state $x_{\mathrm{LQ}}(0) = \bar{x}$, the condition $\dot{x}^1_{\mathrm{LQ}}(0) = Ax_{\mathrm{LQ}}(0) + Bu^1_{\mathrm{LQ}}(0) = 0$ cannot hold along optimal solutions. In other words, we prove that the open-loop optimal prediction leaves the steady state immediately with non-zero velocity, such that the closed-loop system instantaneously moves away from the optimal steady state. The proof proceeds in two steps: first we recall the implications of the stated assumptions for $\mathrm{OCP}^1_{\mathrm{LQ}}$, then we prove the assertion for $\mathrm{OCP}^1_{\mathrm{LQ}}$.

Step 1: By assumption there are no active constraints at the optimal steady state. Hence the optimality conditions of $\mathrm{SOP}^1_{\mathrm{LQ}}$ read

$$0 = \begin{bmatrix} A & 0 & B \\ -Q^\top & -A^\top & -S \\ S^\top & B^\top & R \end{bmatrix} \begin{bmatrix} \bar{x}^1_{\mathrm{LQ}} \\ \bar{\lambda}^1_{\mathrm{LQ}} \\ \bar{u}^1_{\mathrm{LQ}} \end{bmatrix} + \begin{bmatrix} 0 \\ -q \\ r \end{bmatrix}. \tag{17}$$

Setting w.l.o.g. $\bar{z}^1_{\mathrm{LQ}} = 0$, we obtain

$$0 = A^\top \bar{\lambda}^1_{\mathrm{LQ}} + q, \qquad 0 = \bar{u}^1_{\mathrm{LQ}} = -R^{-1}(r + B^\top \bar{\lambda}^1_{\mathrm{LQ}}). \tag{18a}$$

Note that regularity, i.e. $\det\, H_{uu}(\bar{z}^1_{\mathrm{LQ}}) \neq 0$, implies that R^{-1} exists. Moreover, for $\mathrm{OCP}^1_{\mathrm{LQ}}$ regularity entails that

$$u^1_{\mathrm{LQ}} = 0 \qquad \Leftrightarrow \qquad r + B^\top \lambda^1_{\mathrm{LQ}} = 0. \tag{18b}$$

Starting at $x_{\mathrm{LQ}}(0) = \bar{x}^1_{\mathrm{LQ}} = 0 \in \operatorname{int}\mathcal{X}$, the optimality conditions of $\mathrm{OCP}^1_{\mathrm{LQ}}$ entail

$$-\dot{\lambda}^1_{\mathrm{LQ}} = Qx^1_{\mathrm{LQ}} + Su^1_{\mathrm{LQ}} + q + A^\top \lambda^1_{\mathrm{LQ}}, \tag{19a}$$
$$u^1_{\mathrm{LQ}} = -R^{-1}(r + B^\top \lambda^1_{\mathrm{LQ}}), \tag{19b}$$

and the transversality condition $\lambda^1_{\mathrm{LQ}}(T) = 0$.

Step 2: For the sake of contradiction, assume that for $x_{\mathrm{LQ}}(0) = \bar{x}^1_{\mathrm{LQ}} \in \operatorname{int}\mathcal{X}$, the optimal pair $z^1_{\mathrm{LQ}}(\cdot, \bar{x}^1_{\mathrm{LQ}})$ remains at the steady state $(\bar{x}^1_{\mathrm{LQ}}, \bar{u}^1_{\mathrm{LQ}})$ for some non-vanishing interval $[0,\tau]$. Combining (18) with (19) and $x_{\mathrm{LQ}}(0) = \bar{x}^1_{\mathrm{LQ}} = 0$, we obtain that the optimal pair $z^1_{\mathrm{LQ}}(\cdot, \bar{x}^1_{\mathrm{LQ}})$ remains at $(\bar{x}^1_{\mathrm{LQ}}, \bar{u}^1_{\mathrm{LQ}})$ if and only if

$$-\dot{\lambda}^1_{\mathrm{LQ}} = A^\top \lambda^1_{\mathrm{LQ}} + q, \tag{20a}$$
$$-r = B^\top \lambda^1_{\mathrm{LQ}}, \tag{20b}$$

holds on $[0,\tau]$. Observe that $\bar{x}^1 \in \operatorname{int}\mathcal{X}$ implies that no state constraint can be activated immediately. In other words, we can do the analysis as if $\mathrm{OCP}^1_{\mathrm{LQ}}$ does not involve any active state constraint on $[0,\tau]$.

Regarding (20b) as the linear output of (20a), conditions (20) require that the output of a linear system stays constant for some non-vanishing interval $[0,\tau]$. Taking into account that controllability of (A,B) implies observability of $(A^\top, B^\top)$, we have that the only solution to (20) is a constant solution. Note that any constant solution $\tilde{\lambda}_{\mathrm{LQ}}$ to (20) combined with $\bar{x}^1_{\mathrm{LQ}}, \bar{u}^1_{\mathrm{LQ}}$ satisfies the NCOs (17). By Assumption 2 we have that $\tilde{\lambda}_{\mathrm{LQ}} = \bar{\lambda}^1_{\mathrm{LQ}} \neq 0$ is the unique constant solution to (20).

As we consider time-invariant OCPs, this also implies $\lambda^1_{\mathrm{LQ}}(T) = \bar{\lambda}^1_{\mathrm{LQ}} \neq 0$, which contradicts the boundary (transversality) condition $\lambda^1_{\mathrm{LQ}}(T) = 0$. Hence, we arrive at a contradiction, i.e. starting at the optimal steady state $\bar{x}^1_{\mathrm{LQ}}$, the system immediately leaves the optimal steady state with non-zero velocity $\dot{x}^1_{\mathrm{LQ}}(0) = Bu^1_{\mathrm{LQ}}(0) \neq 0$. □

PROOF (THEOREM 3). By virtue of Theorem 2, we restrict the proof to OCP^1. Temporarily assume that EMPC based on OCP^1 stabilizes the system to $\bar{x} = 0$. Then, this implies that starting at $x_0 = \bar{x}$, there exists $\tau > 0$ such that for all $t \in [0,\tau]$, $x^1(t) \equiv \bar{x}$ and $\lambda^1(t) \equiv \bar{\lambda}^1$.

Observe that due to the regularity assumption on OCP^1, in a small neighborhood around $\bar{z} \in \operatorname{int}\mathcal{Z}$, $u^1(t)$ is a continuous function of $x^1(t)$ and $\lambda^1(t)$. In other words, for $x^1(t)$ to leave $\bar{x}$ for $t > \tau$, we need to have that $u^1(\tau) \neq \bar{u}$, which in turn only happens if $\lambda^1(\tau) \neq \bar{\lambda}^1$.

Lemma 2 implies $\lambda^1(t) = \lambda^1_{LQ}(t) + O(\|x_0 - \bar{x}\|^2)$, $\forall\, t \in [0,\tau]$ and we assumed $\|x_0 - \bar{x}\| = 0$. In Lemma 3 we have shown that $\bar{\lambda}^1 \neq 0$ implies $\lambda^1_{LQ}(t) \not\equiv \text{const.}$ for all $t \in [0,\tau]$. In turn this yields $u^1(\tau) \not\equiv \bar{u}$ on $[0,\tau)$. Thus the optimal solution $x^1(t)$ leaves $\bar{x}$ immediately, i.e. $x^1(t) \neq \bar{x}$, $\forall\, t > 0$. □

This result has several consequences, which we elaborate in the following corollaries and remarks. The first important consequence regards closed-loop performance, evaluated via the asymptotic average, defined as

$$\mathrm{Av}[v(\cdot)] := \liminf_{t\to\infty} \frac{1}{t}\int_0^t v(\tau)\,\mathrm{d}\tau. \tag{21}$$

Henceforth, we consider the closed-loop performance as measured by the asymptotic average cost, i.e. $\mathrm{Av}[F(z(\cdot))]$. Note that we will use the economic cost F also when referring to closed-loop trajectories obtained by tracking MPC schemes.

Corollary 2 (TMPC performing better than EMPC).
Consider a stabilizing TMPC (in the sense of Definition 3, with $\bar{z}$ the economic optimum given by (6)*) and a stabilizing EMPC based on* OCP^i*, $i \in \mathcal{I}$ with $T < \infty$, $E(x) = 0$ and $\bar{\lambda}^1 \neq 0$. We define $z^{\mathrm{cl}}_{\mathrm{TMPC}}$ and $z^{\mathrm{cl}}_{\mathrm{EMPC}}$ the closed-loop state and control trajectories obtained with TMPC and EMPC respectively. Then*

$$\mathrm{Av}[F(z^{\mathrm{cl}}_{\mathrm{TMPC}}(\cdot))] \leq \mathrm{Av}[F(z^{\mathrm{cl}}_{\mathrm{EMPC}}(\cdot))],$$

i.e. the TMPC controller yields a better average closed-loop performance than the EMPC controller. Moreover, if $\bar{z}$ is a strict global optimum, then the inequality is strict.

PROOF. This is an immediate consequence of Theorem 3: whenever $\bar{\lambda} \neq 0$, EMPC stabilize s the system to a steady state $\bar{z}_{\mathrm{EMPC}} \neq \bar{z}$. Therefore, $F(\bar{z}_{\mathrm{EMPC}}) > F(\bar{z})$. TMPC on the other hand, does stabilize the system to $\bar{z}$, such that it yields a better average closed-loop performance, i.e.

$$\mathrm{Av}[F(z_{\mathrm{TMPC}}(\cdot))] \leq \mathrm{Av}[F(z_{\mathrm{EMPC}}(\cdot))].$$

Finally, consider the case when $\bar{z}$ is a strict global optimum. Because $z^{\mathrm{cl}}_{\mathrm{TMPC}}$ asymptotically tends to $\bar{z}$, while $\bar{z}_{\mathrm{EMPC}} \neq \bar{z}$, for t sufficiently large, $F(z^{\mathrm{cl}}_{\mathrm{TMPC}}(t)) < F(z^{\mathrm{cl}}_{\mathrm{EMPC}}(t))$, which yields $\mathrm{Av}[F(z^{\mathrm{cl}}_{\mathrm{TMPC}}(\cdot))] < \mathrm{Av}[F(z^{\mathrm{cl}}_{\mathrm{EMPC}}(\cdot))]$. □

Lemma 4 (Linear cost rotation in TMPC).
Let there be a stabilizing TMPC with $F^{\mathrm{t}}(x,u)$ satisfying (11) *and $E(x) = 0$. Consider a linear rotation of the stage cost, i.e. consider using the cost defined by $\hat{F}(x,u) = F^{\mathrm{t}}(x,u) + a^\top f(x,u)$, $a \neq 0$ and $M(x_0, x(T)) = 0$.*

Then, the obtained MPC scheme is economic in the sense of Definition 3 and does not stabilize the system to the origin.

PROOF. We define $\hat{L}^i$, $i \in \mathcal{I}$ analogously to the definition of L^i, but by replacing F^{t} with $\hat{F}$. Accordingly, we use the definitions $\mathrm{S\hat{O}P}^i$, $\hat{\bar{z}}^i$ and $\hat{\bar{\lambda}}^i$. By assumption, $\bar{z}^i = 0$. Moreover, by using the same arguments of Theorem 2, one immediately obtains that $\hat{\bar{z}}^i = \bar{z}^i = 0$ and $\hat{\bar{\lambda}}^i = \bar{\lambda}^i - a$. Therefore, $\hat{F}_z(0) \neq 0$ and, by Definition 3 the obtained MPC scheme is economic.

Finally, by Theorem 3, $\hat{\bar{\lambda}}^1 \neq 0$ implies that the EMPC scheme does not stabilize the closed-loop system to the optimal steady-state. □

Remark 1 (No exact turnpikes in regular OCPs).
Recently, it has been shown in [13, 15] that under certain technical assumptions EMPC without terminal constraints and without terminal penalty implies (i) finite-time convergence to the optimal steady state and (ii) recovering infinite-horizon optimal performance via MPC receding horizon optimization. The core assumption of [13, 15] is that the underlying OCP admits an exact turnpike*, which implies that, for long horizons, the open-loop optimal solutions have to be exactly at steady state during the largest part of the optimization horizon. In [15], it is furthermore shown for a specific class of singular OCPs that turnpikes, if they appear, have to be exact. In this context, Lemma 3 allows the immediate conclusion that turnpikes of regular OCPs (Definition 1), if they exist, are never exact.*

3.3. Storage Function Geometry and Optimal Steady-State Multiplier

We turn towards the investigation of the relation between the Lagrange multipliers of $\mathrm{SOP}^i, i \in \mathcal{J}$ and the local geometry of the storage function.

Theorem 4 (Storage function slope at $\bar{x}$).
Let S be a storage function which satisfies the strict dissipation inequality (12) *along any optimal pair $z^i(\cdot), i \in \mathcal{I}$. Suppose that S is continuously differentiable on some open neighborhood $\mathcal{B}(\bar{x})$ of the optimal steady state $\bar{x}$.*

Then the slope of S at $\bar{x}$ is given by the Lagrange multiplier of SOP^1*, i.e.*

$$S_x(\bar{x}) = -\bar{\lambda}^1.$$

PROOF. On the open set $\mathcal{B}(\bar{z}) := \mathcal{B}(\bar{x}) \times \mathcal{B}(\bar{u})$, consider the the rotated cost function given by

$$\hat{F}(z) := F(z) + S_x f(z).$$

As F and f are assumed to be continuously differentiable on $\mathcal{Z}$, $\hat{F}$ is so on $\mathcal{B}(\bar{z})$. Strict dissipativity implies

$$\hat{F}(z) - F(\bar{z}) \geq \alpha(\|z - \bar{z}\|), \quad \forall z \in \mathcal{B}(\bar{z}). \tag{22}$$

Hence, $\bar{z}$ is a strict local minimizer of $\hat{F}$ on $\mathcal{B}(\bar{z})$. Differentiability of $\hat{F}$ on $\mathcal{B}(\bar{z})$ implies that $\hat{F}_x(\bar{z}) = 0$.

Consider now the SOP (6) formulated using (a) the original cost F, i.e. SOP^1 and (b) the rotated cost $\hat{F}$, i.e. SOP^2 with S as specified above. Statement (iii) of Theorem 2 implies that $\hat{\bar{\lambda}} = \bar{\lambda}^1 + S_x(\bar{x})$. Because $\hat{F}_x(\bar{z}) = 0$, we have $\hat{\bar{\lambda}} = 0$ and, therefore, $S_x(\bar{x}) = -\bar{\lambda}^1$. □

At first glance the assumption of local differentiability of S close to $\bar{x}$ might appear to be a strict condition. However, close to $\bar{z} \in \operatorname{int}\mathcal{Z}$, one may approximate OCP^i by means of (8) as a linear quadratic problem (Lemma 2). Furthermore, storage functions for linear systems subject to quadratic supply rates can be computed via Linear Matrix Inequalities (LMIs) as quadratic forms [38, 47, 48]. Thus, the local differentiability assumption imposed on S does not appear to be overly restrictive.

Note that the last result connects the stability proof of [10], which makes use of a linear storage function, with the result of [1], which uses a nonlinear storage function. In the former publication, the connection to the Lagrange multiplier of the SOP is explicitly made. In the latter one instead, this connection has not been investigated.

Remark 2 (Gradients of value and storage functions). *It has been shown in [44] that whenever the optimal pairs $z(\cdot)^i$ stay close to the turnpike $\bar{z} \in \operatorname{int}\mathcal{Z}$, then also the adjoint $\lambda^i(t)$ is close to its turnpike value $\bar{\lambda}^i$. Provided that the horizon is long enough such that the turnpike can be observed, combining these two observations with Theorem 4 yields*

$$V_x^1(x)\big|_{x \approx \bar{x}} = \lambda^1(0) \approx \bar{\lambda}^1 = -S_x(\bar{x}), \tag{23}$$

i.e. the negative gradient of any locally differentiable storage function approximates the gradient of the optimal value function of OCP^1 *at $\bar{x}$.*

3.4. Recovering Stability at the Optimal Steady State

In the following, we show how closed-loop stability of the optimal steady state can be recovered and, consequently, optimal average performance can be achieved also in the absence of terminal cost or constraints.

Lemma 5 (Nonlinear rotation of cost functions). *Any EMPC scheme based on* OCP^i*, $i \in \mathcal{I}$ with $T < \infty$, and $E(x) = -S(x)$ is a TMPC scheme in the sense of Definition 3, provided (i) that $S(x)$ is a storage function which satisfies the strict dissipation inequality* (12) *and (ii) that $S(x)$ is absolutely continuous in the sense of Definition 2.*

PROOF. By Theorem 2, all OCPs yield the same primal solution and, therefore, induce the same EMPC stability properties. Hence, we focus on OCP^2. By construction, $E(x) = -S(x)$ implies that $M(x_0, x(T)) = 0$. Moreover, absolute continuity of S implies that the strict dissipation inequality (12) can be written in its differential form almost everywhere on $[0, T]$. In turn, this gives

$$F(z) - F(\bar{z}) - S_x f(z) \geq \alpha(\|z - \bar{z}\|).$$

Recalling, that w.l.o.g. we have set $F(\bar{z}) = 0$ and $\bar{z} = 0$, this proves that the rotated stage cost is positive definite. □

Provided that a almost everywhere differentiable storage function S is known, the immediate consequence of this lemma is the applicability of sufficient TMPC stability conditions such as [21, 29, 20, 42]. In other words, EMPC falls back to TMPC penalizing the deviation from the optimal steady state $\bar{z}$ and, by an appropriate choice of sampling period and prediction horizon, convergence to and/or stability of $\bar{x}$ can be concluded. We refer to [1] for the counterpart for EMPC with terminal constraints.

Unfortunately, although it provides a condition which enforces stability without terminal cost nor constraints, Lemma 5 is impractical, as it requires explicit knowledge of a storage function. The computation of storage functions is in general as difficult as the computation of Lyapunov functions for uncontrolled systems [11, 18]; i.e., one typically applies sum-of-squares techniques to polynomial problems of rather small dimensions.

Next, we analyze how to tackle this issue by means of end penalties in the linear-quadratic setting.

Lemma 6 (Properties of stabilizing LQ EMPC). *Consider* OCP$^1_{\mathrm{LQ}}$ *with the problem data from* (16)*, and such that $\bar{\lambda}^1_{\mathrm{LQ}} \neq 0$. Let (A, B) be stabilizable, consider an EMPC formulation with $E(x) = \frac{1}{2} x^\top P_T x$, with $P_T = P_T^\top$. Suppose that, with the chosen T and P_T, the EMPC with instantaneous feedback, i.e. $\delta = 0$, asymptotically stabilizes the system to some $\bar{z}_{\mathrm{EMPC}} \neq \bar{z} = 0$. Then the following statements hold:*

(i) For increasing prediction horizons T, the closed-loop steady-state $\bar{z}_{\mathrm{EMPC}}$ tends to $\bar{z}$ with an exponential decay in T.

(ii) The EMPC formulation with $E(x) = \frac{1}{2} x^\top P_T x + x^\top p_T$ stabilizes the closed-loop system to a steady-state $\bar{z}_{\mathrm{EMPC}}$, which tends to the optimal steady-state $\bar{z}_{\mathrm{LQ}} = 0$ linearly as p_T tends to $\bar{\lambda}^1_{\mathrm{LQ}}$.

PROOF. Recall that the fact that the EMPC formulation with $E(x) = 0$ does not stabilize the system to $\bar{z}_{\mathrm{LQ}} = 0$ is a consequence of Theorem 3. In order to prove Claims (i) and (ii) we rely on the characterization of the optimal solution to OCP^1 provided in Lemma 1.

The optimality conditions of SOP^1_{LQ} entail, cf. (18a),

$$\begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \bar{\lambda}^1_{\text{LQ}} + \begin{bmatrix} q \\ r \end{bmatrix} = 0. \tag{24}$$

Henceforth, the subscript $\cdot_\infty$ denotes steady-state solutions for $T = \infty$. Consider the optimal feedback (9) from Lemma 1. As we assume that EMPC with instantaneous feedback stabilizes the system, in the limit for $T \to \infty$, we obtain

$$\begin{aligned} 0 &= A^\top P_\infty + P_\infty A + Q - (PB + S)K_\infty, \\ K_\infty &= R^{-1}(B^\top P_\infty + S^\top), \\ 0 &= \begin{bmatrix} I & -K_\infty^\top \end{bmatrix} \left(\begin{bmatrix} A^\top \\ B^\top \end{bmatrix} p_\infty + \begin{bmatrix} q \\ r \end{bmatrix} \right). \end{aligned} \tag{25}$$

Then, because $\begin{bmatrix} I & -K_\infty^\top \end{bmatrix}$ is full row rank, we have

$$\begin{aligned} p_\infty &= \bar{\lambda}^1_{\text{LQ}}, \qquad B^\top p_\infty + r = 0, \\ u_\infty &= -K_\infty x. \end{aligned} \tag{26}$$

Using (26), Equation (9a) can be written as

$$u^1_{\text{LQ}}(0,T) = -K(0,T)x^1_{\text{LQ}} - R^{-1}B^\top (p(0,T) - p_\infty).$$

By assumption, T and P_T are chosen such that $A - BK(0,T)$ is Hurwitz and thus invertible. Using the last equation the steady-state $\bar{x}_{\text{EMPC}}$ satisfies

$$0 = (A - BK(0,T))\bar{x}_{\text{EMPC}} - BR^{-1}B^\top (p(0,T) - p_\infty). \tag{27}$$

Using (24) and $p_\infty = \bar{\lambda}^1_{\text{LQ}}$, and pre-multiplying by $[I \; -K(t,T)^\top]$ we obtain

$$\begin{bmatrix} I & -K(t,T)^\top \end{bmatrix} \left(\begin{bmatrix} A^\top \\ B^\top \end{bmatrix} p_\infty + \begin{bmatrix} q \\ r \end{bmatrix} \right) = 0.$$

Therefore, $q - K(t,T)^\top r = -(A - BK(t,T))^\top p_\infty$. Hence, we rewrite (9d) as

$$\dot{p}(t,T) = (BK(t,T) - A)^\top (p(t,T) - p_\infty), \; p(T,T) = p_T. \tag{28}$$

By assumption, T is large enough such that $A - BK(0,T)$ is asymptotically stable. Then, $p(0,T) - p_\infty$ decays exponentially with increasing T. Moreover, for a fixed T, $p(0,T)$ depends linearly on $p_T - p_\infty$. Hence, $\bar{x}_{\text{EMPC}}$ depends linearly on $p_T - p_\infty$ and decays exponentially for increasing T. □

Remark 3 (Primal interpretation of $E(x) = x^\top \bar{\lambda}^1$). *The end penalty $E(x) = x^\top \bar{\lambda}^1$ is equivalent to a gradient-correcting linear rotation of the stage cost. The EMPC problem of Lemma 6 can be expressed in its rotated form, i.e.* OCP^2, SOP^2, *by setting $S_x(x) = -\bar{\lambda}^1$. For $P_T = 0$ this yields $L^2(z) = F(z) + (Ax + Bu)^\top \bar{\lambda}^1$ and $M(x(0), x(T)) = 0$. This formulation can be seen as the form $\widehat{\text{OCP}}^1$, $\widehat{\text{SOP}}^1$ of a problem formulated using the same system dynamics, but the cost defined by $\widehat{F}(z) = L^2(z)$ and $\widehat{E}(x) = 0$, which implies that $\hat{\bar{\lambda}}^1 = 0$. Moreover, the optimality conditions of the* $\text{SOP}^2 \equiv \widehat{\text{SOP}}^1$ *imply that* $\widehat{F}$ has no gradient at the optimal steady-state pair $\bar{z} = 0$. *Nevertheless, the problem does not necessarily define a TMPC scheme, since the cost $\widehat{F}$ is in general not positive definite.*

Remark 4 (Adjoint interpretation of $E(x) = x^\top \bar{\lambda}^1$). *Note that the end penalty can be motivated not only as a local gradient correction of $\hat{F}(z) = L^2(z)$ at $\bar{z}$. In the view of the NCOs of* OCP^1, *we observe that $E(x) = x^\top \bar{\lambda}^1$ implies the boundary/transversality condition $\lambda^1(T) = \bar{\lambda}^1$. Having in mind that, for OCPs without terminal constraints, leaving arcs of turnpikes are driven by $\lambda^1(T) = 0$, $\bar{\lambda}^1 \neq 0$—i.e. they do not occur whenever the optimal steady state corresponds to the unconstrained minimum of $F(z)$—we can interpret $E(x) = x^\top \bar{\lambda}^1$ as a simple way of enforcing a terminal constraint on the adjoint at $\bar{\lambda}^1$, which corresponds to the optimal steady state $\bar{z}$.*

Remark 5 (Stabilizing indefinite LQR feedback). *A sufficient condition for the LQR feedback to be stabilizing is $S = 0$, $Q = Q^T \succeq 0$, such that $C^\top C = Q$ with (A, C) detectable, cf. [2]. However, in many relevant EMPC applications, this is not the case. We remark that if the set $\mathbb{Z}$ is not compact, then strict dissipativity does not automatically imply stability of infinite horizon optimal solutions. A simple example is given by $\dot{x} = x + u$, $F(x,u) = u^2$. Strict dissipativity holds with e.g. $S(x) = x^2$, but the optimal solution is $u = 0$ and the system is unstable. If, on the other hand, $\mathbb{Z}$ is compact and the problem is feasible, the optimal solution stabilizes the system to the origin. For more insight on this problem see [46] and, for a discrete-time counterpart, [22].*

Remark 6 (Case $P_T = 0$). *The case of $P_T = 0$ is particularly interesting because it corresponds to the case of a formulation without quadratic terms in the terminal penalty. That $P_T \neq 0$ is not necessary to guarantee stability is readily seen in the case $Q = Q^T \succeq 0$, such that $C^\top C = Q$ with (A, C) detectable, cf. [2]. Unfortunately, a characterization of stability conditions in the generic case when $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \not\succeq 0$ is, to the best of the authors' knowledge, not available.*

Before we state our main result, we introduce the following closed-loop dynamics

$$\dot{x} = Ax + Bu^1_{\text{LQ}}(\tau, x(t_k)), \quad x(0) = x_0 \tag{29a}$$

$$\tau := t - t_k, \quad k = \max\{k \in \mathbb{N} \,|\, \tau = t - k\delta \in [0, \delta)\}. \tag{29b}$$

generated by the sampled-data EMPC based on OCP^1_{LQ}.

Theorem 5 (Asymptotic stability of EMPC). *Consider an EMPC controller based on the regular positive* OCP^i, *$i \in \mathfrak{I}$ with $T < \infty$. Let Conditions A1-A3 of Theorem 1 hold. Then, the following holds:*

(i) If $E(x) = x^\top \bar{\lambda}^1$ and $T, \delta \geq 0$ are chosen such that (29) *is uniformly exponentially stable at $x = \bar{x}$, then there exists $\tilde{T} \geq T, \tilde{\delta} \in (0, \delta)$ such that for all $x_0 \in \mathcal{X}_0$ the closed EMPC loop is uniformly exponentially stable at $\bar{x}$.*

(ii) If, $E(x) = 0$, $\bar{\lambda}^1 \neq 0$, $P_T = 0$, $\delta = 0$, and T is chosen such that the instantaneous feedback (9) *asymptotically stabilizes the linear system (A, B), then there exist a finite horizon $\tilde{T} \geq T$ and $\bar{x}_{EMPC} \neq \bar{x}$ such that for all $x_0 \in \mathcal{X}_0$ the closed EMPC loop is exponentially stable at $\bar{x}_{EMPC}$.*

PROOF. Conditions A1–A3 ensure that Theorem 1, respectively Corollary 1, holds for instantaneous ($\delta = 0$) or sampled-data ($\delta > 0$) EMPC. Hence, we can conclude that, by choosing T sufficiently large and δ sufficiently small, OCP^i, $i \in \mathcal{I}$ is recursively feasible and the closed-loop system will converge to some small neighborhood of $\bar{x}$. Thus, it suffices to locally analyze the closed-loop dynamics generated by EMPC based on OCP^1

$$\dot{x} = f(x, u^1(\tau, x(t_k))), \tag{30}$$

where τ is defined by (29b). Part (i) considers the sampled-data case $\delta > 0$ and Part (ii) deals with the instantaneous case $\delta = 0$.

Part (i): Note that in the analysis we have to acknowledge the fact that the rhs of the sampled-data system (30) does not necessarily evolve continuously with time.

Consider the initial condition $x_0 = \bar{x}$. Due to the terminal penalty $E(x) = x^\top \bar{\lambda}^1$, the NCO (5) admit the steady-state solution $(\bar{x}, \bar{u}, \bar{\lambda}^1)$. Moreover, as by assumption OCP^1 is regular positive at $\bar{z}$, the triple $(\bar{x}, \bar{u}, \bar{\lambda}^1)$ satisfies local second-order sufficient conditions of optimality for OCPs, cf. [35, Thm. 2.2]. Due to the LQ approximation properties of Lemma 2, we may even conclude that $(\bar{x}, \bar{u}, \bar{\lambda}^1)$ is the unique optimal solution originating at $x(0) = \bar{x}$.[10] Thus, $(\bar{x}, \bar{u})$ is a steady state of (30).

Now, linearizing (30) around $(\bar{x}, \bar{u})$ yields

$$\dot{x} = Ax + Bu^1_{\text{LQ}}(\tau, x(t_k)) + O(\|x_0 - \bar{x}\|^2), \tag{31}$$

whereby we employ Lemma 2 to approximate $u^1(\cdot)$ by $u^1_{\text{LQ}}(\cdot)$. Invoking Lemma 2 and (9a) we have that

$$u^1_{\text{LQ}}(\tau, x(t_k)) = -K(t, T)x^1_{LQ}(t) - R^{-1}(B^\top p(\tau, T) + r)$$

Using (10), (28), which implies that $p(\tau, T) \equiv \bar{\lambda}^1$, and (26), which implies that $B^\top \bar{\lambda}^1 + r = 0$ we obtain

$$u^1_{\text{LQ}}(\tau, x(t_k)) = -K(\tau, T)x^1(t) + O(\|x_0 - \bar{x}\|^2). \tag{32a}$$

Moreover, for $\tau \in [0, \delta)$, the triangle inequality gives that

$$\|x_0 - \bar{x}\| \leq \|x(t) - \bar{x}\| + \|x(t) - x_0\| \leq (1 + \delta L_f)\|x(t) - \bar{x}\|, \tag{32b}$$

where the bound on the right follows from $\|x(t) - x_0\| \leq \delta L_f \|x(t) - \bar{x}\|$ and L_f is a uniform Lipschitz constant of $f(x, u^1(\tau, x(t_k)))$. Using (32) to rewrite (31) yields

$$\dot{x}(t) = (A - BK(\tau, T))x(t) + O(\|x(t) - \bar{x}\|^2).$$

Now invoking a standard result [28, Thm. 3.3.41], we conclude that $\bar{x}$ is a locally uniformly exponentially stable equilibrium of (30).

[10] Any competing optimal solution would need to satisfy (10), which however states that for $x_0 = \bar{x}$ the unique LQ solution is met.

Part (ii): On a sufficiently small neighborhood of $\bar{x}$, we again characterize the optimal solution by the corresponding LQ approximation (8). Due to $\delta = 0$ the closed-loop dynamics of the local approximation turn out to be the LTI system

$$\dot{x} = (A - BK(0, T))x - BR^{-1}(B^\top p(0, T) + r). \tag{33}$$

Hence, by assumption within a sufficiently small neighborhood the LQR solution will be asymptotically stabilizing and thus the EMPC will converge to $\bar{x}_{EMPC}$, which differs from $\bar{x}$ if $\bar{\lambda}^1 \neq 0$. $\square$

Remark 7 (Limit-cycles in sampled-data EMPC). *The subtle difference between Part (i) and Part (ii) of the above theorem is that a sampled-data $\delta > 0$ local LQ-approximation with $E(x) = x^\top p_T$, $p_T \neq \bar{\lambda}^1$ cannot be expected to be stabilizing. This is easy to see in* (9a), (28)*: the fact that $p_T \neq \bar{\lambda}^1$ implies that $p(t, T) \not\equiv const$, for all $t \in [0, \delta)$. In turn this implies that the LQ-approximation has the closed-loop dynamics*

$$\dot{x} = (A - BK(\tau, T))x - BR^{-1}(B^\top p(\tau, T) + r),$$

with τ from (29)*. Note that this system differs from* (33) *by the periodic forcing $-BR^{-1}(B^\top p(\tau, T) + r)$. In other words, whenever $p_T \neq \bar{\lambda}^1$ and $\delta > 0$, the closed-loop system will approach a limit cycle in-between two sampling instants. However, in typical EMPC implementations with piecewise constant inputs one will not observe this as one often computes the solutions only at the sampling instants. In Section 4.3 we present a numerical example exhibiting the predicted limit-cycle behavior in-between sampling instants.*

Finally, without further elaboration, we remark that uniform asymptotic stability of (29) *does not suffice to guarantee local uniform exponential stability of* (30) *, cf. [28, Rem. 3.3.42]. Thus the assumption of uniform exponential stability of* (29) *at $\bar{x}$ in Part (i) is crucial.*

Remark 8 (Asymptotic stability in instantaneous EMPC). *Recall that Lemma 6 derives a relation between $p_T \neq \bar{\lambda}^1$ and the closed-loop steady-state attained by the instantaneous LQR feedback. Combining Theorem 5 Part (ii) with Lemma 6, we obtain that if instantaneous EMPC practically stabilizes a neighborhood of $\bar{x}$, then (a) the closed EMPC loop converges to some steady state $\bar{x}_{EMPC} \neq \bar{x}$ inside this neighborhood, and (b) considering $E(x) = x^\top \bar{\lambda}^1$ will lead to stability of $\bar{x}$.*

We conclude the discussion with a direct consequence of Theorem 5.

Corollary 3 (Recovering average performance for EMPC). *The average performance of the EMPC scheme from Theorem 5 is no worse than that of any TMPC scheme.*

4. Simulation Examples

In this section, we provide three numerical examples illustrating the theoretical developments of the paper.

4.1. *A Linear System with Quadratic Cost*

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad F(z) = \frac{1}{2} z^\top H z + h^\top z, \tag{34a}$$

$$A = \begin{bmatrix} -2.4 & 0 \\ 1.2 & 1.2 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0.2 \end{bmatrix}, \tag{34b}$$

$$B = \begin{bmatrix} 0.05 \\ -0.05 \end{bmatrix}, \qquad h = \begin{bmatrix} 0.0 \\ -24.0 \\ -0.5 \end{bmatrix}. \tag{34c}$$

Note that the stage cost does have a gradient at the optimal steady-state. Moreover, the Hessian of the stage cost is indefinite, such that a linear rotation of the cost is not sufficient to yield a positive-definite stage cost. Because the system is linear, it is possible to compute the storage function by solving an SDP [48]. Note that the technique has been developed for discrete-time systems but can readily be adapted to the case of continuous-time systems. Moreover, sampled-data systems can alternatively be considered as discrete-time systems within the framework of [48]. Finally, we stress here that, because of the absence of active constraints at steady-state, the linear rotation of the stage cost corresponds to setting $h = 0$.

We use a prediction horizon $T = 0.5$ s, a sampling time $\delta = 0.1$ s, and we use an explicit Runge-Kutta scheme of order 4 with a fixed time grid based on 50 identical integration steps per sampling interval. We consider the three initial conditions $(0, 0.3)$, $(-0.3, -0.1)$, $(0, 0)$. The closed-loop trajectories obtained by the original formulation and by the formulation with the linearly rotated cost, i.e. using $E(x) = x^\top \bar{\lambda}^1$, are displayed in Figure 1. As predicted by the theory, the linearly rotated scheme stabilizes the system to the optimal steady-state, while the original scheme does not. Indeed, while in the first case the MPC predictions do not leave the optimal steady-state, in the second case, they first bring the system close to the steady-state but afterwards they move away from it.

4.2. *A Simple Nonlinear System*

Consider the nonlinear system

$$\dot{x} = \begin{bmatrix} 0.1u(1 - x_1) - 1.2x_1 \\ 0.1u(1 - x_2) + 1.2x_1 \end{bmatrix}, \tag{35a}$$

$$F(x, u) = -2ux_2 + 0.5u + 0.1(u - 12)^2, \tag{35b}$$

which has an optimal steady-state at $\bar{x} = (0.5, 0.5)$, $\bar{u} = 12$. We use a prediction horizon $T = 0.5$ s, a sampling time $\delta = 0.1$ s, we use an explicit Runge-Kutta of order 4 with a fixed time grid based on 50 identical integration steps per sampling interval. We consider the three initial conditions $(0.5, 0.8)$, $(0.2, 0.4)$, $(0.5, 0.5)$. The closed-loop trajectories obtained by the original formulation and by the formulation with the linearly rotated cost are displayed in Figure 2.

We remark that (34) is the linear quadratic approximation of (35), computed at the optimal steady-state, cf. (8).

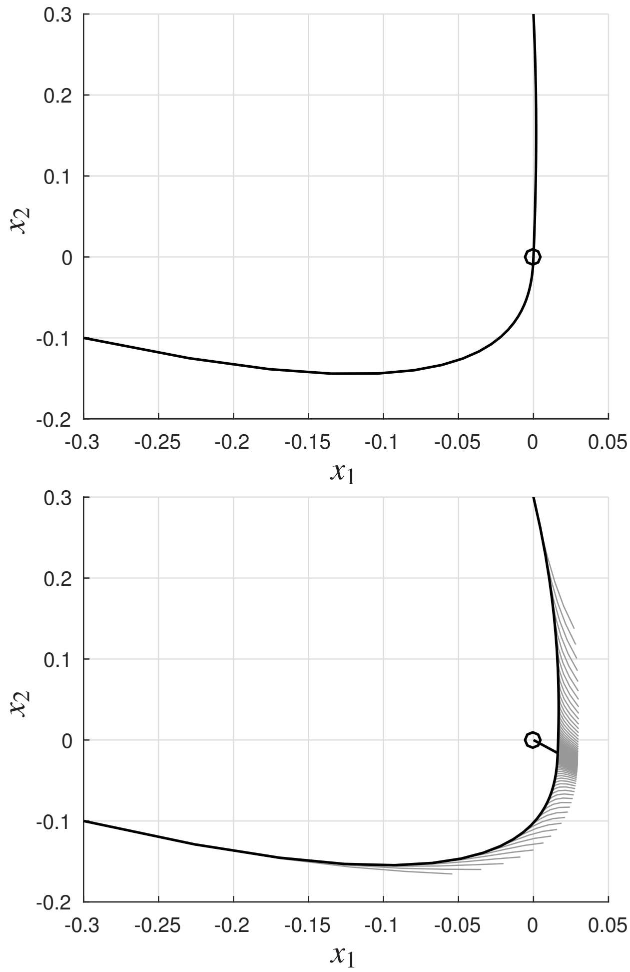

Figure 1: Linear system with quadratic cost. Closed-loop simulations starting from three different initial conditions. The MPC prediction at each sampling instant are displayed in grey lines. The optimal steady-state is displayed as a black circle. Left graph: formulation with linear rotation; right graph: formulation without linear rotation.

4.3. *Convergence to the Economically Optimal Steady-State*

In this subsection, we verify the results of Lemma 6 numerically. We define the closed-loop steady state $\bar{z}^{\mathrm{cl}}_{\mathrm{EMPC}} = (\bar{x}^{\mathrm{cl}}_{\mathrm{EMPC}}, \bar{u}^{\mathrm{cl}}_{\mathrm{EMPC}})$ obtained with the EMPC formulation. We show the linear dependence of $\bar{z}^{\mathrm{cl}}_{\mathrm{EMPC}}$ on the linear rotation $E(x) = x^\top p_T$, and the exponential dependence of $\bar{z}^{\mathrm{cl}}_{\mathrm{EMPC}}$ on the prediction horizon T. Moreover, in order to measure average performance in the nominal case, we use the metric

$$G_{\mathrm{EMPC}} = \frac{F(\bar{x}^{\mathrm{cl}}_{\mathrm{EMPC}}, \bar{u}^{\mathrm{cl}}_{\mathrm{EMPC}}) - F(\bar{x}, \bar{u})}{F(\bar{x}, \bar{u})}. \tag{36}$$

In Figure 3, the closed-loop steady-state $\bar{z}_{\mathrm{EMPC}}$ is displayed for several choices of cost rotations, obtained by using $E(x) = \sigma x^\top \bar{\lambda}^1$, $\sigma \in [0, 1]$. For the linear-quadratic case, one obtains that the closed-loop steady-state drifts away from the optimal steady-state with a linear relation to σ, as predicted by Lemma 6. For the nonlinear case, instead, the drift is present but nonlinear.

In Figure 4, the distance of the closed-loop steady-state $\bar{z}_{\mathrm{EMPC}}$ is displayed for an increasing prediction horizon T. As

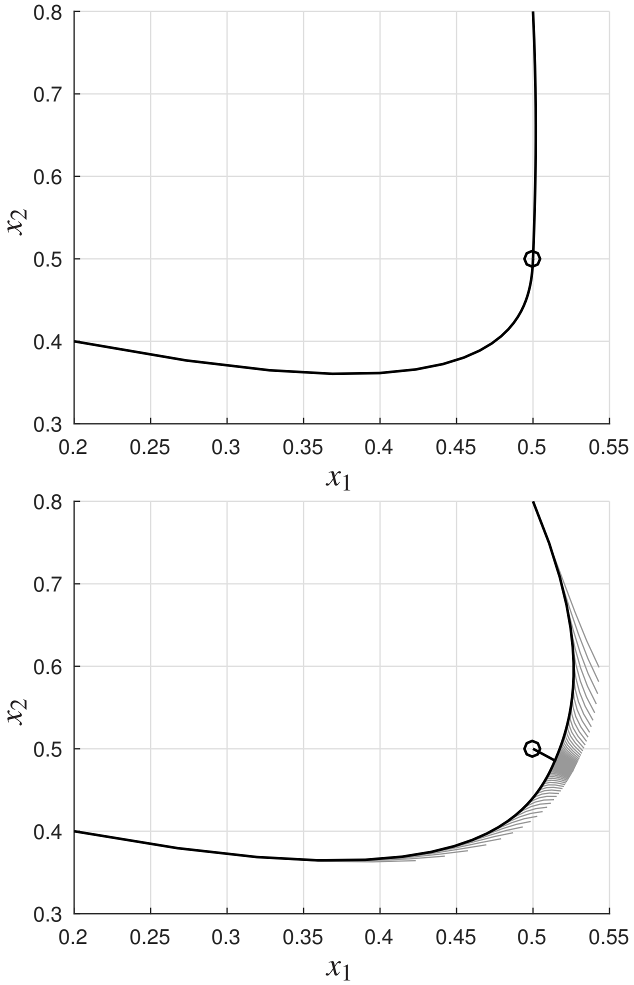

Figure 2: Nonlinear system. Closed-loop simulations starting from three different initial conditions. The MPC prediction at each sampling instant are displayed in grey lines. The optimal steady-state is displayed as a black circle. Left graph: formulation with linear rotation; right graph: formulation without linear rotation.

predicted by Lemma 6, for the linear-quadratic case $\bar{x}^{\mathrm{cl}}_{\mathrm{EMPC}}$ converges exponentially to $\bar{x}$. Moreover, we observe a similar behaviour also for the nonlinear case. Finally, also the average performance converges exponentially with increasing prediction horizons.

In Figure 5 we display the sampled-data LQR formulation for the considered linear-quadratic example. It can be seen that the formulation without gradient correction has an oscillatory behaviour, as predicted by Remark 7. We remark that in Figures 1 and 2 we only displayed the states at the sampling instants and the oscillations are therefore not visible.

4.4. Continuously Stirred Tank Reactor

We consider the example of a continuously stirred tank reactor (CSTR) [43], also used in [17, 18] to investigate turnpike and dissipativity properties of OCPs. A model of the reactor, including the concentration of species A and B, c_A, c_B in mol/l

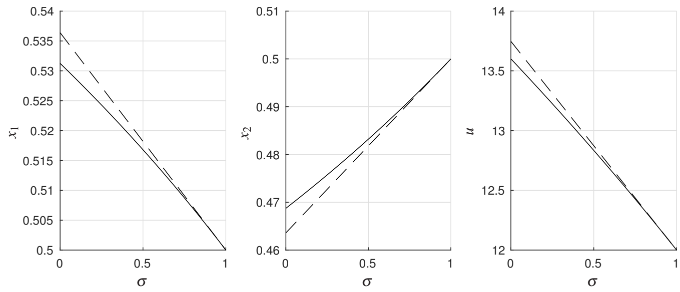

Figure 3: Closed-loop steady state obtained with piecewise constant inputs and several cost rotations, obtained by using $E(x) = \sigma x^\top \bar{\lambda}^1$, $\sigma \in [0,1]$. Comparison of the nonlinear system (continuous line) and its local linear-quadratic approximation (dashed line).

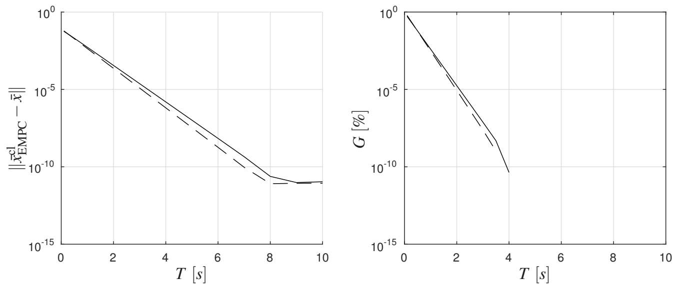

Figure 4: Left graph: convergence of the closed-loop steady state to the economically optimal steady-state for increasing horizon length T and piecewise constant inputs. Right graph: convergence of the average performance to the economically optimal performance for increasing horizon length T. The nonlinear example is displayed in continuous line and the linear-quadratic example in dashed line.

and the reactor temperature ϑ in °C as state variables, reads

$$\begin{aligned} \dot{c}_A &= -r_A(c_A,\vartheta) + (c_{in} - c_A)u_1 \\ \dot{c}_B &= \ \ r_B(c_A,c_B,\vartheta) - c_B u_1 \\ \dot{\vartheta} &= \ \ h(c_A,c_B,\vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1, \end{aligned}$$

where $r_A = k_1 c_A + k_3 c_A^2$, $r_B = k_1 c_A - k_2 c_B$, $h = -\delta(k_1 c_A \Delta H_{AB} + k_2 c_B \Delta H_{BC} + k_3 c_A^2 \Delta H_{AD})$ and $k_i = k_{i0} e^{\frac{-E_i}{\vartheta + \vartheta_0}}$, $i = 1,2,3$. The system parameters can be found in [43]. The states and inputs are subject to the constraints $c_A \in [0,6]\frac{mol}{l}, c_B \in [0,4]\frac{mol}{l}, \vartheta \in [70,150]°C$ and $u_1 \in [3,35]\frac{1}{h}, u_2 \in [0,200]°C$. We consider the problem of maximizing the production rate of c_B; thus F in (3) and (6) is

$$F(c_B,u_1) = -\beta c_B u_1, \quad \beta > 0.$$

In [18], the globally optimal steady state is given as

$$\bar{x} = [2.1756, 1.1049, 128.53]^\top, \quad \bar{u} = [35, 142.76]^\top.$$

The original formulation yields a singular OCP with a turnpike that seems not to be exact, though no formal proof of its non-exactness is currently available. In this paper, we regularize the problem in order to avoid chattering of the actuators by adding the term $0.001\|u - \bar{u}\|_2^2$ to the stage cost. This makes the OCP regular positive, which implies that the turnpike cannot be exact.

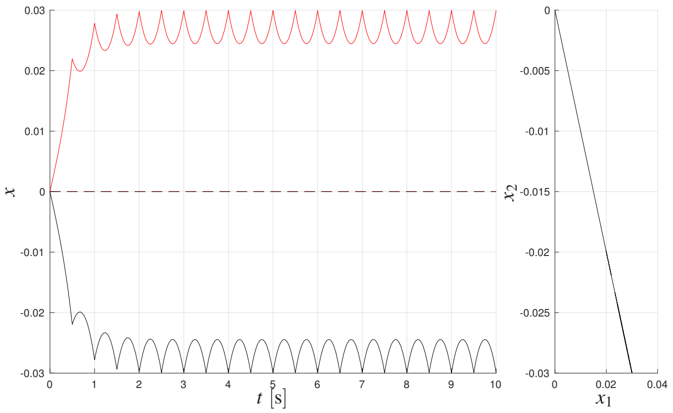

Figure 5: Sampled-data LQR formulation with and without gradient correction, respectively in continuous and dashed line. Prediction horizon $T = 0.5$ s, sampling time $\delta = 0.1$ s. Left graph: state evolution in time, x_1 in red, x_2 in black. Right graph: state-space plot.

The closed-loop trajectories obtained with and without the linear rotation of the cost are displayed in Figure 6. It can be seen that, also in this case, when the cost is not rotated the closed-loop system converges to a steady-state which is not economically optimal, which is in agreement with the results in [18]. Without linear rotation of the cost, one obtains the closed-loop average loss $G_{\text{EMPC}} = 1.49$ %.

5. Conclusion

This paper has investigated how the addition of a gradient correcting linear end penalty to EMPC formulations without stabilizing terminal conditions affects closed-loop stability properties in sampled-data formulations. We have highlighted how different OCP definitions used in the literature can be related to the same MPC formulation. Put differently, we have shown that the proposed linear end penalties are equivalent to a linear rotation of the stage cost. We have then proven that, whenever the Lagrange multiplier of the corresponding SOP is nonzero, economic MPC based on regular OCPs cannot be stabilizing to the economically optimal steady state. Under the assumption of strict dissipativity, rotating the cost using the storage function solves this issue. However, computing storage functions for nonlinear systems is in general difficult. Our main result alleviates this problem as it establishes a strong connection between the storage function and the Lagrange multiplier of the SOP. Using this relation, we prove that, under mild conditions, a linear rotation of the cost is sufficient to enforce local uniform exponential stability of the economically optimal steady state. Moreover, we have highlighted that in sampled-data EMPC one should expect limit cycle behavior in-between sampling instants whenever the gradient correcting end penalty is not employed. Several simulations underpin the efficacy of linear gradient correcting rotations.

Ongoing research is aiming at extending our results to the discrete-time case. Future investigations will include a thorough analysis of the connection between regular OCPs, turnpikes and leaving arcs. Moreover, the impact of a linear rotation of the cost on the transient performance will be the subject of future research.

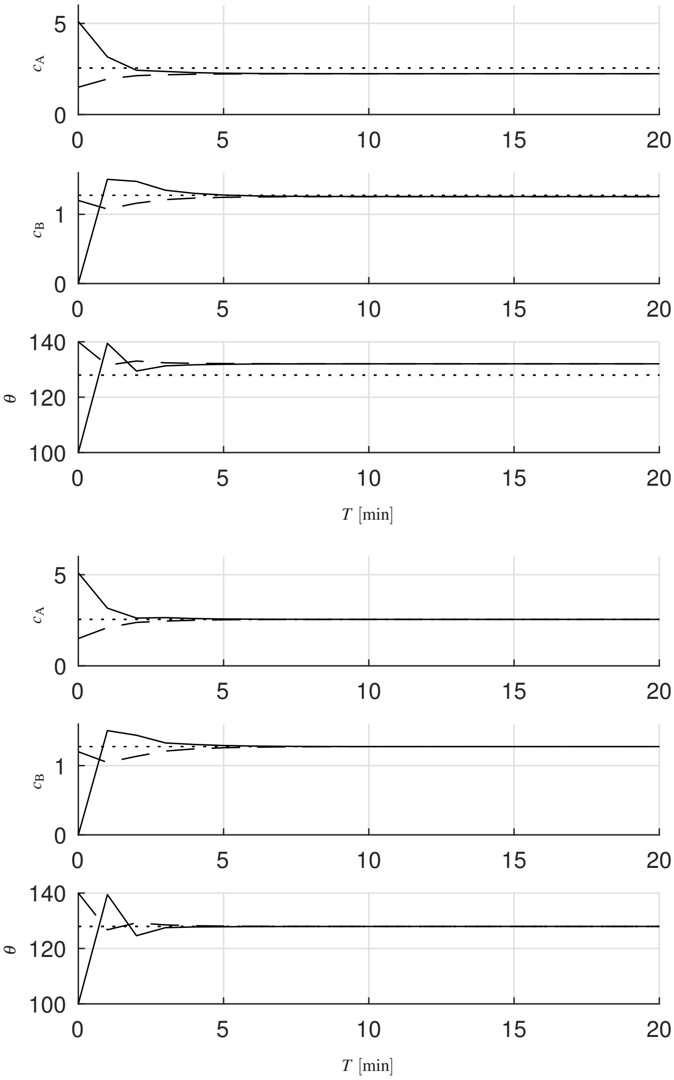

Figure 6: CSTR example discretized using 50 steps of explicit RK 4 over a sampling time $T_s = 1$ min and prediction horizon $T = 3$ min. Top graph: standard implementation, without cost rotation. Bottom graph: cost rotation. The closed-loop trajectories are displayed in continuous and dashed line for two test scenarios. The economically optimal steady state is displayed in dotted line.

Acknowledgements

The authors acknowledge the very helpful comments of the reviewers of this paper.

Appendix A. Proof of Lemma 1

The NCO for the considered LQ OCP read

$$\begin{bmatrix}\dot{x}\\ -\dot{\lambda}\\ 0\end{bmatrix} = \begin{bmatrix} A & 0 & B\\ Q & A^\top & S\\ S^\top & B^\top & R\end{bmatrix}\begin{bmatrix}x\\ \lambda\\ u\end{bmatrix} + \begin{bmatrix}0\\ q\\ r\end{bmatrix}, \tag{A.1}$$

$$\begin{bmatrix} x(0)\\ \lambda(T)\end{bmatrix} = \begin{bmatrix} x_0\\ P_T x(T) + p_T\end{bmatrix}. \tag{A.2}$$

As in standard LQR theory, we consider a backsweep ansatz, where $\lambda(t) = P(t)x(t) + p(t)$ is motivated by the terminal constraint for the adjoint [6, 2]. Differentiating this ansatz function yields

$$0 = -\dot{\lambda} + \dot{P}x + P\dot{x} - \dot{p}. \tag{A.3}$$

Solving the NCO for u as a function of x and λ, and substituting $\lambda(t) = P(t)x(t) + p(t)$, one obtains (9a). By substituting $\lambda(t) = P(t)x(t) + p(t)$, (A.3) and u from (9a) into the second equation in (A.1), one obtains

$$\begin{aligned} 0 &= \Big(Q + A^\top P + PA - (PB+S)R^{-1}(B^\top P + S^\top) + \dot{P}\Big)x \\ &+ \Big(-(R^{-1}(B^\top P + S^\top)^\top B^\top + A^\top)p - R^{-1}(B^\top P + S^\top)^\top r + q - \dot{p}\Big). \end{aligned}$$

Using $K := R^{-1}(B^\top P + S^\top)$, we can rewrite the above equation

$$\begin{aligned} 0 = \Big(Q + A^\top P + PA - (PB+S)K + \dot{P}\Big)x \\ + \Big((A^\top - K^\top B^\top)p + q - K^\top r - \dot{p}\Big). \end{aligned}$$

As the above equation has to hold for all x, Equations (9) are readily obtained.

Furthermore, it is known from optimal control theory that the adjoint λ is directly related to the optimal value function V by $\lambda(t) = V_x(x(t))$. As this has to hold at $t = 0$, we obtain

$$V_x^1(x(0)) = \lambda^1(0) = P(0,T)x(0) + p(0,T).$$

Integration with respect to $x(0)$ yields the desired optimal value function.

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