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Lakatosian and Euclidean populations: a pluralist approach to conceptual change in mathematics

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Abstract
Lakatos' (Lakatos, 1976) model of mathematical conceptual change has been criticized for neglecting the diversity of dynamics exhibited by mathematical concepts. In this work, I will propose a pluralist approach to mathematical change that re-conceptualizes Lakatos' model of proofs and refutations as an ideal dynamic that mathematical concepts can exhibit to different degrees with respect to multiple dimensions. Drawing inspiration from Godfrey-Smith's (Godfrey-Smith, 2009) population-based Darwinism, my proposal will be structured around the notion of a conceptual population, the opposition between Lakatosian and Euclidean populations, and the spatial tools of the Lakatosian space. I will show how my approach is able to account for the variety of dynamics exhibited by mathematical concepts with the help of three case studies.

Keywords Conceptual change in science · Mathematical change · Lakatos · Conceptual populations · Lakatosian space · Godfrey-Smith

1 Introduction

Lakatos (Lakatos, 1976) famously argued that conceptual change and concept formation are among the main engines of mathematical progress. Despite the historical significance of Lakatos' seminal book, several philosophers (e.g., Fine 1978; Feferman 1978; Corfield 2002; Werndl 2009) have stressed how his dialectics of tentative proofs and proposed counterexamples does not seem adequate as a general model of mathematical conceptual change.

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The main aim of this article is to propose a pluralist approach to mathematical conceptual change that re-conceptualizes Lakatos' model as an ideal dynamic that mathematical concepts can exhibit to different degrees with respect to multiple dimensions. My approach is directly inspired by Godfrey-Smith's (Godfrey-Smith, 2009) population-based Darwinism. I will show how the inadequacies of Lakatos' model of conceptual change stressed by its critics are analogous to the ones of recipes approaches (Godfrey-Smith, 2007) to natural selection in modeling biological populations. I will then argue that Godfrey-Smith's Darwinism, a framework designed to improve recipes-approaches to natural selection, offers a blueprint for a pluralist approach to mathematical conceptual change with analogous advantages.

My approach will be centered around the notion of a conceptual population, i.e., a set of conceptual variants and a set of mathematical problems together with an heuristic power ordering of conceptual variants. I will then argue that Lakatos' ideal of proofs and refutations correctly describes the change typical of a specific kind of conceptual populations, i.e., what I will call Lakatosian populations. In opposition to Lakatosian populations, I will define Euclidean populations, i.e., a kind of conceptual population exhibiting a dynamic of conceptual change radically different from what Lakatos prescribes. I will structure the opposition between Lakatosian and Euclidean populations along the three dimensions of (what I will call) the Lakatosian space: conceptual variation, reproductive competition, and continuity. Depending on how they score on these dimensions, conceptual populations can be judged to be more Lakatosian or more Euclidean, occupying different regions of the Lakatosian space. Different parts of the Lakatosian space will then correspond to populations exhibiting different dynamics of conceptual change. I will analyze this variety of dynamics by means of three case studies: Lakatos' own example of Euler's conjecture and the concept of polyhedron (Lakatos, 1976), Hamilton's invention of the quaternions (Hamilton, 1843a, b, 1853; Pickering, 1995), and the pre-abstract group concepts (Wussing, 1984).

In Section 2, I will describe Lakatos' seminal model of conceptual change and the many critiques and counterexample that have been presented against it. In Section 3, I will present the debate over recipes approaches to natural selection, focusing on Godfrey-Smith's population-based Darwinian framework, the pluralist approach to natural selection that will constitute the blueprint for my proposal. In Section 4, I will present my approach centered around the notion of a conceptual population, the opposition between Lakatosian and Euclidean populations, and the Lakatosian space. In Section 5, I will show how my approach can be fruitfully applied to three case studies. Finally, I will draw some general conclusions about what my proposal achieves and sketch some possible directions for future work.

2 Lakatos and his discontents

In this section, I will describe Lakatos' seminal model of conceptual change and I will discuss several critiques that have been raised to it.

2.1 Lakatos' concept-stretching

Lakatos (Lakatos, 1976) presents his model of conceptual change through a rational reconstruction of the notion of polyhedron in connection with Euler's conjecture.[1] In Lakatos' reconstruction, the problem starts with the conjecture, supported by Cauchy's thought-experiment, that Euler's formula connecting the number of vertices, edges and faces of regular polyhedra ($V - E + F = 2$) holds for any polyhedron whatsoever. In a series of reconstructed steps, the conjecture and its alleged proof get challenged by a series of counterexamples, after every one of which an attempt to defend or improve the conjecture and its proof is made.

Lakatos discusses several methods for defending the conjecture from counterexamples. First, there are so-called barring-methods, i.e., methods that defend the conjecture from a counterexample by modifying the conjecture in such a way that the counterexample no longer applies to it. This can be done either by redefining what counts as a polyhedron (i.e., the method of monster-barring, cf., Lakatos, 1976, p. 25) or by restricting the class of polyhedra to which the conjecture applies (i.e., the method of exception-barring, cf., Lakatos, 1976, p. 28). Lakatos' preferred method for defending a conjecture from a counterexample is instead the method of lemma-incorporation (Lakatos, 1976, p. 36), a method that plays a central role in Lakatos' view of conceptual change. This method consists in finding a hidden conjecture-lemma (e.g., polyhedra are stretchable onto a plane) refuted by a given counterexample (e.g., the nested cube) and inscribe this 'guilty' lemma into the conjecture as a condition for its applicability. In this way, this method saves the conjecture by restricting its domain to a narrower one (e.g., Euler's conjecture for 'simple' polyhedra, i.e., the stretchable ones). Differently from barring-methods, then, this method saves the conjecture by improving its proof. Lakatos (Lakatos, 1976, p. 53) then gives some heuristic rules for how to correctly apply this method. These rules prescribe a repeated search for heuristic counterexample of a given conjecture and an iterated application of the method of lemma-incorporation, in order to improve the proof of the conjecture. Lakatos (Lakatos, 1976, p. 61) warns us against the abuse of the method of lemma-incorporation, though. Lemma-incorporation saves the conjecture via restricting its intended domain. If this retreat to a narrower domain is repeated too many times, we may be left with a lack of content in our theorem. This impoverishment of content can be countered trying to replace lemmas that are refuted by heuristic counterexamples with unfalsified ones, thereby increasing the content of the theorem. Another way of countering the decrease of content is, according to Lakatos, the more general deductive guessing (Lakatos, 1976, p. 71) for deeper theorems to which given counterexamples do not apply anymore.

Lakatos' rational reconstruction of Euler's conjecture paradigmatically exemplifies this interplay of conjectures, counterexamples, methods, and heuristic strategies. In this dynamic, the concept of a polyhedron changes consistently with the interplay of the counterexamples and the conjecture-saving methods. The search for heuristic counterexamples drastically expands the domain of the conjecture out of the paradigmatic examples that constituted its starting point. In this expanded domain, it is not

[1] For in-depth analyses of Lakatos' seminal book, see (Larvor, 1998; Kadvany, 2001).

clear how to apply the concept of a polyhedron correctly, warranting the use of barring-methods against counterexamples. Lemma-incorporation and deductive guessing then offer proof-generated definitions of what a polyhedron is, inscribing proof-lemmas into the definition of the concept in order to shield the conjecture against counterexamples (the former) or to boost its content (the latter). This process creates several proof-generated concepts of a polyhedron, each one of them theoretically stretched by an underlining tentative proof of the conjecture. Lakatos refers to these dynamics as *concept-stretching*, i.e., the stretching of the concept under focus produced by this repeated interactions between conjectures, proofs, and counterexamples.[2]

2.2 Critiques and counter-examples to Lakatos' model

We saw how Lakatos' concept-stretching gives us a recipe for mathematical conceptual change based on the multi-level interactions of conjectures, counterexamples, barring-methods, and heuristics.

This seminal model of conceptual change in mathematics attracted many critiques. Some of the critiques are of a general methodological character. Logicians (e.g., Feferman 1978; Priest and Thomason 2007), for instance, felt that Lakatos' model lacked formal clarity in its recipe for change, while several philosophers (cf. Fine 1978; Corfield 2003; Werndl 2009) complained about the extremely proof-centric account of mathematics that Lakatos gave. Especially with the rise of the philosophy of mathematical practice (cf. Mancosu 2008), a movement that repeatedly stressed the multi-faceted character of mathematical activity (cf. Ferreirós 2015), Lakatos' picture of mathematical conceptual change as driven solely by proofs and refutations appears indeed too narrow as a general model of how concepts in mathematics change.

In addition to these methodological critiques, a common critique moved to Lakatos' model of conceptual change is its inability to reconstruct paradigmatic cases of mathematical conceptual change that differ from Lakatos' case studies. As several philosophers stressed, the history of mathematics exhibits a plurality of conceptual dynamics that cannot be arguably reconstructed as concept-stretching. Feferman (Feferman, 1978), for instance, points to the existence of crystal-clear mathematical concepts, such as the concept of natural number. Nowhere in the history of these concepts one can find the multiple, alternative, definitions of the same mathematical concept prescribed by Lakatos' model of conceptual change. Feferman moreover stressed how, even in historical cases where many alternative definitions of a mathematical concept are indeed present, the conceptual dynamics do not always resemble the alternations of proofs and refutations described by Lakatos. Some cases of mathematical conceptual change seem, in fact, mostly driven by organization and generalization worries, underlying a more collective conceptual evolution. This seems often the case especially in late nineteenth and twentieth century mathematics, where organizational and foundational ideas took central stage (cf. Kitcher 1984). Another weak point of Lakatos' model of conceptual change is that it does not seem to be directly applicable to

[2] Some scholars use other terms to refer to Lakatos' model of conceptual change such as 'concept-trafficking' (Mormann, 2002). I follow Fine (Fine, 1978) in using concept-stretching to refer to the whole model of conceptual change presented by Lakatos.

axiomatic mathematics. As Mormann (Mormann, 2002) stressed, conceptual variation in axiomatized mathematics seems to involve also the change of the proof-problem under focus, a possibility that is not clearly licensed by Lakatos' model.[3]

Faced with these counterexamples to Lakatos' concept-stretching, supporters of a broadly Lakatosian view of mathematics proposed generalizations of Lakatos' model of conceptual change. Examples of such Neo-Lakatosian models are Fine's model of conceptual refinement (Fine, 1978), Hallett's (Hallett, 1979a, b) and Corfield's (Corfield, 2003) sketches of a methodology of mathematical research programs, and Mormann's (Mormann, 2002) evolutionary selection theory for mathematical concepts. All these generalizations of Lakatos' model seek to replace concept-stretching with a more general dynamic that mathematical concepts (or more abstract units of change such as research programs) allegedly ought to follow. In this paper, I will improve Lakatos' model in a different way, not by generalizing its account, but instead by contextualizing it. As we will see in Section 4, I will in fact propose a pluralist approach to mathematical conceptual change that understands the dynamic of proofs and refutations described by Lakatos as one of the many ways in which mathematical concepts can change.

3 Recipes vs pluralist approaches to natural selection

We have seen how several philosophers have criticized Lakatos' model of mathematical conceptual change, proposing alleged counterexamples to its dynamics of proofs and refutations. In this section, we will look at the debate in philosophy of biology concerning the correct conceptualization of natural selection. As we will see in the following pages, this debate exhibits structural analogies to the discussions over mathematical conceptual change that we witnessed in Section 2. I will argue that, thanks to these analogies between the two debates, recent solutions to the problem of conceptualizing natural selection offer a blueprint for improving models of mathematical conceptual change.

Despite the pivotal role that natural selection plays in Darwinian theory, philosophers have discussed at length the most adequate way of expressing the principles behind it. Of special interest for the present paper is the fate of the so-called recipe approaches to natural selection. These approaches try to give an abstract summary of evolutionary dynamics behind natural selection in the form of a recipe for change. Recipes approaches have been heavily criticized in philosophy of biology (cf. Brandon 1978; Godfrey-Smith 2007). In order to understand what recipe approaches to natural selection are, let us take a look at a paradigmatic example of them, i.e., Lewontin's mature formulation of evolution by natural selection:

> "A sufficient mechanism for evolution by natural selection is contained in three propositions:
>
> 1. There is variation in morphological, physiological, and behavioral traits among members of a species (the principle of variation).

[3] Note that whether Lakatos regarded conceptual variation to be exclusive of non-axiomatized mathematics is a controversial point in Lakatosian scholarship. For different takes on this question, see (Corfield, 2002; Feferman, 1978; Priest and Thomason, 2007).

2. The variation is in part heritable, so that individuals resemble their relations more than they resemble unrelated individuals and, in particular, offspring resemble their parents (the principle of heredity).
3. Different variants leave different numbers of offspring either in immediate or remote generations (the principle of differential fitness).

All three conditions are necessary as well as sufficient conditions for evolution by natural selection …Any trait for which the three principles apply may be expected to evolve." (Lewontin, 1985, p. 76)

In this formulation, as well as in other recipes-like ones, variation, heritability and fitness differences are meant to be necessary and sufficient ingredients for producing evolution by natural selection. The problem with these approaches is that these ingredients are arguably neither necessary nor sufficient to cover all the different cases of actual evolution by natural selection. That is, there are cases where all the ingredients are present but change does not occur and cases where change does occur without all the ingredients (Brandon, 1978; Godfrey-Smith, 2007). Godfrey-Smith diagnoses this problem as caused by the attempt of traditional recipes approaches to perform two contrasting tasks at the same time. These recipes are, on the one hand, meant to describe all genuine cases of evolution by natural selection and, on the other hand, expected to consist of a simple, causally transparent mechanism for change. Abstract recipes like Lewontin's are then the result of an uncomfortable trade-off between these two tasks, trying to squeeze all the diverse forms in which natural selection produces evolutionary change into one neat, encompassing mechanism.

As a solution to these issues, Godfrey-Smith (Godfrey-Smith, 2009) proposes a gradual and plural Darwinian framework centered around the family of concepts of a Darwinian population. In Godfrey-Smith's framework, we can talk about a Darwinian population in three senses, a minimal, a paradigm, and a marginal one:

- "A *Darwinian population in the minimal sense* is a collection of causally connected individual things in which there is variation in character, which leads to differences in reproductive output (…) and which is inherited to some extent" (Godfrey-Smith, 2009, p. 39);
- A *Darwinian population in the paradigm sense* is a minimal Darwinian population that has reliable inheritance mechanisms, unbiased and slight variation, reproductive competition, reproductive differences highly dependent on intrinsic features of the individuals, and that exhibits continuity.[4]
- A *Darwinian population in the marginal sense* is a population which does not fully satisfy the requirements for a minimal Darwinian population, but only approximates them.

The minimal concept is supposed to be applicable to very different biological phenomena, requiring only a minimal locality constraint on the members of the population. The members of a Darwinian population in the minimal sense, i.e., the *Darwinian individuals*, must exhibit the three ingredients of recipe-like Darwinism (variation,

[4] This is a rough summary of this concept. What Godfrey-Smith actually requires from a paradigm Darwinian population is more nuanced and gradient. For a full-account of this notion see (Godfrey-Smith, 2009, pp. 41-59)

inheritance, fitness differences) only to some extent. The other two senses in which one can speak of a Darwinian populations are instead designed to stress the extent to which evolution by natural selection is central to the dynamics of a given population. Populations approximating the ideal dynamic of evolution by natural selection are the paradigm ones. These are the Darwinian populations representing significant Darwinian processes, i.e., processes that exhibit all the paradigmatic features of a truly Darwinian process. Paradigm Darwinian populations not only exhibit all the ingredients of recipe-like Darwinism, but they instantiate ‘the right kind’ of variation, fitness differences, and inheritance. These populations exhibit reliable inheritance mechanisms, slight and unbiased variation, reproductive differences highly dependent on intrinsic individual features and other extra features that contribute to make the perfect scenario for evolution by natural selection. Finally, the concept of a marginal Darwinian population allows one to stretch Darwinian concepts onto biological phenomena whose dynamics are not really Darwinian, but in which one can discern aspects that are partially Darwinian in character.

Godfrey-Smith adds structure to his population-based set-up of evolutionary theory with the aid of the *Darwinian space*, i.e., a space the dimensions of which are parameters tracing how much a population is paradigmatically Darwinian with respect to a given feature. This spatial structure is meant to split into different dimensions the extent to which a given evolutionary process has a Darwinian character, allowing a gradual representation of all the possible types of Darwinian processes. The Darwinian space has five dimensions, representing five different parameters: fidelity of inheritance, abundance of variation, reproductive competition, continuity, and dependence of reproductive differences on intrinsic character (Godfrey-Smith, 2009, p. 63). Fidelity of inheritance tracks how much the state of a parent is predictive of the state of the offspring. Abundance of variation measures the amount of variation amongst the individuals of a population at a time. Reproductive competition indicates the extent to which the reproductive success of a given individual reduces the success of others members of the population. Continuity is a measure of the overall extent to which similar members of the populations have similar fitness. Dependence of reproductive differences on intrinsic character tracks how much differences in reproductive output are caused by intrinsic features of the members of the population (and not by extrinsic ones).

Each of these parameters represents an aspect with respect to which a given population can be more or less Darwinian. Different regions of the Darwinian space, i.e., different combinations of these parameters, represent different types of biological phenomena. Paradigm Darwinian populations occupy then the part of the Darwinian space where all five parameters take high values, while marginal Darwinian populations are at the opposite side. Minimal Darwinian populations occupy instead a large portion of the space, including the part where paradigm Darwinian populations are. Moreover, specific regions of the space (representing specific combinations of the parameters) are able to explicate phenomena underlying specific dynamics of populations such as the concept of drift and error catastrophe (Godfrey-Smith, 2009, pp. 59-64).

These spatial tools enrich the family of concepts of a Darwinian population with a more fine-grained structure, enabling Godfrey-Smith’s framework to adequately represent the plurality of evolutionary dynamics. The diversity of ways in which evolution by natural selection occurs is not squeezed anymore into a one-size-fits-all abstract recipe, but it is reflected by all the possible combinations of parameters allowed by the Darwinian space. Thanks to this rich structure, Godfrey-Smith's framework is able to account for several issues faced by recipe-based account of Darwinism, such as the problem of units of selection, the relationship between reproduction and individuality, or the explication of evolutionary drift.

4 Lakatosian populations, Euclidean populations, and the Lakatosian space

We have seen how Godfrey-Smith offers a pluralist Darwinian framework that overcomes the issues faced by recipe-like accounts of evolution by natural selection. Natural selection is understood by Godfrey-Smith not as a general model that all evolutionary episodes of change must instantiate in the same way, but as an ideal that biological populations can exhibit to various degrees with respect to several aspects.

In what follows, I am going to propose an analogous pluralist framework for modeling mathematical conceptual change that evades the problems faced by Lakatos' concept-stretching. We have in fact seen in Section 2 that there seem to be many counterexamples to Lakatos' model of conceptual change, i.e., many cases where mathematical concepts change without exhibiting the dialectics of proofs and refutations prescribed by Lakatos. These counterexamples, analogously to the ones to recipes-like approaches to natural selection, show the impossibility of squeezing a vast range of conceptual episodes into one single encompassing mechanism for change. My solution to this problem is, in analogy with Godfrey-Smith's solution for natural selection, to rethink Lakatos' concept-stretching as an ideal of conceptual change that conceptual histories in mathematics can exhibit to various degrees with respect to several aspects.

Let me stress that, by building upon Godfrey-Smith's Darwinism, I do not seek to develop a Darwinian model of conceptual change. What I will do is, instead, to use the rich structure of Godfrey-Smith's framework, i.e., a family of population concepts the dynamics of which is structured via multiple dimensions, to develop a model of mathematical conceptual change. Specifically, I will center my framework around the opposition between two, radically different, ideals of conceptual change, i.e., the Lakatosian and the Euclidean ideal, respectively represented by two types of conceptual populations, i.e., Lakatosian and Euclidean populations. I will structure the relationship between these two types of populations via a spatial framework, i.e., the Lakatosian space. The Lakatosian space will be made of the following three parameters: conceptual variation, reproductive competition, and continuity. These parameters will trace how much a given conceptual population is more Lakatosian or more Euclidean with respect to a given aspect of its conceptual change dynamic. Different regions of the Lakatosian space will then represent different dynamics of conceptual change that conceptual populations in mathematics can exhibit. In this section, I will present my framework, while in Section 5 I will apply it to three case studies from the history of mathematics.

4.1 Lakatosian populations and Euclidean populations

My framework is centered around the notion of a *conceptual population* (Toulmin, 1970, 1972), i.e., a group of conceptual variants competing for similar mathematical problems. More specifically, in my framework, conceptual populations are entities made of three components: conceptual variants, mathematical problems, and a heuristic power ordering. Conceptual variants are specific versions of a mathematical concept. Mathematical concepts are the main actors of my framework. I am not assuming any specific psychological or philosophical theory about the specific structure or ontological status of concepts. What I am assuming is a broadly Lakatosian view of mathematical concepts, according to which concepts in mathematics are elastic, often indefinite, entities that are modifiable by the problem-solving practice of mathematicians (cf. Schlimm 2012). My framework is thus ***prima facie*** incompatible with a Fregean view of mathematical concepts, where concepts are fixed and definite third-world entities that cannot change. [5] A ***conceptual variant*** is then a specific version of a given mathematical concept that has been used by a mathematician in an attempt to solve (one of) the mathematical problem(s) faced by a given population. Different variants of a given mathematical concept are often identifiable with the different definitions of this concept that mathematicians put forward, but this is not always the case. In conceptual populations corresponding to very informal mathematical practices, in fact, different understandings and uses of a mathematical concept could represent different conceptual variants without necessarily involving different definitions or, alternatively, different definitions could be just a matter of terminological difference (and, therefore, not corresponding to different variants). What constitutes a specific version, and thus a conceptual variant, of a mathematical concept is thus not something that can be specified in a general definition, but it is instead dependent on the specific pragmatic and disciplinary characteristics of the mathematical practice that a given conceptual population represents. *Mathematical problems* are abstract problems that can be instantiated by many token-like specific questions. Mathematical problems are not restricted only to searches for proofs but they can be problems of different sorts, such as classification problems (e.g., the search for ordering principles in nineteenth-century geometry), definitional problems (e.g., Hamilton's search for higher-complex numbers with a suitable geometrical and algebraic reading), and many others.[6] Proof-problems such as Euler's conjecture, are only a proper subset of my understanding of mathematical problems. Conceptual variants of the same conceptual population have to face similar mathematical problems, competing against each other for starring in successful solution-attempts, i.e., valid solutions.

This interaction between conceptual variants and mathematical problems produces a ranking of conceptual variants of a given conceptual population that I call *heuristic power ordering*. The heuristic power of a given conceptual variant of a given conceptual population tracks the disposition of this variant to successfully interact with the related

[5] That said, enthusiasts of a Fregean view of mathematical concepts can perhaps have a deflationary, merely linguistic reading of my framework, seeing conceptual variants as different modes of references or linguistic items connected to a mathematical concept.

[6] Some philosophers might consider these kinds of mathematical problems as implicit proof-problems. Consistently with the pluralist spirit of the whole framework, I chose to include different kinds of problems.

mathematical problem, i.e., its propensity to figure in valid solutions to that problem. The more promising a variant is, i.e., the higher its heuristic power or ranking in the population, the more likely this variant will become the accepted definition of the concept. Symmetrically, variants with low heuristic power are more likely to appear weird and artificial definitions of a given concept. The heuristic power can be thought as a kind of ordinal fitness ranking among variants of a mathematical concept, intuitively understandable as the propensity of a given variant of being used in fruitful solutions by mathematicians working to solve the given mathematical problem.[7]

Just like actual biological Darwinian populations exhibit different evolutionary dynamics, conceptual populations may exhibit different kinds of dynamics of conceptual change. If, in Godfrey-Smith's Darwinian framework, Darwinian populations can instantiate the Darwinian ideal to a different extent (thereby qualifying to be Darwinian populations in the marginal, minimal, or paradigmatic sense), in my framework conceptual populations can instantiate two opposite ideals to a different extent. The central idea of my framework is, in fact, that certain kinds of dynamics make conceptual populations approximate Lakatos' ideal of proofs and refutations, while others are typical of populations closer to the opposite Euclidean ideal of (absence of) change. I will make more precise this idea by defining two opposite kinds of conceptual populations, namely Lakatosian and Euclidean populations:

> **Lakatosian Population**: a conceptual population in which there is *high conceptual variation* and *high reproductive competition* between the conceptual variants, which lead to differences in *heuristic power continuously* distributed.

> **Euclidean Population**: a conceptual population in which there is *low conceptual variation* and *low reproductive competition* between the conceptual variants, which lead to differences in *heuristic power discretely* distributed.

These two types of conceptual populations are defined around three notions: conceptual variation, reproductive competition, and continuity in the distribution of heuristic power. These notions denote three different aspects with respect to which conceptual populations can be more Lakatosian or more Euclidean. Conceptual variation denotes the amount of variation amongst conceptual variants of a given population. Reproductive competition denotes the extent to which conceptual variants of a population are competing for the same problem. Finally, the continuity in the distribution of heuristic power denotes, in analogy with the concept of fitness-landscape (Wright, 1932) in evolutionary biology, whether similar conceptual variants of a given population have similar heuristic power. If this condition occurs, I will say that a given conceptual population has a continuous distribution of heuristic power; otherwise I will call that distribution discrete.

Lakatosian populations are then conceptual populations with high conceptual variation, high reproductive competition, and a continuous distribution of heuristic power. This combination of these three aspects makes the kind of conceptual change exhibited

[7] Even though biological fitness is usually measured on an absolute scale, it has been argued that an ordinal scale would suffice (Okasha, 2018, pp. 168-170).

by a conceptual population approach the ideal of Lakatos' concept-stretching. Examples of Lakatosian populations are Lakatos' own case studies, i.e., the polyhedron population and the continuity population (Lakatos, 1976, 1978). As we have seen in Section 2, Lakatos' dance of proof-attempts and counter-examples involves a plethora of different definitions of the concept under focus, competing against each other with the aim of solving the same mathematical problem. From the perspective of my framework, then, Lakatosian populations are conceptual populations with high conceptual variation. Furthermore, Lakatosian populations must exhibit also high reproductive competition amongst the variants, i.e., the variants of a population are not just competing for similar problems, but for the same one(s). In other words, there have to be many conceptual variants and few problems. Finally, Lakatosian populations enjoy a continuous distribution of heuristic power, i.e., similar conceptual variants have similar heuristic power. The similarity in the heuristic power of similar conceptual variants is in fact paramount for the repeated series of proof-attempts and counterexamples that is the engine of Lakatos' concept-stretching.

Euclidean populations are instead conceptual populations that exhibit low conceptual variation, low reproductive competition, and a discrete distribution of heuristic power. This combination makes a conceptual population approach the ideal of Euclidean absence of conceptual change. Examples of Euclidean populations are the so-called crystal-clear mathematical concepts such as the concept of natural number. The evolution of concepts like natural number does not seem in fact to involve any form of conjecture and refutations whatsoever, but just a series of rigorizations and conceptual analyses of a well-understood concept. From the perspective of my framework, then, Euclidean populations must exhibit a low conceptual variation and a low reproductive competition. In these populations, in fact, one can find few conceptual variants that are often meant to tackle different problems, thereby not really competing with each other. Finally, Euclidean populations exhibit a discrete distribution of heuristic power, i.e., small variations in the definition of a concept may cause significant differences in the heuristic power of the variants involved.

Lakatosian populations and Euclidean populations are then two opposite kinds of conceptual populations, respectively describing opposite dynamics of conceptual change. In a Lakatosian population many conceptual variants with similar heuristic power compete against each other for the same problem(s). In a Euclidean population, instead, few variants with different heuristic power cope with different problems.

4.2 Lakatosian space

I will now add more structure to the opposition between Lakatosian and Euclidean populations. I will present three parameters that track the degree to which a conceptual population exhibits one of the aspects through which I discussed Lakatosian and Euclidean populations, i.e., conceptual variation, reproductive competition, and continuity in the heuristic power distribution. These three parameters can be understood as dimensions constituting the *Lakatosian space*. Points in this space are possible combinations of these parameters, representing a possible kind of conceptual population. Conceptual populations of the same kind, i.e., having the same combination of these three parameters, occupy the same point of the Lakatosian space. This additional spatial structure provides my framework with a fine-grained way of understanding to which degree a given conceptual population exhibit a more Lakatosian or a more Euclidean character with respect to one of these three aspects. Let us survey the three parameters constituting the dimensions of the Lakatosian space, one by one.

Conceptual Variation (CV): This parameter represents the amount of variation among the conceptual variants of a given conceptual population. It can be measured by tracking how many different variants a given conceptual population exhibits. Conceptual populations with high CV are representing (parts of) mathematical conceptual histories in which many possible definitions of a concept are proposed and discussed. This situation is typical of stages of generalization of accepted concepts, where several properties of the concept in the wider context are open to discussion (cf. Waismann 1948), such as the case of the quaternions (Hamilton, 1853). Conceptual populations with low CV represent instead (parts of) mathematical conceptual histories in which a (group of) definition(s) is accepted and therefore not truly questioned. This situation is typical of periods in the history of a given mathematical field in which a natural or a very fruitful definition of a concept is found (Tappenden, 2008a, b) and, using a game-theoretic notion, it becomes evolutionary stable against mutations (Weibull, 1995). An example of this situation is the abstract concept of group.

Reproductive Competition (RC): This parameter tracks the extent to which conceptual variants have to compete for the same mathematical problems, i.e., how much different definitions of a given concept have to 'fight against each other to survive'. It can be measured by looking at the ratio between the number of conceptual variants and the number of mathematical problems of a given conceptual population. The higher the ratio, the more competitive the conceptual population is. The lower the ratio gets, instead, the less a given conceptual population resembles a 'struggle for existence'. Populations with low RC often have an environment made of several mathematical problems for which 'specialized' variants evolved in parallel, defusing the struggle amongst the variants. This situation is typical of (periods of) bodies of mathematics in which the conceptual variants are very well adapted to specific problems, such as the many pre-calculus 'analysis' techniques. Populations with high RC have instead many conceptual variants competing for the same problem(s). This situation is typical of (periods of) bodies of mathematics centered around a general problem, such as the Newtonian and Leibnizian calculus.

Continuity ($Cont$): This parameter tests whether the distribution of heuristic power among conceptual variants of a given population is continuous, i.e., whether similar variants have a similar heuristic power. In analogy to the fitness-landscape biological metaphor, if a given conceptual population exhibits continuity, then its heuristic power landscape is smooth. Otherwise, the distribution of heuristic power among the variants of the population is somewhat discrete, i.e., small changes in the definition of a variant can lead to enormous differences in terms of fruitfulness. As we will see in the next section, this continuity (or the lack thereof) in the distribution of heuristic power is connected with the degree of axiomatization of the related body of math-

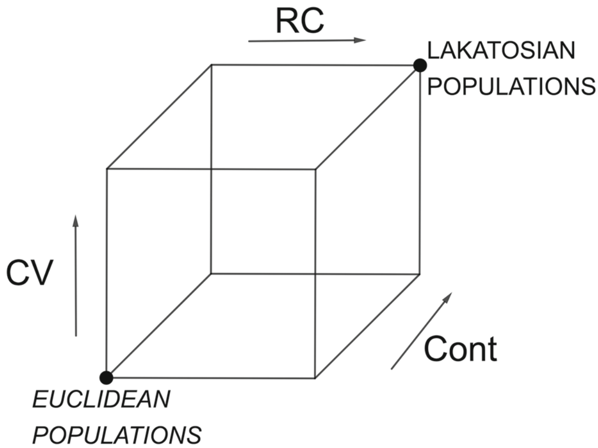

Fig. 1 A representation of the Lakatosian space, showing the parts of the space corresponding to Lakatosian and Euclidean populations

ematics. Axiomatized bodies of mathematics constrain in fact the possible choices of conceptual variants to the ones produced by the tinkering of the axioms, i.e., the manipulation of an existing axiomatic characterization of a concept aimed at conceptual change (Pickering, 1995; Schlimm, 2013). Small variations in a given axiom may have then enormous repercussions on the heuristic power of the conceptual variants so defined. Highly axiomatized bodies of mathematics typically exhibit therefore a discrete distribution of heuristic power, while conceptual histories that are not (fully) axiomatized usually enjoy a continuous one. Examples of the former kind of conceptual populations are the quaternions or the abstract group concept, whereas the pre-abstract group concepts exemplify the latter.

These three parameters then can be understood as the dimensions of the Lakatosian space. We can then assign to both Lakatosian and Euclidean populations a given region of the Lakatosian space (Fig. 1).

5 Three case studies

I have presented my pluralist approach to mathematical conceptual change, centered around the notion of a conceptual population, the opposition between Lakatosian and Euclidean populations, and the Lakatosian space. In this section, I will show how my framework can be applied to three historical episodes of mathematical conceptual change: Lakatos' own example of Euler's conjecture and the concept of polyhedron (Lakatos, 1976), Hamilton's invention of the quaternions (Hamilton, 1843a, b, 1853; Pickering, 1995), and the pre-abstract group concepts (Wussing, 1984). We will see how these three case studies exhibit three different conceptual change dynamics, typical of different kinds of conceptual populations. We will see that these different kinds of conceptual population occupy different parts of the Lakatosian space, having a more

Lakatosian or a more Euclidean character with respect to the three dimensions of the space (i.e., conceptual variation, reproductive competition, and continuity).

Before starting to look at the case studies, let me stress that, in order to apply my framework to historical case studies, one needs to represent a certain mathematical practice as a conceptual population, that is, as we saw in the last section, one needs to identify a set of conceptual variants, a set of mathematical problems, and a heuristic power ordering. Such identification involves many theoretical and historical assumptions on the mathematical practice that is the object of the case study, including which mathematical problems were the mathematicians of that practice actively trying to solve, what counted as a properly different conceptual variant, and which variants were involved in fruitful solutions. As such, the following reconstructions of historical episodes of mathematical conceptual change as conceptual populations, and the related attributions of high and low score on the dimensions of the Lakatosian space, should be considered dependent on the historical reconstruction through which they are carried out, and not as absolute in character.

5.1 Lakatos' polyhedron example

Lakatos' own master example of the dynamic of proofs and refutations can be easily represented as a conceptual population. A conceptual population, in my framework, has three components: a set of conceptual variants, a (set of) mathematical problem(s), and a heuristic power ordering.

In Lakatos' book (Lakatos, 1976), one can find no less than thirteen different definitions of a polyhedron, all facing the same proof-problem of Euler's conjecture. These different definitions of polyhedron will constitute the set of conceptual variants of (what I will call) the polyhedron population. Specifically, we can take, as the first conceptual variant (p_1) of our polyhedron population the naive concept of polyhedron which is used in the proof-experiment of Euler's conjecture (Lakatos, 1976, pp. 6-10). Seven other conceptual variants are given by the definitions of a polyhedron generated via the method of monster-barring: (p_2) "a solid whose surface consists of polygonal faces" (Lakatos, 1976, p. 15); (p_3) "a surface consisting of a systems of polygons" (Lakatos, 1976, p. 16); (p_4) "a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and (2) it is possible to get from the inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex" (Lakatos, 1976, p. 17); (p_5) "in the case of a genuine polyhedron, through any arbitrary point in space there will be at least one plane whose cross-section with the polyhedron will consist of one single polygon" (Lakatos, 1976, p. 23); (p_6) definition of p_5 plus the further condition that "edges have two vertices" (Lakatos, 1976, p. 24). Two other variants of a polyhedron are obtained via the method of exception-barring: (p_7) "polyhedra that have no cavities, tunnels, or 'multiple structure'" (Lakatos, 1976, p. 29), (p_8) "convex polyhedra" (Lakatos, 1976, p. 30). Then, we have two variants of a polyhedron generated by the method of lemma-incorporation: (p_9) "simple polyhedra, i.e., those which, after having had a face removed, can be stretched onto a plane" (Lakatos, 1976, p. 36), (p_{10}) "simple polyhedron with all its faces simply-connected" (Lakatos, 1976, p. 38). Finally, content-increasing methods give us other three variants of a polyhedron, i.e., (p_{11}) “Gergonne-polyhedra" (Lakatos, 1976, p. 63), (p_{12}) “Legendre-polyhedra" (Lakatos, 1976 p. 63), and (p_{13}) “closed normal polyhedra" (Lakatos, 1976 p. 81).[8]

The mathematical problem of the polyhedron population is the Euler’s conjecture, i.e., the constant proof-problem that polyhedron variants have to face in Lakatos’ reconstruction. The heuristic power ordering amongst the variants then simply corresponds to their order of appearance in Lakatos’ discussion, because each one of them is introduced as a way of dealing with a given counterexample affecting the previous variants or via a content-increasing method. Thus, one can assume that the conceptual variants appear in Lakatos’ reconstruction in increasing order of fitness to Euler’s conjecture (i.e., in increasing order of heuristic power).

Now that we reconstructed Lakatos’ example as a conceptual population, we can assess its conceptual change dynamics from the perspective of my framework. Not surprisingly, the polyhedron population is an example of a Lakatosian population, i.e., a conceptual population exhibiting high conceptual variation, high reproductive competition, and a continuous distribution of heuristic power. We have in fact seen above the remarkable number of polyhedron variants that appear in Lakatos’ reconstruction. In my terminology, then, the polyhedron population clearly exhibits high conceptual variation. Moreover, we saw that all the different variants of a polyhedron compete against each other in the context of proving Euler’s conjecture. The polyhedron population exhibits thus high reproductive competition. Finally, the polyhedron population arguably shows a continuous distribution of heuristic power among its conceptual variants, i.e., similar definitions of a polyhedron have similar heuristic power. This can be seen by looking at pairs of very similar definitions of a polyhedron such as (p_2, p_3). These pairs of variants that differ only for a minor tweak in their definition cope similarly with Euler’s conjecture, i.e., they face (almost) the same counterexamples.

We can now appreciate how the specific character of the polyhedron population allows Lakatos to describe the history of the polyhedron concept as a paradigmatic example of concept-stretching. In order for a concept to be stretched via Lakatos’ succession of proofs and refutations, in fact, there is a need of a stable mathematical problem and a plethora of tentative definitions of a mathematical concept. All these tentative definitions have to compete against each other for solving the same problem and there cannot be significant discrepancies of heuristic power among similar definitions. In other words, Lakatos’ concept-stretching model needs a specific kind of conceptual populations with high conceptual variation, high reproductive competition, and a continuous distribution of heuristic power. Lakatos’ concept-stretching is thus perfect to describe the dynamics of conceptual change typical of Lakatosian populations. My framework is then able to specify the right domain of application of Lakatos’ model of mathematical conceptual change: mathematical conceptual histories that can be reconstructed as Lakatosian populations. In the next two subsection, we will see in fact how the quaternion and the pre-abstract group populations, i.e., two examples

[8] In some sense, Lakatos discusses another definition of a polyhedron, formalized in terms of vector algebra (Lakatos, 1976 pp. 112-126), but this definition is of a completely different kind than previous ones and thus should not be considered as a variant of the same conceptual population. It is a tentative formalization of both the concept of a polyhedron and Euler’s conjecture in terms of vector algebra and thus pertains to a different conceptual population than the polyhedron one.

that have been claimed to defy Lakatos' concept-stretching model (Feferman, 1978; Mormann, 2002), are not examples of Lakatosian populations.

5.2 Hamilton's invention of the quaternions

As my second case study I will reconstruct the conceptual history behind Hamilton's invention of the quaternions. This historical case has been discussed by Mormann (Mormann, 2002) as an example of axiomatic variation that defies Lakatos' model of concept-stretching. I will show how, when reconstructed as a conceptual population, the quaternion population exhibits indeed different features than ones typical of Lakatosian populations. As an historical basis for my reconstruction I will follow Hamilton's own memoirs (Hamilton, 1843a, b, 1853), together with Pickering's detailed analysis of Hamilton's practice (Pickering, 1995).

Hamilton's search for quaternions started with the idea of generalizing complex numbers to triplets. Before going into Hamilton's repeated tries into developing systems of triplets, I need to stress some basic facts about algebraic and geometric properties of complex numbers needed to understand Hamilton's generalization attempts. Central to Hamilton's research is the geometrical understanding of complex numbers, where the real and the ideal component of a number are not seen as quantities, but as coordinates of the end-point of a line segment starting from the origin in a two-dimensional plane. In this interpretation, the x-axis of the plane represents the real component of a given number, while the y-axis the imaginary one. This correspondence between algebraic entities and line segments extends also to the operations between complex numbers, so that algebraic operations can be given a meaningful geometrical reading. Multiplication between complex numbers can be thus defined equivalently algebraically as

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

or geometrically as the conjunction of two rules: "the product of two line segments is another line segment that (1) has the length given by the product of the lengths of the two segments to be multiplied, and that (2) makes an angle with the x-axis equal to the sum of the angles made by the two segments" (Pickering, 1995, p. 123).

Hamilton's search for higher complex numbers started by generalizing this geometrical reading of complex numbers to the three-dimensional case. He started thinking about another imaginary component j, geometrically represented as a line perpendicular to the two-dimensional complex plane (Hamilton, 1843b, p. 107). He also naturally assumed that $j^2 = -1$. We can take this first vague idea of a triplet as constituting the first conceptual variant (q_1) of the quaternion population.

Hamilton then focused on the algebraic operations that can be performed on this conceptual variant. Addition and subtraction were easily extended to the triplet case. Multiplication, instead, provided the newborn quaternion population with a stable mathematical problem. Hamilton started from the restricted case:

$$(x + iy + jz)^2 = x^2 - y^2 - z^2 + 2ixy + 2jxz + 2ijxz$$

The problem was how to understand the last term of the equation, $2ijxz$ and the product ij there contained. Hamilton's first natural choices, giving rise to two new conceptual variants of the quaternion populations, were $(q_2)\ ij = 1$ and $(q_3)\ ij = -1$. These two variants were both equally understandable from a purely algebraic point of view, but they both failed to have a reasonable geometric interpretation. Both variants were in fact still understood from the geometrical perspective of a line perpendicular to the complex plane and thus Hamilton's geometrical understanding of the multiplication operation was that "its real part ought to be $x^2 - y^2 - z^2$ and its two imaginary parts ought to have for coefficients $2xy$ and $2xz$" (Hamilton, 1843a, p. 103). The term $2ijxz$ contained in the algebraic understanding of the multiplication needed to vanish.

A new conceptual variant (q_4) arises exhibiting $ij = 0$, thus making the algebraic understanding of the multiplication in superficial agreement with Hamilton's geometrical intuitions. Only superficially, though, because to make the product of two arbitrary segments equal to zero violates the geometrical rule that wants the length of the segment product equal to the lengths of the segments multiplied (Pickering, 1995, p. 132).

Hamilton then let the commutativity assumption common to all the aforementioned conceptual variants go and assumed the more general $(q_5)\ ij = k$ and $ji = -k$, leaving undefined the value of k. This new conceptual variant achieved for the first time complete agreement between the algebraic and the geometric interpretation of multiplication for the aforementioned restricted case of the mathematical problem. Hamilton was then led to the general case of the multiplication of two arbitrary triplets and there the new conceptual variant was of little use. How one should understand the orientation of the product triplet for non-coplanar triplets? Hamilton thus dropped the perpendicularity to the complex plane assumption, together with the orientation-part of the geometrical understanding of multiplication for complex numbers. He first returned to the idea of $ij = 0$, this time not restricted by these two assumptions (q_6) and started working the general case only in terms of the length-part of the geometrical understanding of multiplication. Again, the algebraic and the geometrical understanding of the multiplication operation did not agree with each other, forcing Hamilton to a more radical departure from his original intuitions.

Hence, Hamilton started considering k not only as the undefined product of i and j, like it was in the variants q_5 and q_6, but as a whole new imaginary, thus obtaining the first conceptual variant in the quaternion populations with three different imaginary components (q_7). This new conceptual variant was still too unspecified to cope successfully with the multiplication problem in its general setting, since k^2 was still undefined. Three different choice for specifying this quantity naturally presented themselves to Hamilton, namely $(q_8)\ k^2 = 0$, $(q_9)\ k^2 = 1$, $(q_{10})\ k^2 = -1$. The first variant, i.e., q_8, was quickly discharged for breaking again the geometrical reading of multiplication. Finally, Hamilton saw that q_{10} was the only choice that coped successfully with the multiplication environment:

> And since the order of these imaginaries is not indifferent, we cannot infer that $k^2 = ijij$ is $+1$, because $i^2 \times j^2 = -1 \times -1 = +1$. It is more likely that $k^2 = ijij = -iijj = -1$. And in fact this last assumption is necessary, if we would conform the multiplication to the law of multiplication of moduli. (Hamilton, 1843b, p. 108)

We can now model this conceptual history as a conceptual population. The set of conceptual variants of the quaternion population is made of all the 10 different triplet and quaternion variants singled out in the historical narrative above. The mathematical problem that all these variants had to face was the problem of having a fully satisfactory multiplication operation, i.e., what I called the multiplication problem. Like in the case of the polyhedron population, we can assume that the heuristic power ordering among the quaternion variants corresponds to their order of appearance in my recollection of Hamilton's search for quaternions, since any quaternion variant that I presented is superior to the precedent ones in terms of counterexamples or success with the restricted domain of the multiplication problem.

Let us look at what kind of conceptual population the quaternion population exemplifies. The reconstruction of Hamiliton's invention of the quaternions provides us with many different conceptual variants. Thus, the quaternion population can be said to exhibit high conceptual variation. Furthermore, we saw that all the quaternion variants compete against each other in coping with the multiplication problem, making the quaternion population a conceptual population with high reproductive competition. With respect to the parameters of conceptual variation and reproductive competition, the quaternion population exhibits a Lakatosian character, just like the polyhedron population. Intuitively, however, the reconstruction of Hamilton's invention of the quaternions tells us a different story than Lakatos' fictional classroom. Quaternion variants do not exhibit the same dynamics of conceptual change that polyhedron variants have. The appearance of quaternion variants is somehow constrained by the possible ways in which the related axioms can be manipulated. The story of Hamilton's research is a story of axiomatic tinkering, a story of a painstaking succession of small modifications to the definition of hyper-complex numbers needed to produce a suitable multiplication operation for this extended number domain. In this story, we saw that small modifications to the definition of a quaternion, such as the steps from q_3 to q_4 and from q_9 to q_{10}, produced huge discrepancies of effectiveness in coping with the multiplication problem. The latter case is particularly striking, since the last two variants considered by Hamilton differ only in the polarity of their specification of k^2, which is $+1$ in q_9 and -1 in q_{10}. This small difference is enough to cause a very significant hiatus in terms of heuristic power between the two variants, making q_{10} the only quaternion variant to cope with the general multiplication problem in a successful way. From the perspective of my framework, the quaternion population clearly exhibits a discrete distribution of heuristic power. With respect to this specific aspect, the quaternion population has an Euclidean character and therefore it cannot be considered a Lakatosian population. In this way, the quaternion population shows us a different dynamic of conceptual change than Lakatos' polyhedron example. The difference between the two examples lies in the discreteness of the heuristic power distribution among the quaternion variants. The present case study shows how this discreteness is a symptom of a highly axiomatized body of mathematics. We saw, in fact, how Hamilton's manipulation of axiomatic systems shaped the possible quaternion definitions into a set of mutually exclusive options with very different heuristic power.

We have then see how Hamilton's invention of the quaternions represents a different example of conceptual change than Lakatos' polyhedron case. Thanks to the fine-grained structure of my framework, we saw how this difference can be traced to a specific feature of the quaternion population, namely the discreteness in how the heuristic power is distributed among the quaternion variants. This discreteness is the reason why Hamilton's search for an adequate quaternion concept cannot be reconstructed as exemplifying the Lakatosian ideal of proofs and refutations. The specific features of the quaternion populations (i.e., a high conceptual variation, a high reproductive competition, and a discrete distribution of heuristic power) make this population exemplifying instead a different model of mathematical conceptual change centered around the mathematical activity known as *axiomatic tinkering* (Pickering, 1995; Schlimm, 2013). In an episode of conceptual change driven by axiomatic tinkering, the dialectic of tentative proofs and supposed refutations envisaged by Lakatos is heavily constrained by the axiomatic character of the mathematical problem under focus. Conceptual change becomes then a byproduct of the mathematician's repeated struggle to manipulate some specific mathematical axioms in a satisfactory manner.

5.3 Pre-abstract group theory

As my third case study, I will focus on the history of pre-abstract group concepts, using as historical reference the detailed reconstruction of Wussing (Wussing, 1984). I will show how this conceptual history represents a kind of conceptual population different from both the polyhedron and the quaternion population.

Pre-abstract group concepts were developed between 1770 and 1880 in relation to three connected but independent fields of mathematical inquiry: number-theory, algebra, and geometry. Specifically, they arose in the context of three mathematical problems: the classification of number forms in number theory, the general solvability of algebraic equations in algebra, and the search for ordering principles in geometry.

The first mathematical problem in relation to which some group variant implicitly appeared was the problem of developing a general theory of forms (such as binary quadratic forms) in number theory. Wussing shows how Euler's theory of power residues involved in its partitioning of reminders "a clear example of group-theoretic thinking" (Wussing, 1984, p. 49). We can use the implicit, vague and heavily underdefined group-theoretic notion at work in Euler's paper as our first conceptual variant of the pre-abstract group population (g_1). Gauss' work gives us the next two group variants that emerged in relation to this problem. The first one is his notion of 'congruence' (g_2) that he used to structure and extend Euler's theory of power residue (Wussing, 1984, pp. 52-54). The second one is the notion of 'composition of forms' (g_3), which constituted the center of Gauss' general theory of forms (Wussing, 1984, pp. 55-61). The final conceptual variants within the pre-abstract group population that emerged in the context of number theory was Kronecker's axiomatization of a finite abelian group (g_4) (Wussing, 1984, pp. 61-67).

The second set of group variants is the one related to the problem of solving algebraic equations of higher degree. Lagrange was the first to undertake a structural study of algebraic equations (Wussing, 1984, pp. 71-79). The central offspring of his seminal study was the connection between the solvability of algebraic equations and the concept of permutation. Specifically, Lagrange realized that the degree of the resolvent of a given equation is the number of different values that the roots of the original equation take when permuted in all the possible ways. This implicit notion of permutation constitutes another group variant (g_5), the first that emerged in the context of algebra. The next steps in the theory of permutations give us two other conceptual variants. Ruffini built on Lagrange's theory, asserting for the first time the unsolvability by radicals of equations with degree higher than four. In his work one can find a general classification of permutations, simple and various kinds of complex ones, where he used the notion of permutation with implicit group-theoretic character (g_6) (Wussing, 1984, pp. 80-84). Cauchy improved further the theory of permutation with his concept of 'system of conjugate substitution' (g_7), with which he implicitly defined (a version of) the permutation-theoretic concept of group in terms of its generator. Finally, Galois was the first to define explicitly the permutation-theoretic concept of a group (g_8), understood as necessarily closed under multiplication (Wussing, 1984, pp. 111-117). He used this notion for defining the 'Galois group of an equation', which together with the pivotal property of the normality of a subgroup, allowed him to assign at every equation a permutation group whose structure reveals all the essential properties of the equation, including whether it is solvable by radicals.

The last problem connected to the pre-abstract group variants is the search for ordering principles in geometry. As our first variant that emerged in this context we can take Möbius' notion of 'affinity' (g_9) used in his intuitive classification of geometric relations (Wussing, 1984, pp. 35-42). The next conceptual step in ordering geometries is Cayley's notion of 'invariant' (g_{10}), which he used in his abstract classificatory efforts. These steps in the search for ordering principles led famously to the Erlangen Program and its group-theoretic classification of geometries. In regards to new pre-abstract group variants, we owe to the Erlangen Program a new explicit definition of group (Wussing, 1984, pp. 187-193). Klein defined a group not in terms of permutations (like Galois did), but he spelled out his variant of the group concept in terms of transformations (g_{11}). After the Erlangen Program, Klein and Lie respectively developed two other variants of the group concept, obtained by extending and sharpening the still quite under-defined notion of transformation group (Wussing, 1984, pp. 205-223). We owe to Klein the notion of an infinite discrete group of transformations (g_{12}) and to Lie the notion of a continuous group of transformation (g_{13}).

We can now model this conceptual history as a conceptual population. The set of variants of the pre-abstract group population consists of all the 13 conceptual variants singled out so far. The set of mathematical problems consists of three different mathematical problems, i.e., the classification of number forms, the solvability of algebraic equations, and the search for ordering principles in geometry. With regards to heuristic power, in this case study we have three different, partial orderings. In fact, as Wussing stresses in his historical reconstruction, the three aforementioned mathematical problems gave rise to three different sets of group variants, each one of them with its own preferred definition of a group, i.e., finite abelian groups, permutation groups, transformation groups. All these three notions were selected as the culmination of a series of implicit and explicit group-theoretic notions, each one with a more general intended domain or more successfully adapted to a restricted version of the problem than the precedent one. Then, the three different mathematical problems give rise to three different partial heuristic power orderings.

We can then assess which kind of conceptual population the pre-abstract group population is. With respect to conceptual variation, the reconstruction of the history of the pre-abstract group concepts gives us many different conceptual variants. Just like the polyhedron and the quaternion population, the pre-abstract group population exhibits then high conceptual variation. With respect to the distribution of heuristic power, like the polyhedron population and unlike the quaternion population, the pre-abstract group population exhibits a continuous distribution of heuristic power. In all three partial heuristic power ordering, in fact, similar pre-abstract group variants, such as g_6 and g_7 for instance, have very similar heuristic power.[9] With respect to the parameters of conceptual variation and distribution of heuristic power, then, the pre-abstract group population has a Lakatosian character. However, this population has a Euclidean character with respect to the other dimension of the Lakatosian space, i.e., reproductive competition. As we saw in the historical reconstruction, the history of pre-abstract group concepts proceeded as a series of generalizations and further applications, without any significant dialectic between proofs and refutations. Moreover, the plurality of mathematical problems of the pre-abstract group population allowed the coexistence of different pre-abstract group variants, each one of them very successful in coping with its own related problem. The three aforementioned locally preferred pre-abstract group variants, i.e., finite abelian groups, permutations groups, transformation groups, evolved collectively, each one of them improving their respective predecessors within the context of a specific mathematical problem.

We can now appreciate the specific kind of conceptual population exhibited by this case study. The pre-abstract group population has a high conceptual variation and a continuous distribution of heuristic power, together with a low reproductive competition. This combination of parameters represents an episode of conceptual change where many similar variants evolve collectively as a series of domain-specific generalization related to different mathematical problems. This dynamic of conceptual change is typical of mathematical conceptual histories that can be modeled as *domain-specific generalizations*, i.e., subsequent generalizations of concepts related to specific mathematical problems. In such cases, conceptual change is not driven by proofs and refutation, but instead by “internal organization" (cf. Feferman 1978, p. 174) or “systematization" (Kitcher 1984, pp. 217-225).

More generally, the three case studies I presented demonstrate how my framework allows a very fine-grained classification of the dynamics of conceptual change exhibited by mathematical conceptual histories. Examples of conceptual change can be rationally reconstructed as conceptual populations and classified to be more Lakatosian or more Euclidean in character with respect to the dimensions of the Lakatosian spaces. The opposition between the Lakatosian and the Euclidean ideal of conceptual change is then broke down into a plurality of conceptual features that mathematical conceptual histories can exhibit to a different degree. Different combinations of these features give rise to different dynamics of conceptual change, such as the axiomatic tinkering of the quaternion populations or the domain-specific generalizations of the pre-abstract

[9] Note that this continuity is due to the lack of axiomatization of the related parts of mathematics. The situation was about to change for group theory in a few years with the development of the abstract group concept (and the related, different conceptual population) (Wussing, 1984, pp. 230-254)

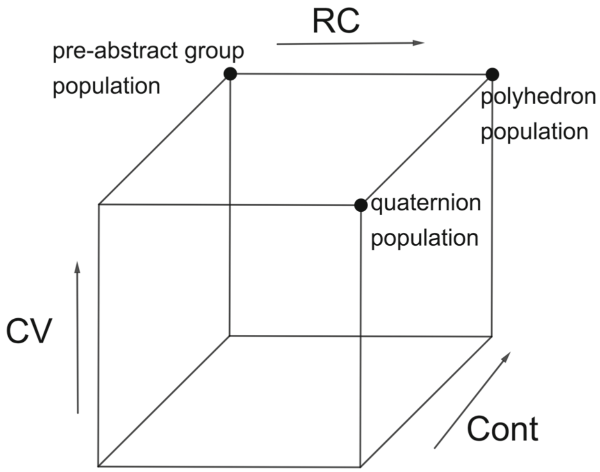

Fig. 2 A representation of the Lakatosian space showing the parts of the space corresponding to the dynamics of conceptual change exhibited by the three case studies

group population, that occupy parts of the Lakatosian space in between Lakatosian and Euclidean populations (Fig. 2).

6 Conclusion

Let me recall the main steps of the present work. We first saw Lakatos' seminal model of mathematical conceptual change and the various critiques and counterexamples to it that can be found in the philosophical literature. I then pointed out to an analogous dialectic in philosophy of biology related to the problems of recipes approaches to natural selection. Building upon a recent framework developed to solve these issues, i.e., Godfrey-Smith's population-based Darwinism, I proposed a pluralist framework for conceptual change in mathematics. My framework is made of three main ingredients: the notion of a conceptual population, the opposition between Lakatosian and Euclidean populations, and the spatial tools of the Lakatosian space. After presenting my framework, I showed how it can be applied to three different episodes of conceptual change from the history of mathematics. We saw how different mathematical conceptual histories can be reconstructed in my framework as different kinds of conceptual populations, exhibiting to a different degree the three parameters of the Lakatosian space: conceptual variation, reproductive competition, and distribution of heuristic power. Depending on how it scores on these parameters, a conceptual population can instantiate an episode of conceptual change closer to the Lakatosian ideal or more similar to Euclidean populations. Moreover, different combinations of these parameters represent specific dynamics of conceptual change, such as the axiomatic tinkering of the quaternion population or the domain-specific generalizations of the pre-abstract group population.

My proposal is open to several extensions and additions. A natural follow-up to this work would involve the analysis of episodes of conceptual change from the history of mathematics exhibiting further kinds of conceptual populations. Moreover, one could add further dimensions to the Lakatosian space. A time-dimension, for instance, would allow the reconstruction of time-dependent aspects of mathematical conceptual change such as the emergence and the reproduction of conceptual variants. This extension would also make possible to mirror several inter-practice transitions (Kitcher, 1984; Ferreirós, 2015) as specific kinds of movements, from one conceptual population to another one, along the time-dimension of the Lakatosian space so augmented.

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