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Convex Enclosures for Constrained Reachability Tubes

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Abstract: This paper is concerned with computing enclosures for the constrained reachable set of uncertain nonlinear dynamic systems. Our main contribution is a nontrivial extension of the generalized differential inequality, proposed in Villanueva et al. (2015), for the case that an a priori enclosure, of the reachable set is available. A practical implementation is worked out in detail for the case of ellipsoidal enclosures. The applicability of the proposed method is illustrated using a Lotka-Volterra system, for which a nonlinear solution invariant is known.

Keywords: Reachability analysis, differential inequalities, set-based computing

1. INTRODUCTION

Computing enclosures for the reachable set of uncertain dynamic systems is a central task in various model-based methodologies. These include, global dynamic optimization [Papamichail and Adjiman 2002], robust optimal control [Mitchell et al. 2005], guaranteed state and parameter estimation [Jaulin 2002, Paulen et al. 2016], and fault-detection [Tulsyan and Barton 2016].

This paper focuses on continuous-time set-propagation methods for reachability analysis. These are based on the construction of an auxiliary dynamic system whose flow describes a pointwise-in-time enclosure of the reachable set of the original system. Walter's theory of differential inequalities (DIs) [Walter 1970] provides a theoretical basis for continuous-time interval methods. This approach has also been used for polyhedral enclosures [Harwood and Barton 2016], Taylor models [Chachuat and Villanueva 2012], and ellipsoidal enclosures [Kurzhanski and Vályi 1997, Houska et al. 2012].

Some systems admit an a priori enclosure of their reachable sets, which can be used to reduce conservatism. An extension of Walter's theory enabling the use of linear and nonlinear solution invariants for interval-based DIs can be found in [Scott and Barton 2013a] and [Shen and Scott 2018]. A method for constructing polytopic enclosures under polytopic a priori enclosures can be found in [Harwood and Barton 2018]. These methods have found applications in chemical kinetics [Scott and Barton 2010], chemical reactors [Tulsyan and Barton 2017] and reachability of differential algebraic equations [Scott and Barton 2013b].

This paper extends the differential inequality based enclosure methods from [Villanueva et al. 2015] for systems with known a priori enclosures. After formalizing the problem in Section 2, we provide sufficient conditions for a set-valued function to be an enclosure of the reachable set of an uncertain dynamic system for which a time-varying a priori enclosure is known. These conditions, introduced in Section 3, are given in the form of a differential inequality for the support function of the set-valued function. In Section 4 the special case of ellipsoidal enclosures is presented. Section 5 presents the implementation details of the ellipsoidal approach, and its application to a Lotka-Volterra system. Section 6 concludes the paper.

Notation: L_1^n is the set of n-dimensional L_1-integrable functions, while the Sobolev space of weakly differentiable functions with L_1^n derivatives is denoted by $W_{1,1}^n$. Weak derivatives of functions $z \in W_{1,1}^n$ are denoted by $\dot{z}$. The set of compact and convex compact sets in $\mathbb{R}^n$ are denoted respectively by $\mathbb{K}^n$ and $\mathbb{K}_{\mathrm{C}}^n$. The power set of a set $Z \subseteq \mathbb{R}^n$ is denoted by $\mathcal{P}(Z)$. Its interior and closure are denoted by $\mathrm{int}(Z)$ and $\mathrm{cl}(Z)$. The support function of $Z \subseteq \mathbb{R}^{n_x}$ is denoted by $V[Z](c) = \sup_{z\in Z}\ c^\intercal z$, for all $c \in \mathbb{R}^{n_x}$. The set of $n \times n$ positive semidefinite and definite matrices are denoted by $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ respectively. An ellipsoid with center $q \in \mathbb{R}^n$ and shape matrix $Q \in \mathbb{S}_+^n$ is denoted by

$$\mathcal{E}(q,Q) = \left\{ q + Q^{\frac{1}{2}} v \;\middle|\; v^\intercal v \leq 1 \right\} ,$$

where $Q^{\frac{1}{2}}$ is the positive semidefinite square root of Q.

2. PROBLEM FORMULATION

We consider uncertain dynamic systems of the form

$$\begin{array}{ll} \forall t \in [0,T], & \dot{x}(t) = f(t, x(t), p) \,, \\ \text{with} & x(0) \in X_0 \in \mathbb{K}_{\mathrm{C}}^{n_x}. \end{array} \quad (1)$$

The function $x : [0,T] \to \mathbb{R}^{n_x}$ denotes the state trajectory and X_0 is a given set of initial conditions. Moreover,

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$p \in \mathbb{R}^{n_p}$ denotes a time-invariant disturbance, whose value is unknown but assumed to be bounded by given set, $p \in \mathbb{P} \in \mathbb{K}_C^{n_p}$. The function $f : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ is assumed to be jointly continuous in all its variables as well as locally Lipschitz continuous in x, uniformly on $[0,T]\times\mathbb{P}$.

Let $\chi(\cdot, x_0, p)$ be the solution of (1) for a given $x_0 \in X_0$ and a given $p \in \mathbb{P}$. We assume that a set-valued function $\Theta : \mathbb{R} \to \mathbb{K}_C^{n_x}$ is given, such that

$$\forall t \in \mathbb{R}_+, \quad \chi(t, x_0, p) \in \Theta(t) \quad \text{and} \quad \text{int}(\Theta(t)) \neq \varnothing \ , \quad (2)$$

for all $x_0 \in X_0$ and all $p \in \mathbb{P}$. We call this function a robust positive invariant tube.

An example of a dynamic system with invariants is the index-1 semi-explicit differential-algebraic equation

$$\begin{aligned} \dot{x}_D(t) &= g(t, x_D(t), x_A(t), p) \\ 0 &= h(t, x_D(t), x_A(t), p) \ . \end{aligned}$$

Here, $x_D(t) \in \mathbb{R}^{n_D}$ and $x_A(t) \in \mathbb{R}^{n_A}$ are the differential and algebraic variables, respectively. Differentiating the algebraic equation $0 = h(t, x_D, x_A, p)$ with respect to the algebraic variable, one gets

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_D}\frac{\mathrm{d}x_D}{\mathrm{d}t} + \frac{\partial h}{\partial x_A}\frac{\mathrm{d}x_A}{\mathrm{d}t} = 0 \ .$$

Since the system is of index-1, the matrix $\frac{\partial h}{\partial x_A}$ is invertible. If $x = (x_D, x_A)$, one can write

$$f = \begin{pmatrix} g \\ -\left(\frac{\partial h}{\partial x_A}\right)^{-1}\left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_D} g\right) \end{pmatrix} \ . \quad (3)$$

Now, the state of the dynamic system defined through the right-hand side (3) must lie on the manifold given by $h(t, x_D(t), x_A(t), p) = 0$. This relation can be used to define a set-valued function Θ with

$$\Theta(t) \supseteq \left\{ \xi \in \mathbb{R}^{n_D + n_A} \ \middle| \forall p \in \mathbb{P}, \ h(t, \xi, p) = 0 \right\} \ .$$

In this paper we focus in the following problem. Given (1) and Θ, we are interested in characterizing the constrained reachability tube $X : [0,T] \to \mathcal{P}(\mathbb{R}^{n_x})$ given by

$$X(t) = \left\{ \xi \in \mathbb{R}^{n_x} \ \middle| \begin{array}{l} \exists x \in W_{1,1}^{n_x}, \exists p \in \mathbb{R}^{n_p} : \\ \forall \tau \in [0,t], \\ \dot{x}(\tau) = f(\tau, x(\tau), p), \\ x(0) \in X_0, \ p \in \mathbb{P}, \\ x(t) = \xi \end{array} \right\} \ . \quad (4)$$

Here, it is important to recall that, as stated in (2), the set-valued function Θ is assumed to satisfy $X(t) \subseteq \Theta(t)$ for all $t \in [0,T]$. Unfortunately, an exact characterization of X is impossible except in very few instances. Thus, the focus of this paper is on constructing tractable outer approximations of X.

3. CONVEX ENCLOSURES USING DIFFERENTIAL INEQUALITIES

This section focuses on conditions for a set-valued function $Y : [0,T] \to \mathbb{K}_C^{n_x}$ to be an outer approximation (enclosure) of X on $[0,T]$, i.e. such that it satisfies $Y(t) \supseteq X(t)$ for all $t \in [0,T]$.

The conditions given in this paper are generalizations of those given in Thm. 3 in Villanueva et al. [2015]. This theorem establishes sufficient conditions for a set-valued function Y to be an enclosure of the reachability tube of (1) in the absence of constraints.

The theorem is reproduced next—with a slightly different notation—for the sake of completeness. In the following, we use the the short-hand notation

$$\Gamma(c, \tau, Z_1, Z_2) = \left\{ f(\tau, \xi, \rho) \ \middle| \begin{array}{c} c^\mathsf{T}\xi = V[Z_1](c) \\ \xi \in Z_1 \cap \text{int}(Z_2) \\ \rho \in \mathbb{P} \end{array} \right\} \ ,$$

which is defined for all $Z_1 \in \mathbb{K}_C^{n_x}$, all closed $Z_2 \in \mathcal{P}(\mathbb{R}^{n_x})$, all $\tau \in \mathbb{R}$, and all $c \in \mathbb{R}^{n_x}$.

Theorem 1. Consider system (1). Let $Y : [0,T] \to \mathbb{K}_C^{n_x}$ be a set-valued function such that

(1) the function $V[Y(\cdot)](c)$ is, for all $c \in \mathbb{R}^{n_x}$, Lipschitz continuous on $[0,T]$; and
(2) the inequalities

$$\begin{array}{ll} \forall t \in (0,T], & \dot{V}[Y(t)](c) \geq V\left[\Gamma\left(c, t, Y(t), \mathbb{R}^{n_x}\right)\right](c) \\ \text{with} & V[Y(0)](c) \geq V[X_0](c) \end{array}$$

hold for all $c \in \mathbb{R}^{n_x}$. Then, Y is an enclosure of X, i.e. we have $Y(t) \supseteq X(t)$ for all $t \in [0,T]$.

Proof. See Villanueva et al. [2015] for a proof. □

The next theorem presents the main contribution of this section, namely, a generalization of Thm. 1 for enclosing reachable sets when an a priori enclosure is known. In particular, it introduces a differential inequality which—unlike that in Thm. 1—takes advantage of Θ, to reduce the conservatism of Y.

Theorem 2. Let $Y : [0,T] \to \mathbb{K}_C^{n_x}$ be a set-valued function satisfying $Y(t) \cap \text{int}(\Theta(t)) \neq \varnothing$ for all $t \in [0,T]$. Moreover, let Y be such that

(1) the functions $V[Y(\cdot)](c)$ and $V[Y(\cdot) \cap \Theta(\cdot)](c)$ are, for all $c \in \mathbb{R}^{n_x}$, Lipschitz continuous on $[0,T]$; and
(2) the differential inequalities

$$\begin{array}{ll} \forall t \in (0,T], & \dot{V}[Y(t)](c) \geq V[\Gamma(c, t, Y(t), \Theta(t))](c) \\ \text{with} & V[Y(0)](c) \geq V[X_0](c) \end{array}$$

hold for all $c \in \mathbb{R}^{n_x}$. Then, the convex set-valued function $Y_\Theta : \mathbb{R} \to \mathbb{K}_C^{n_x}$ given by $Y_\Theta(t) = Y(t) \cap \Theta(t)$, for all $t \in [0,T]$, satisfies $Y_\Theta(t) \supseteq X(t)$ for all $t \in [0,T]$.

A proof of this theorem is included in Appendix A.

4. ELLIPSOIDAL ENCLOSURES FOR CONSTRAINED REACHABILITY

In the following, we introduce a practical construction for ellipsoidal enclosures satisfying the differential inequality from Theorem 2. We consider ellipsoidal-valued enclosures,

$$Y(t) = \mathcal{E}(q(t), Q(t)) \ .$$

Here, $q : \mathbb{R} \to \mathbb{R}^{n_x}$ and $Q : \mathbb{R} \to \mathbb{S}_{++}^{n_x}$ denote the central path of the tube and its time-varying shape matrix.

Let the differentiable functions $\overline{\theta} : \mathbb{R} \to \mathbb{R}^{n_x}$ and $\overline{\Theta} : \mathbb{R} \to \mathbb{S}_{++}^{n_x}$, as well as the pairs $(q_0, Q_0) \in \mathbb{R}^{n_x} \times \mathbb{S}_{++}^{n_x}$ and $(\overline{p}, \overline{P}) \in \mathbb{R}^{n_p} \times \mathbb{S}_{++}^{n_x}$ be given, such that

$$\forall t \in \mathbb{R}, \quad \Theta(t) \subseteq \mathcal{E}\left(\overline{\theta}(t), \overline{\Theta}(t)\right) \ , \quad X_0 \subseteq \mathcal{E}\left(q_0, Q_0\right) \ ,$$
$$\text{and} \quad \mathbb{P} \subseteq \mathcal{E}\left(\overline{p}, \overline{P}\right) \ .$$

We assume that a nonlinearity bound

$$p : \mathbb{R} \times \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_x \times n_p} \times \mathbb{R}^{n_x} \times \mathbb{S}_+^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{S}_+^{n_x}$$

is given such that

$$f(t,\xi,p) - A(\xi - r) - B\left(\rho - \overline{P}\right) \in \mathcal{E}(0, p(t,A,B,r,R))$$

for all $t \in \mathbb{R}$, all $\xi \in \mathcal{E}(r,R)$; all vectors $r \in \mathbb{R}^{n_x}$, and $\rho \in \mathbb{R}^{n_p}$; and all matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_p}$, and $R \in \mathbb{S}_{++}^{n_x}$. Notice that since f is Lipschitz continuous in ξ, such a function can always be constructed [Houska et al. 2012]. Furthermore, we introduce the shorthand notations

$$\varphi(t,r,R,\sigma) = f\left(t,r,\overline{p}\right) + \sigma R\left(\overline{\Theta}(t)^{-1}\right)\left(\overline{\theta}(t) - r\right) \quad (5)$$

and

$$\begin{aligned}\Phi(t,r,R,A,B,\kappa,\lambda,\sigma) &= AR + RA^{\intercal} \\ &+ (\kappa+\lambda)R + \frac{1}{\kappa}B\overline{P}B^{\intercal} + \frac{1}{\lambda}p(t,A,B,r,R) \\ &+ \sigma\left(I - \left\|\overline{\Theta}(t)^{-\frac{1}{2}}\left(r - \overline{\theta}(t)\right)\right\|_2^2 I - R\left(\Theta(t)^{-1}\right)\right)R\,.\end{aligned} \quad (6)$$

which are, again, defined for all vectors r and ρ as well as all matrices A, B, and R of compatible dimensions.

The following theorem summarizes the construction of ellipsoidal outer approximations for $X(t)$.

Theorem 3. Let $q : \mathbb{R} \to \mathbb{R}^{n_x}$, $Q : \mathbb{R} \to \mathbb{S}_{++}^{n_x}$, be any differentiable functions satisfying

$$\begin{aligned}\dot{q}(t) &= \varphi(t,q(t),Q(t),\sigma(t)) \\ \dot{Q}(t) &\succeq \Phi(t,q(t),Q(t),A(t),B(t),\kappa(t),\lambda(t),\sigma(t))\end{aligned}$$

for all $t \in (0,T]$, with $q(0) = q_0$ and $Q(0) \succeq Q_0$ and for any given functions $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$, $B : \mathbb{R} \to \mathbb{R}^{n_x \times n_p}$, $\kappa, \lambda : \mathbb{R} \to \mathbb{R}_{++}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$. Then, the set-valued function $Y : [0,T] \to \mathbb{K}_{\mathrm{C}}^{n_x}$ given by $Y(t) = \mathcal{E}(q(t),Q(t))$, for all $t \in [0,T]$, satisfies the differential inequality of Theorem 2 on $[0,T]$.

The proof of this theorem is rather technical. For the sake of brevity, only an outline is provided in Appendix B.

5. IMPLEMENTATION AND CASE-STUDY

5.1 Implementation details

This section describes the main details behind the implementation of the ellipsoidal outer approximation scheme for the constrained reachability tube X. The main difficulty in using the conditions of Theorem 3 is the construction of the right-hand side in the auxiliary bounding systems, i.e. the functions (5) and (6).

Notice that the right-hand side of ODEs for q and Q is identical to the ones introduced by Kurzhanski and Varaiya [2002], Houska et al. [2012] [Villanueva et al. 2015, see also], except for the terms

$$\sigma R\left(\overline{\Theta}(t)^{-1}\right)\left(\overline{\theta}(t) - r\right)$$

in φ and

$$\sigma\left(I - \left\|\overline{\Theta}(t)^{-\frac{1}{2}}\left(r - \overline{\theta}(t)\right)\right\|_2^2 I - R\left(\Theta(t)^{-1}\right)\right)R$$

in Φ. For example, in [Villanueva et al. 2015] the implementation of the auxiliary bounding system—for the unconstrained reachability problem—is based on Taylor model arithmetics. Notice that the use of Taylor model arithmetics provides a means to have a fully systematic and automatic way for evaluating the right-hand side functions φ and Φ. Due to the dependence of Taylor models on interval arithmetics, which are only Lipschitz continuous, such an implementation could not be used within gradient-based algorithms for optimal control.

In this paper, we choose the functions A and B given by

$$A(t) = \frac{\partial f}{\partial x}(t,q(t),\overline{p}) \quad \text{and} \quad B(t) = \frac{\partial f}{\partial p}(t,q(t),\overline{p})\,.$$

The nonlinearity estimate can be constructed using in a semi-automated manner, using Hessian bounds [see, e.g. Villanueva et al. 2017a,b]. In this paper, we use an analytical approach, exploiting globally valid algebraic relations within a function [Houska et al. 2012, Villanueva et al. 2018, see, e.g.] .

Another difficulty is in choosing the functions λ, κ, and σ introduced in Theorem 3. In a robust control context, such functions can be interpreted as additional degrees of freedom, and are left to be chosen by the optimizer. In this paper, we choose fixing these degrees of freedom using heuristic relations. In particular, we choose these functions such that they minimize, e.g. the trace of $\dot{Q}(t)$. In this case, we have

$$\kappa(t) = \frac{\sqrt{\mathrm{Tr}(Q(t))}}{\mathrm{Tr}(B(t)Q_w B(t)^{\intercal})}$$

and

$$\lambda(t) = \frac{\sqrt{\mathrm{Tr}(Q(t))}}{\mathrm{Tr}(p(t,A(t),B(t),Q(t),r(t)))}\,.$$

For more details, see Villanueva et al. [2015, 2017b]. For σ, we can also introduce the following heuristic

$$\sigma(t) = \begin{cases} 0 & \text{if}\,\mathrm{Tr}(Q(t)) \geq \mathrm{Tr}(Q(t)\overline{\Theta}(t)^{-1}Q(t)) \\ \infty & \text{otherwise}\end{cases}$$

which minimizes $\mathrm{Tr}(\dot{Q}(t))$, whenever $\mathcal{E}(q(t),Q(t)) \subseteq \mathcal{E}(\overline{\theta}(t),\overline{\Theta}(t))$. In order to avoid discontinuities and stiffness in the integration routine, we use the smooth approximation

$$\sigma(t) = \frac{\overline{\sigma}}{\pi}\left(\frac{\mathrm{atan}\left(\mathrm{Tr}(Q(t)) - \mathrm{Tr}\left(Q(t)\overline{\Theta}(t)^{-1}Q(t)\right)\right)}{\epsilon} + \frac{\pi}{2}\right),$$

with $\overline{\sigma} \in (0,\infty)$ and $\epsilon > 0$.

5.2 Case Study: Lotka-Volterra System

Consider the Lotka-Volterra system

$$\begin{aligned}\dot{x}_1(t) &= p_1 x_1(t)(1 - x_2(t)) \\ \dot{x}_2(t) &= p_2 x_2(t)(x_1(t) - 1)\end{aligned} \quad (7)$$

with initial conditions $x_0 = (1.2, 1.1)^{\intercal}$. The uncertain parameters are bounded by the ellipsoid $\mathcal{E}(\overline{p},\overline{P})$ with $\overline{p} = (3,1)^{\intercal}$ and $\overline{P} = 2\,\mathrm{diag}(10^{-4},10^{-4})$.

Equation (7) satisfies an invariant of the form

$$\psi(x(t),p) = p^{\intercal}\begin{pmatrix}\log\dfrac{x_1(t)}{1.2} - (x_1(t) - 1.2) \\ \log\dfrac{x_2(t)}{1.1} - (x_2(t) - 1.1)\end{pmatrix} = 0$$

for every, fixed, $p \in \mathbb{R}^2$. This can be verified by checking that $\nabla_x \psi f = 0$. Figure 1(left) shows the sets

$$\left\{\xi \in \mathbb{R}^2 \,\middle|\, \begin{array}{l}\exists \rho \in \mathbb{P}: \\ \psi(\xi,\rho) = 0\end{array}\right\} \subseteq \overline{\Psi} = \left\{\xi \in \mathbb{R}^2 \,\middle|\, \max_{\rho \in \mathbb{P}} \psi(\xi,\rho) \leq 0\right\},$$

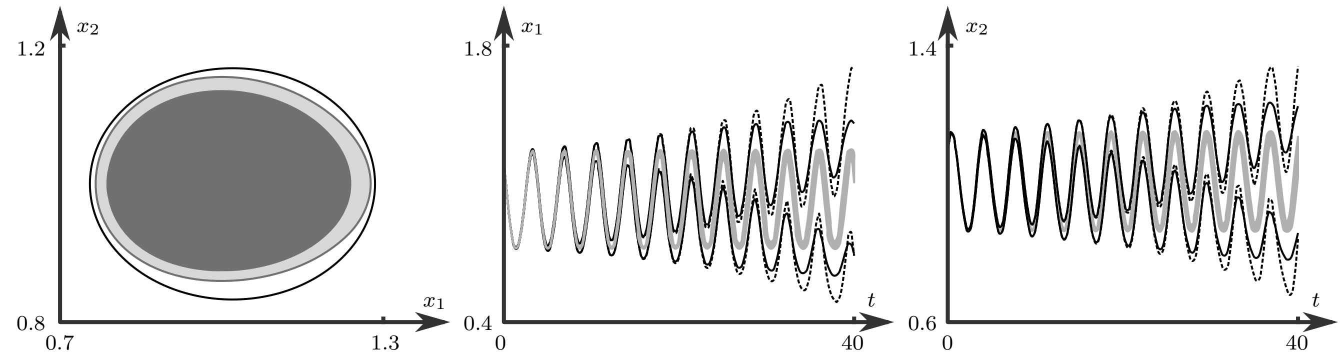

Fig. 1. Invariant and state bounds for the Lotka-Volterra system. Left: set of solution invariants (dark gray), $\overline{\Psi}$ (light gray) and its ellipsoidal enclosure (boundary shown in black). Center and right: Projections of $X(\cdot)$ onto the state-time coordinates (gray), unconstrained ellipsoidal bounds (dashed lines) and projections of $Y(\cdot)$ (solid lines) .

in dark and light gray, respectively. Let $\overline{\theta} = (1,1)^\mathsf{T}$ and $\overline{\Theta} = 10^{-2}\,\mathrm{diag}(6, 2.5)$, then the ellipsoid $\mathcal{E}\left(\overline{\theta}, \overline{\Theta}\right)$—whose boundary is depicted as a black line—is an enclosure of $\overline{\Psi}$. We then set $\Theta : t \mapsto \mathcal{E}\left(\overline{\theta}, \overline{\Theta}\right)$.

The bounding system from Theorem 3 was implemented in MATLAB and solved forward in time using its adaptive-step Runge-Kutta 4-5. The nonlinearity estimate p is given by

$$\begin{aligned} &p(t, A(t), B(t), q(t), Q(t)) = \\ &\qquad \mathrm{diag}\left(n_1(q(t), Q(t))^2, n_2(q(t), Q(t))^2\right)\,, \end{aligned}$$

with

$$\begin{aligned} n_1(r,R) = \;& \sqrt{R_{1,1}\overline{P}_{1,1}} + r_1\sqrt{R_{2,2}\overline{P}_{1,1}} \\ & + r_2\sqrt{R_{1,1}\overline{P}_{1,1}} + \sqrt{R_{1,1}R_{2,2}\overline{P}_{1,1}} \\ n_2(r,R) = \;& r_1\sqrt{R_{2,2}\overline{P}_{2,2}} + r_2\sqrt{R_{1,1}\overline{P}_{2,2}} \\ & + \sqrt{R_{1,1}R_{2,2}\overline{P}_{2,2}} + \sqrt{R_{2,2}\overline{P}_{2,2}}\,. \end{aligned}$$

The functions n_1 and n_2, were obtained by the procedure outlined in Villanueva et al. [2018, Appendix 2].

The central and right panels in Figure 1 show projections of $X(\cdot)$ onto the (x_1, t)- and (x_2, t)-spaces (in gray), as well as projections of ellipsoidal enclosures. The dashed lines, were obtained without taking intersections, i.e. $\sigma(t) = 0$, for all $t \in [0, T]$. The solid black lines denote the projections of $Y(t)$, computed via Thm. 3. None of these methods leads to bound explosion over $[0, 40]$, while the bounds obtained using Taylor models [Chachuat and Villanueva 2012, Lin and Stadtherr 2007], and polyhedral enclosures [Harwood and Barton 2016] diverge before $T = 40$.

6. CONCLUSION

This paper has presented a framework for reachability analysis of continuous-time nonlinear systems, whenever a time-varying a priori enclosure—namely, a robust positive invariant tube, is available. The methodology uses a differential inequality which provides a sufficient condition for a time-varying support function to describe a convex enclosure for constrained reachability tubes. In addition, we provide tractable conditions for the case of ellipsoidal-valued enclosures. The applicability of this theory is demonstrated by means of the construction of enclosures for the reachable sets of a Lotka-Volterra system.

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Appendix A. PROOF OF THEOREM 2

Before providing a proof for Thm. 2, we prove the following intermediate result.

Proposition 4. Let $Y : [0,T] \to \mathbb{K}^{n_x}_{\mathrm{C}}$ be a set-valued function whose images, $Y(t)$, are strictly convex and have a nonempty intersection $Y(t) \cap \Theta(t) \neq \emptyset$ for all $t \in [0,T]$. Moreover, let Y be such that the functions $V[Y(\cdot)](c)$ and $V[Y(\cdot) \cap \Theta(\cdot)](c)$ are, for all $c \in \mathbb{R}^{n_x}$, Lipschitz continuous on $[0,T]$. If in addition, Y satisfies the differential inequalities

$$\dot{V}[Y(t)](c) \geq V\left[\left\{ f(t,\xi,\rho) \,\middle|\, \begin{array}{c} c^{\mathsf{T}}\xi = V[Y(t)](c) \\ \xi \in Y(t) \cap \Theta(t) \\ \rho \in \mathbb{P} \end{array} \right\}\right](c),$$

for all $c \in \mathbb{R}^{n_x}$ and all $t \in [0,T]$, then Y is an enclosure of X, i.e. we have $Y(t) \supseteq X(t)$ for all $t \in [0,T]$.

Proof. Our proof is indirect. Let us assume that the conditions in the theorem hold, but Y is not an enclosure of $X(t)$. Then, by the Lipschitz continuity of $V[Y(\cdot)](c)$, we can assert the existence of a time instant $t \in [0,T]$; a vector $c \in \mathbb{R}^{n_x}$ with $c \neq 0$ and some $\epsilon > 0$ such that

$$X(t) \subseteq Y(t) \tag{A.1}$$

$$V[X(t)](c) = V[Y(t)](c) \tag{A.2}$$

and

$$\forall \tau \in (t, t+\epsilon): \quad V[X(\tau)](c) > V[Y(\tau)](c)\,. \tag{A.3}$$

Notice that by the sufficiency of the conditions in Thm 1, and our assumptions for the proof, this is only possible if the unconstrained differential inequality is violated at (t,c). That is, we must have

$$\dot{V}[Y(t)](c) < V\left[\Gamma\left(c,t,Y(t),\mathbb{R}^{n_x}\right)\right]\,. \tag{A.4}$$

Let us introduce the constraint set

$$F[Z_1,Z_2](c) = \left\{ \xi \in \mathbb{R}^{n_x} \,\middle|\, \begin{array}{c} c^{\mathsf{T}}\xi = V[Z_1](c) \\ \xi \in Z_1 \cap Z_2 \end{array} \right\},$$

defined for all $Z_1, Z_2 \in \mathcal{P}(\mathbb{R}^{n_x})$ and all $c \in \mathbb{R}^{n_x}$. Since $Y(t)$ is assumed to be strictly convex, $F\left[Y(t),\mathbb{R}^{n_x}\right](c)$ the supporting facet of $Y(t)$ in the direction c is a singleton. Recall that Θ is, by assumption, an enclosure of X. Hence, as a consequence of (A.1) and (A.2), the implication

$$\xi \in F\left[Y(t),\mathbb{R}^{n_x}\right](c) \Longrightarrow \xi \in F\left[X(t),\Theta(t)\right](c) \tag{A.5}$$

must hold. Using (A.1) and (A.2) again, shows that

$$\xi \in F\left[X(t),\Theta(t)\right](c) \Longrightarrow \xi \in F\left[Y(t),\Theta(t)\right](c)\,. \tag{A.6}$$

also holds. Thus, (A.5), (A.6) together with (A.4) imply

$$\dot{V}[Y(t)](c) < V\left[\left\{ f(t,\xi,\rho) \,\middle|\, \begin{array}{c} c^{\mathsf{T}}\xi = V[Y(t)](c) \\ \xi \in Y(t) \cap \Theta(t) \\ \rho \in \mathbb{P} \end{array} \right\}\right](c)\,,$$

which is a contradiction to our assumption that all the conditions of the proposition hold. This, in turn, yields the statement of the proposition. □

We can now proceed to give a proof of Theorem 2.

Proof. Notice that Proposition 4 considers:

- set-valued functions Y with strictly convex images for all $t \in [0,T]$; and
- a larger set $Y(t) \cap \Theta(t)$—compared to the tighter intersection $Y(t) \cap \mathrm{int}(\Theta(t))$ in Thm. 2.

Our proof proceeds in two steps: we relax the conditions of Proposition 4 and claim that its conclusion still holds.

Step 1. Let us show that the statement holds without strict convexity of $Y(t)$. Consider the Hausdorff metric

$$d_{\mathrm{H}}(A,B) = \max\left\{ \max_{a\in A}\min_{b\in B}\|a-b\|, \max_{b\in B}\min_{A\in A}\|a-b\| \right\},$$

defined for all $A, B \in \mathbb{K}^{n_x}$. Now, we construct a family of enclosures $Y_\epsilon : [0,T] \to \mathbb{K}^{n_x}_{\mathrm{C}}$ of Y, such that for all $\epsilon > 0$:

- for all $t \in [0,T]$, the sets $Y_\epsilon(t)$ are strictly convex; and
- There exists a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$, satisfying

$$d_{\mathrm{H}}(Y_\epsilon(t), Y(t)) \le \alpha(\epsilon) \quad \text{and}$$
$$\dot V[Y_\epsilon](c) \ge \dot V[Y(t)](c) + L\alpha(\epsilon)$$

for all $t \in [0, T]$ with $L = \frac{1}{T}$.

For a proof of the existence of such functions, see Lemmata 1 and 2 in Villanueva et al. [2015]. Assume for a moment that L is large enough such that

$$\begin{aligned}\dot V[Y_\epsilon](c) &\ge \dot V[Y(t)](c) + L\alpha(\epsilon)\\ &\ge V[\Gamma(c,t,Y(t),\Theta(t))](c) + L\alpha(\epsilon)\\ &\ge V[\Gamma(c,t,Y_\epsilon(t),\Theta(t))](c)\end{aligned}$$

holds for a sufficiently small $\epsilon > 0$. Then, we can apply Proposition 4, to show that Y_ϵ is an enclosure of X. The first claim, namely that the result holds without the strict convexity assumption, follows after taking the limit as $\epsilon \to 0$—invoking a continuity argument. Notice that L can always be as large as needed by partitioning $[0, T]$ into a finite number of sufficiently small subintervals and applying the above procedure.

Step 2. We now show that the result of Proposition 4 still holds if we replace $Y(t) \cap \Theta(t)$ by the tighter intersection $Y(t) \cap \Theta(\mathrm{t})$. Notice that

$$\mathrm{cl}(Y(t) \cap \mathrm{int}(\Theta(t))) = \mathrm{cl}(Y(t) \cap \Theta(t))$$

since $\Theta(t) \neq \varnothing$ and $Y(t) \cap \mathrm{int}(\Theta(t)) \neq \varnothing$. Thus, the statement of the theorem remains unchanged if we replace $\Theta(t)$ in the intersection by its interior. This follows from the fact that the supremum of a continuous function over any bounded set coincides with its maximum over the closure of the set.

This proves the statement of the theorem. □

Appendix B. AN OUTLINE OF THE PROOF OF THEOREM 3

The proof requires ideas from [Kurzhanski and Vályi 1997], Houska et al. [2012], and Villanueva et al. [2015].

Let the set-propagation operator for (1) be denoted by

$$\Pi(t_2, t_1, Z_1) = \left\{ \xi \in \mathbb{R}^{n_x} \;\middle|\; \begin{array}{l} \exists x \in W^{n_x}_{1,1}, \exists p \in \mathbb{R}^{n_p} : \\ \forall t \in [t_1, t_2], \\ \dot x(t) = f(t, x(t), p), \\ x(t) \in \Theta(t),\ p \in \mathbb{P}, \\ x(t_1) = Z_1,\ x(t_2) = \xi \end{array} \right\}.$$

Notice that Π is defined for all $Z_1 \in \mathbb{K}^{n_x}_{\mathrm{C}}$ and all $t_1, t_2 \in \mathbb{R}$ with $t_1 \le t_2$.

The main idea is to use a discretized differential inequality, similar to the one presented in Villanueva et al. [2015, Proposition 1], which requires stronger regularity conditions, on Y. Namely, that

- the sets $Y(t)$ and $Y(t) \cap \Theta(t)$ are strictly convex for all $t \in [0, T]$; and
- the functions $V[Y(\cdot)](c)$ and $V[Y(\cdot) \cap \Theta(\cdot)](c)$ are, for all $c \in \mathbb{R}^{n_x}$, differentiable on $[0, T]$.

Under these conditions, the existence of a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0)$, such that the differential inequality

$$V[\Pi(t, t+h, Y(t))](c) \le V[Y(t+h)](c) + h\alpha(h)\ , \quad \text{(B.1)}$$

holds for all $c \in \mathbb{R}^{n_x}$ with $\|c\| \le 1$ and all $t \in [0, T]$, is sufficient to guarantee that Y satisfies the conditions of Theorem 2. Let

$$\begin{aligned}\tilde x(t,h) = \;& x(t) + hf(t, q(t), \overline{p}) + hA(t)(x(t) - q(t))\\ &+ hB(t)(p - \overline{p}) + hn(t)\end{aligned}$$

be an Euler approximation of (1). Here, $x(t) \in \mathcal{E}(q(t), Q(t))$ and $n(t) \in \mathcal{E}(0, p(t, A(t), B(t), q(t), Q(t)))$. From here on, we use well known results from the field of ellipsoidal calculus for constructing enclosures of the intersection and Minkowski sums of ellipsoids [Kurzhanski and Vályi 1997, Houska 2011]. In particular, since $\Theta(t) \subseteq \mathcal{E}\left(\overline{\theta}(t), \overline{\Theta}(t)\right)$, we have that by setting

$$\tilde Q(t,h) = \left(\sigma_1(t,h)Q(t)^{-1} + \sigma_2(t,h)\Theta(t)^{-1}\right)^{-1}$$
$$\tilde q(t,h) = \left(\sigma_1(t,h)Q(t)^{-1}q(t) + \sigma_2(t,h)\Theta(t)^{-1}\theta(t)\right)\tilde Q(t,h)$$

it follows that $Y(t) \cap \Theta(t) \subseteq \mathcal{E}(\tilde q(t,h), \tilde Q(t,h))$, as long as $\sigma_1(t,h),\ \sigma_2(t,h) \ge 0$ satisfy

$$\begin{aligned}\tilde q(t,h)^{\intercal}\tilde Q(t,h)\tilde q(t,h) = 1 &- \sigma_1(t,h)\left(1 - \left\|Q(t)^{-\frac12}q(t)\right\|_2^2\right)\\ &- \sigma_2(t,h)\left(1 - \left\|\overline{\Theta}(t)^{-\frac12}\overline{\theta}(t)\right\|_2^2\right).\end{aligned} \quad \text{(B.2)}$$

Now, recall that the inclusion

$$\mathcal{E}(q_1, Q_1) \oplus \mathcal{E}(q_2, Q_2) \subseteq \mathcal{E}\left(q_1 + q_2, \frac{Q_1}{\gamma_1} + \frac{Q_2}{\gamma_2}\right)$$

holds, for any pair $(q_1, Q_1), (q_2, Q_2) \in \mathbb{R}^n \times \mathbb{S}^n_{++}$ as long as $\gamma_1, \gamma_2 > 0$ and $\gamma_1 + \gamma_2 = 1$ [Kurzhanski and Vályi 1997]. A repeated application of this statement shows that setting

$$\begin{aligned}\hat Q(t,h) = \;& \frac{1}{\gamma_1(t,h)}(I + hA(t))\tilde Q(t,h)(I + hA(t))^{\intercal}\\ &+ \frac{h^2}{\gamma_2(t,h)}B(t)\overline{P}B(t)^{\intercal}\\ &+ \frac{h^2}{\gamma_3(t,h)}p(t, A(t), B(t), q(t), Q(t)))\\ \hat q(t,h) = \;& \tilde q(t+h) + hf(t, \tilde q(t+h), \overline{p})\ ,\end{aligned}$$

for any function $\gamma_1(t,h), \gamma_2(t,h), \gamma_3(t,h) > 0$ with $\gamma_1(t,h) + \gamma_2(t,h) + \gamma_3(t,h) = 1$, implies

$$V[\Pi(t+h, t, Y(t))](c) \le V[\mathcal{E}(\hat q(t,h), \hat Q(t,h))](c) + h\beta(h)$$

for a continuous function $\beta : \mathbb{R} \to \mathbb{R}$ with $\beta(0) = 0$, for all c with $\|c\| = 1$.

Let $\gamma_1(t,h) = 1 - h\lambda(t) - h\kappa(t)$, $\gamma_2(t,h) = h\kappa(t)$, $\gamma_3(t,h) = h\lambda(t)$, and $\sigma_1(t,h) = 1 - h\sigma(t)$, with $\sigma_2(t,h)$ defined implicitly by (B.2). Differentiating the equations for $\hat q(t,h)$ and $\hat Q(t,h)$ with respect to h yields

$$\frac{\mathrm{d}}{\mathrm{d}h}\hat q(t,0) = \varphi\left(t, \hat q(t,0), \hat Q(t,0), \sigma(t)\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}h}\hat Q(t,0) = \Phi\left(t, \hat q(t,0), \hat Q(t,0), A(t), B(t), \kappa(t), \lambda(t), \sigma(t)\right).$$

Since the right-hand side of these derivatives coincide with the differential equations for q and Q, we must have

$$q(t+h) = \hat q(t,h) + O\left(h^2\right) \quad \text{and} \quad Q(t+h) = \hat q(t,h) + O\left(h^2\right).$$

In turns, this implies the existence of a function $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ and such that the desired inequality, i.e.

$$V[\Pi(t+h, t, Y(t))](c) \le V[\mathcal{E}(q(t+h, Q(t+h)))](c) + h\alpha(h)\ .$$

holds.