Letter

Assessing frustration in real-world signed networks: A statistical theory of balance

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According to the so-called strong version of structural balance theory, actors in signed social networks avoid establishing triads with an odd number of negative links. Generalizing, the weak version of balance theory allows for nodes to be partitioned into any number of blocks with positive internal links, mutually connected by negative links. If this prescription is interpreted rigidly, i.e., without allowing for statistical noise in the observed link signs, then most real graphs will appear to require a larger number of blocks than the actual one, or even to violate both versions of the theory. This might lead to conclusions invoking even more relaxed notions of balance. Here, after rephrasing structural balance theory in statistically testable terms, we propose an inference scheme to unambiguously assess whether a real-world signed graph is balanced. We find that the proposed statistical balance theory leads to interpretations that are quite different from those derived from the current deterministic versions of the theory.

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Introduction. The interest towards signed networks dates back to *balance theory*, first proposed by Heider as a theory of behavior [1]. As the name suggests, the theory revolves around the concept of "balance," i.e., the tendency of people to engage, and remain, in situations that support their beliefs. The practice of adopting signed graphs to model social networks subsequently led Cartwright and Harary [2] to introduce the structural version of balance theory [2–6]: A complete signed graph is said to be balanced if *all triads* have an even number of negative edges, i.e., either zero (in this case, the three edges are all positive) or two. The so-called *structure theorem* states that a complete signed graph is balanced if and only if its set of nodes can be partitioned into $k = 2$ disjoint subsets whose intramodular links are all positive and whose intermodular links are all negative. Cartwright and Harary extended the definition of balance to incomplete graphs [2] by including cycles of length larger than three: A network is now said to be balanced if *all cycles* have an even number of negative edges (although the points of each subset are no longer required to be connected). Taken together, the criteria above define the so-called *strong balance theory*.

In an attempt to make such a framework more applicable, Davis introduced the concept of *k-balanced* networks: According to it, signed graphs are balanced if their set of nodes can be partitioned into $k \geqslant 2$ disjoint subsets with positive intramodular links and negative intermodular links [7]. This generalized definition of balance has led to the formulation of the *weak balance theory*, according to which triads whose edges are all negative are balanced as well, since each node can be thought of as a group on its own. From a mesoscopic perspective, however, both versions of the balance theory require the presence of positive blocks along the main diagonal of the adjacency matrix ($k = 2$, according to the strong variant; $k > 2$, according to the weak variant [8,9]) and of negative off-diagonal blocks. Taken together, the strong and the weak variants of the balance theory define what may be called *traditional balance theory*: Hence, k-balanced networks are traditionally balanced.

Currently, when real-world networks are used to test traditional balance theory, the possible presence of statistical noise in the observed link signs is not taken into account. As a result, a network produced by a k-balanced process might appear as requiring a larger number $k' > k$ of blocks to be consistent with the theory, hence favoring the weak over the strong version of the theory. Even more dramatically, there might not be any partition into blocks with the "ideal" sign assignments compatible with traditional balance theory. The theory might then be erroneously dismissed in favor of looser alternatives such as the so-called *relaxed balance theory* [10], which allows for the blocks of the matrix to be connected with the "wrong" signs, raising, however, the problem that a block structure with arbitrary signs can always be found on any signed graph, thus not being truly informative about the tendency towards balance of real-world networks. Here, we recast the idea of balance theory within a statistical framework, thus allowing for noise in the empirical link signs, while attempting at identifying the underlying "denoised" signed block structure from which more robust conclusions can be

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drawn about the observed level of balance. As expected, we find that the proposed statistical variant of the theory leads to conclusions that are quite different from those derived from the current deterministic variants.

Setting up the formalism. Each edge of a signed graph can be *positive*, *negative*, or *missing*: As we will focus on binary undirected signed networks, a generic entry of the signed adjacency matrix $\mathbf{A}$ will be assumed to read $a_{ij} = -1, 0, +1$, with $a_{ij} = a_{ji}, \forall\, i < j$. To ease mathematical manipulations, let us employ Iverson's brackets (a notation ensuring all quantities of interest to be non-negative—see Supplemental Material Sec. A [11]) and define the quantities $a^{-}_{ij} = [a_{ij} = -1]$, $a^{0}_{ij} = [a_{ij} = 0]$, $a^{+}_{ij} = [a_{ij} = +1]$: The new variables are mutually exclusive, sum to 1, and induce the two matrices $\mathbf{A}^{+}$ and $\mathbf{A}^{-}$, satisfying $\mathbf{A} = \mathbf{A}^{+} - \mathbf{A}^{-}$ and $|\mathbf{A}| = \mathbf{A}^{+} + \mathbf{A}^{-}$. The number of positive and negative links is defined as $L^{+} = \sum_{i=1}^{N} \sum_{j(>i)} a^{+}_{ij}$ and $L^{-} = \sum_{i=1}^{N} \sum_{j(>i)} a^{-}_{ij}$, respectively.

Traditional balance theory. The top-down formulation of the traditional balance theory leads quite naturally to the definition of a score function for quantifying the "degree of compatibility" of a given partition with the theory itself. It is referred to as *frustration*[1] and reads

$$F(\boldsymbol{\sigma}) = \sum_{i=1}^{N} \sum_{j(>i)} a^{-}_{ij}\delta_{\sigma_i,\sigma_j} + \sum_{i=1}^{N} \sum_{j(>i)} a^{+}_{ij}\left(1 - \delta_{\sigma_i,\sigma_j}\right)$$
$$= L^{-}_{\bullet} + L^{+} - L^{+}_{\bullet}$$
$$= L^{-}_{\bullet} + L^{+}_{\circ}, \tag{1}$$

where $\boldsymbol{\sigma} \equiv \{\sigma_i\}$ stands for a vector of labels characterizing a generic partition and $\delta_{\sigma_i,\sigma_j}$ is the Kronecker delta (i.e., $\delta_{\sigma_i,\sigma_j} = 1$ if $\sigma_i = \sigma_j$ and 0 otherwise). In other words, $F(\boldsymbol{\sigma})$ counts the amount of misplaced links according to the traditional balance theory, i.e., the number of negative links within modules (indicated with a solid dot) plus the number of positive links between modules (indicated with an open dot). The simplest operative criterion for singling out a k-balanced partition is based upon the following theorem (see Refs. [7,9,14,15] for similar results).

Theorem I. $F(\boldsymbol{\sigma}) = 0 \Longleftrightarrow$ the partition $\boldsymbol{\sigma}$ is k balanced.

Proof. Sufficiency condition: $F(\boldsymbol{\sigma}) = 0 \Longrightarrow$ the partition $\boldsymbol{\sigma}$ is k balanced. Since $L^{+}_{\circ} \geqslant 0$ and $L^{-}_{\bullet} \geqslant 0$, $F(\boldsymbol{\sigma}) = 0$ implies that $L^{+}_{\circ} = 0$ and $L^{-}_{\bullet} = 0$; hence, the definition of k-balanced partition is satisfied. Necessity condition: The partition $\boldsymbol{\sigma}$ is k-balanced $\Longrightarrow F(\boldsymbol{\sigma}) = 0$. Since a k-balanced partition is defined by the presence of a clustering with k subsets, no negative links within modules and no positive links between modules, $L^{-}_{\bullet} = 0$ and $L^{+}_{\circ} = 0$; hence, $F(\boldsymbol{\sigma}) = 0$. ■

In other words, the bare numerical value $F(\boldsymbol{\sigma})$ can be thought of as acting in a thresholdlike fashion, classifying the configurations characterized by $F(\boldsymbol{\sigma}) = 0$ as balanced and the configurations characterized by $F(\boldsymbol{\sigma}) > 0$ as frustrated. The criterion embodied by the F test is, however, too strict for real-world networks, which are hardly (if ever) found to obey it: As Table I shows, in fact, none of the listed configurations satisfies it. As noticed in Ref. [10], the block structure defining the traditional balance theory is overly restrictive, dooming the vast majority of real-world signed networks to be quickly dismissed as frustrated—in fact, observing one misplaced link is enough to let one conclude that the theory is not obeyed.

TABLE I. Empirical amount of frustration, detected by searching for the partition minimizing $F(\boldsymbol{\sigma})$ that characterizes the listed real-world networks: According to the F test, none of them turns out to satisfy the traditional balance theory. The same result holds true even when employing the generalized definition of the frustration index (here, implemented by posing $\alpha = 0.2$ and $\alpha = 0.8$).

				$G(\boldsymbol{\sigma}\vert\alpha)$	
	N	L	$F(\boldsymbol{\sigma})$	$\alpha = 0.2$	$\alpha = 0.8$
Fraternity [16]	16	40	1	0.2	0.4
New Guinea Highlands (N.G.H.) Tribes [16]	16	58	2	1.4	0.4
Slovenian Parliament [8]	10	45	2	0.4	0.8
Monastery [16]	18	49	5	2.4	1.8
Spanish School 2 [17]	182	866	69	43.4	42.4
Spanish School 1 [17]	359	2048	153	44	61
U.S. Senate [16]	100	2461	247	166.8	56
CoW, 1946–1949 [18]	60	360	12	3.8	5.8
CoW, 1950–1953 [18]	72	437	11	5.6	5.4
CoW, 1954–1957 [18]	80	492	27	7	12.2
CoW, 1958–1961 [18]	101	613	25	6.4	14.6
CoW, 1962–1965 [18]	109	642	32	16.8	24.6
CoW, 1966–1969 [18]	111	607	24	11.8	15.6
Epidermal Growth Factor Receptor (EGFR) [16]	313	755	189	51.2	46.8
Macrophage [16]	660	1897	316	91.4	77.2
Bitcoin Alpha [19]	3775	14120	1399	337.9	585.6
Bitcoin Over-The-Counter (OTC) [19]	5875	21489	3259	540.4	800.4

Softening frustration. In order to overcome what was perceived as a major limitation of the traditional balance theory, Doreian and Mrvar [10] proposed to replace $F(\boldsymbol{\sigma})$ with its softened variant

$$G(\boldsymbol{\sigma}|\alpha) = \alpha L^{-}_{\bullet} + (1-\alpha)L^{+}_{\circ}, \tag{2}$$

allowing (i) the misplaced positive links to be weighted more upon choosing $0 \leqslant \alpha < 1/2$ and (ii) the misplaced negative links to be weighted more upon choosing $1/2 < \alpha \leqslant 1$. Even ignoring the ambiguity due to the lack of a principled way for selecting α (the so-called "α problem" in Ref. [13]), the criterion embodied by the G test is still too strict: As Table I shows, in fact, none of the listed configurations satisfies it either. This is rigorously stated by the following theorem, whose proof is immediate.

Theorem II. If $0 < \alpha < 1$, $F(\boldsymbol{\sigma}) = 0 \Longleftrightarrow G(\boldsymbol{\sigma}|\alpha) = 0$, i.e., the partition $\boldsymbol{\sigma}$ is k balanced.[2]

[1] More formally, *line index of imbalance* [12,13].

[2] Notice that the values $\alpha = 0$ and $\alpha = 1$ would, respectively, lead to the trivially balanced partition characterized by a single community gathering all nodes together and N single-node communities (or singletons).

Relaxed balance theory. In light of previous results, the second attempt pursued by Doreian and Mrvar [10] to overcome the perceived limitations of the traditional balance theory was more radical, as they proposed to relax it by allowing for the presence of positive off-diagonal blocks and negative diagonal blocks, a generalization that has gained the name of *relaxed balance theory* [10]. Such a formulation, however, lacks a proper mathematization, as a score function such as $F(\boldsymbol{\sigma})$ or $G(\boldsymbol{\sigma}|\alpha)$ cannot be easily individuated. In addition, it is affected by the problem highlighted in Ref. [20]: "*[...] if the number of clusters is left unspecified a priori, the best partition is the singleton's partition (i.e., each node in its own cluster) [...].*" A bit provocatively, one may say that "the remedy seems worse than the disease" as no criterion is provided to (i) quantify the extent of the violation of the traditional balance theory and (ii) assess if it is indeed so relevant to justify the introduction of an alternative conceptual framework.

Statistical balance theory. Recasting the theory of balance within a statistical framework solves all the aforementioned problems at once, allowing us to define an inference scheme to unambiguously assess if a signed graph is either traditionally or relaxedly balanced, hence overcoming the limitations of the F-based and G-based tests while providing a proper mathematization of the relaxed balance theory.

In order to define a statistical theory of balance, let us suppose the presence of a probabilistic model behind the appearance of any signed configuration: The traditional balance theory could be then rephrased by posing

$$p^-_{rr} = 0, \quad r = 1 \dots k, \tag{3}$$

and

$$p^+_{rs} = 0, \quad \forall\, r < s, \tag{4}$$

with p^+_{rr} indicating the probability that any two nodes belonging to the same block r are connected by a positive link, p^+_{rs} indicating the probability that any two nodes belonging to the different blocks r and s are connected by a positive link, and analogously for their negative counterparts.

The starting point of our approach is that of softening these positions, replacing them with the milder relationships

$$\text{sgn}[p^+_{rr} - p^-_{rr}] = +1, \quad r = 1 \dots k, \tag{5}$$

which amounts to requiring $p^+_{rr} > p^-_{rr}$, $r = 1 \dots k$, and

$$\text{sgn}[p^+_{rs} - p^-_{rs}] = -1, \quad \forall\, r < s, \tag{6}$$

which amounts to requiring $p^+_{rs} < p^-_{rs}, \forall\, r < s$. A configuration satisfying these relationships will be claimed to support the statistical variant of the traditional balance theory: specifically, its strong variant if $k = 2$ and its weak variant if $k > 2$; otherwise (because $p^+_{rr} \leqslant p^-_{rr}$ for some diagonal blocks or $p^+_{rs} \geqslant p^-_{rs}$ for some off-diagonal blocks), it will be claimed to support the statistical variant of the relaxed balance theory. Additionally, we call a partition *homogeneous* if either $p^+_{rs} = 0$ or $p^-_{rs} = 0, \forall\, r \leqslant s$; otherwise, it will be called *heterogeneous* (see Supplemental Material Sec. B [11]). In other words, the deterministic rules first defined by Cartwright, Harary, and Davis are replaced by a probabilistic criterion individuating "a tendency" to obey, or not to obey, the traditional balance theory.

Tuning the aforementioned parameters on a given signed network requires a generative model to be specified: Here, we will adopt the signed stochastic block model (considered also in Refs. [21–23] but derived within the exponential random graph framework in Supplemental Material Sec. B [11]), defined by the likelihood function

$$\begin{aligned}\mathcal{L}_{\text{SSBM}} = &\prod_{r=1}^{k} (p^+_{rr})^{L^+_{rr}} (p^-_{rr})^{L^-_{rr}} (1 - p^+_{rr} - p^-_{rr})^{\binom{N_r}{2} - L_{rr}} \\ &\times \prod_{r=1}^{k} \prod_{s(>r)} (p^+_{rs})^{L^+_{rs}} (p^-_{rs})^{L^-_{rs}} (1 - p^+_{rs} - p^-_{rs})^{N_r N_s - L_{rs}},\end{aligned} \tag{7}$$

where N_r is the number of nodes constituting block r, L^+_{rr} (L^-_{rr}) is the number of positive (negative) links within block r, L^+_{rs} (L^-_{rs}) is the number of positive (negative) links between blocks r and s, $\forall\, r < s$ and the probability coefficients read $p^+_{rr} = 2L^+_{rr}/N_r(N_r - 1)$, $p^-_{rr} = 2L^-_{rr}/N_r(N_r - 1)$, $r = 1 \dots k$, and $p^+_{rs} = L^+_{rs}/N_r N_s$, $p^-_{rs} = L^-_{rs}/N_r N_s$, $\forall\, r < s$. As maximizing the bare likelihood is a recipe known to be affected by a number of limitations [13], we have opted for the minimization of

$$\text{BIC} = \kappa_{\text{SSBM}} \ln n - 2 \ln \mathcal{L}_{\text{SSBM}}, \tag{8}$$

referred to as the *Bayesian information criterion* (BIC). Derivable as the saddle-point approximation of the (Bayesian) evidence of a model, such a criterion embodies a trade-off between parsimony [accounted for by the first addendum, with κ_{SSBM} being the number of parameters of the model[3] and $n = N(N-1)/2$ proxying the system dimensions] and accuracy (accounted for by the second addendum, i.e., the log-likelihood term) [24,25]. Although the magnitude of both addenda rises with the number of parameters, the log-likelihood term drives BIC towards more negative values while the parsimony term drives BIC towards more positive values: The number of parameters in correspondence of which the minimum is reached is selected and drives the network partition.

Our bottom-up approach is "maximally agnostic" towards any theory of balance, letting the data determine the number and the values of the parameters best fitting a given configuration: BIC is, in fact, sensitive to the "signed density" of the modules "by design," hence capable of spotting the presence of groups of nodes as well as attributing to each of them the sign of the majority of its constituting links. From a purely computational perspective, instead, the complexity of the algorithm to minimize BIC decreases with the link density $c = 2L/N(N-1)$: In other words, the denser the configuration, the faster the algorithm (see Supplemental Material Sec. B [11]).

Results. First, let us test our prescription on a number of synthetic configurations. As Fig. 1 shows, BIC minimization always leads to recovering the planted partition,

[3]To avoid confusion with the number of modules k characterizing k-balanced networks, we have indicated the number of a model parameters as κ: Naturally, $\kappa_{\text{SSBM}} = k(k+1)$ since we need to estimate two parameters per module.

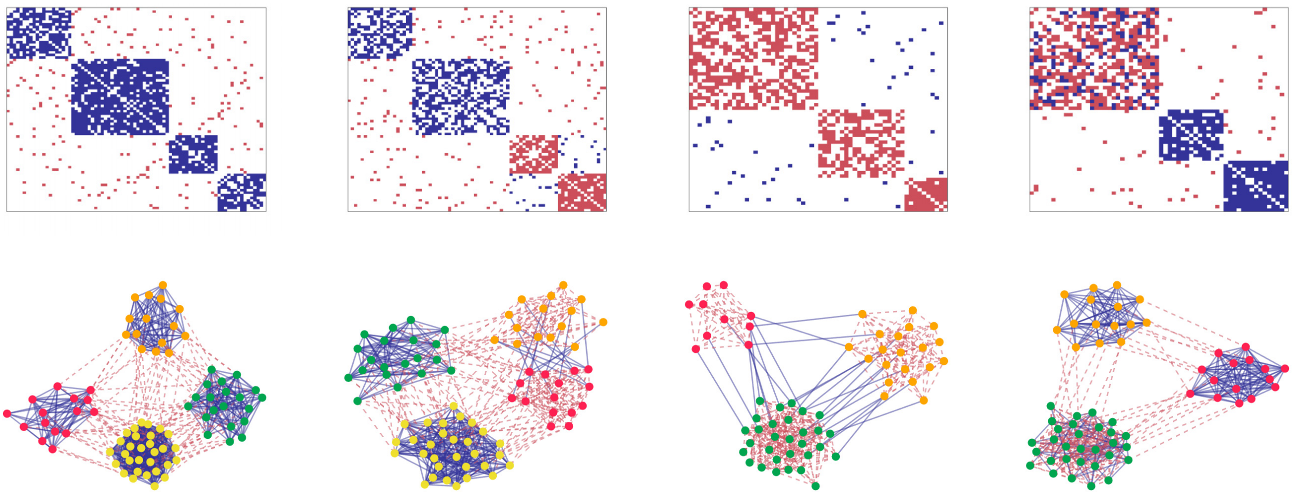


FIG. 1. Consistency checks on four synthetic configurations (positive links are colored in blue; negative links are colored in red): Minimizing BIC always leads to recovering the planted partition, be it homogeneous and balanced according to the traditional balance theory (first panel); homogeneous and balanced according to the relaxed balance theory (second and third panel); or heterogeneous and balanced according to the statistical variant of the relaxed balance theory (fourth panel). Nodes of different colors belong to different groups.

irrespectively from the values of the sets of coefficients $\{p_{rr}^+\}$, $\{p_{rr}^-\}$, $\{p_{rs}^+\}_{r<s}$, $\{p_{rs}^-\}_{r<s}$, i.e., be it a homogeneous partition, balanced according to the weak variant of the traditional balance theory (more precisely, a 4-balanced partition); two homogeneous partitions, balanced according to the relaxed balance theory (e.g., the third adjacency matrix is defined by $p_{11}^+ = p_{22}^+ = p_{33}^+ = 0$ and $p_{12}^- = p_{13}^- = p_{23}^- = 0$); a heterogeneous partition, balanced according to the statistical variant of the relaxed balance theory (i.e., the fourth adjacency matrix, defined by $p_{11}^+ < p_{11}^-$, $p_{22}^+ > p_{22}^-$, $p_{33}^+ > p_{33}^-$ and $p_{12}^+ < p_{12}^-$, $p_{13}^+ < p_{13}^-$, $p_{23}^+ < p_{23}^-$). Additional exercises of the kind are reported in Supplemental Material Sec. B [11], where it is shown that employing BIC leads to robust inference towards perturbations aimed at degrading a given balanced partition, across a wide range of related parameter values.

Second, let us compare our recipe with that prescribed to minimize $F(\boldsymbol{\sigma})$. As Fig. 2 shows, implementing the latter does not lead to recovering the planted partition (in this case, a homogeneous one, compatible with the relaxed balance theory): Instead, it leads to a traditionally balanced configuration where the planted negative cliques have been fragmented into singletons. Minimizing $F(\boldsymbol{\sigma})$ can lead to a number of ambiguous situations, such as (i) returning configurations that are neither traditionally nor relaxedly balanced or (ii) returning more than one frustrated configuration (see Supplemental Material Sec. B [11]). More in general, ignoring the interplay between the signs and the density of connections, solely accounting for the information carried by the first ones may lead to "resolution errors" such as (i) splitting modules (even fully connected ones) into finer regions or (ii) misinterpreting adjacent modules, characterized by the same dominant sign, as single coarser regions.

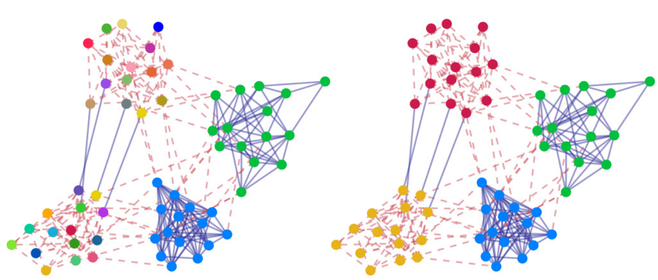
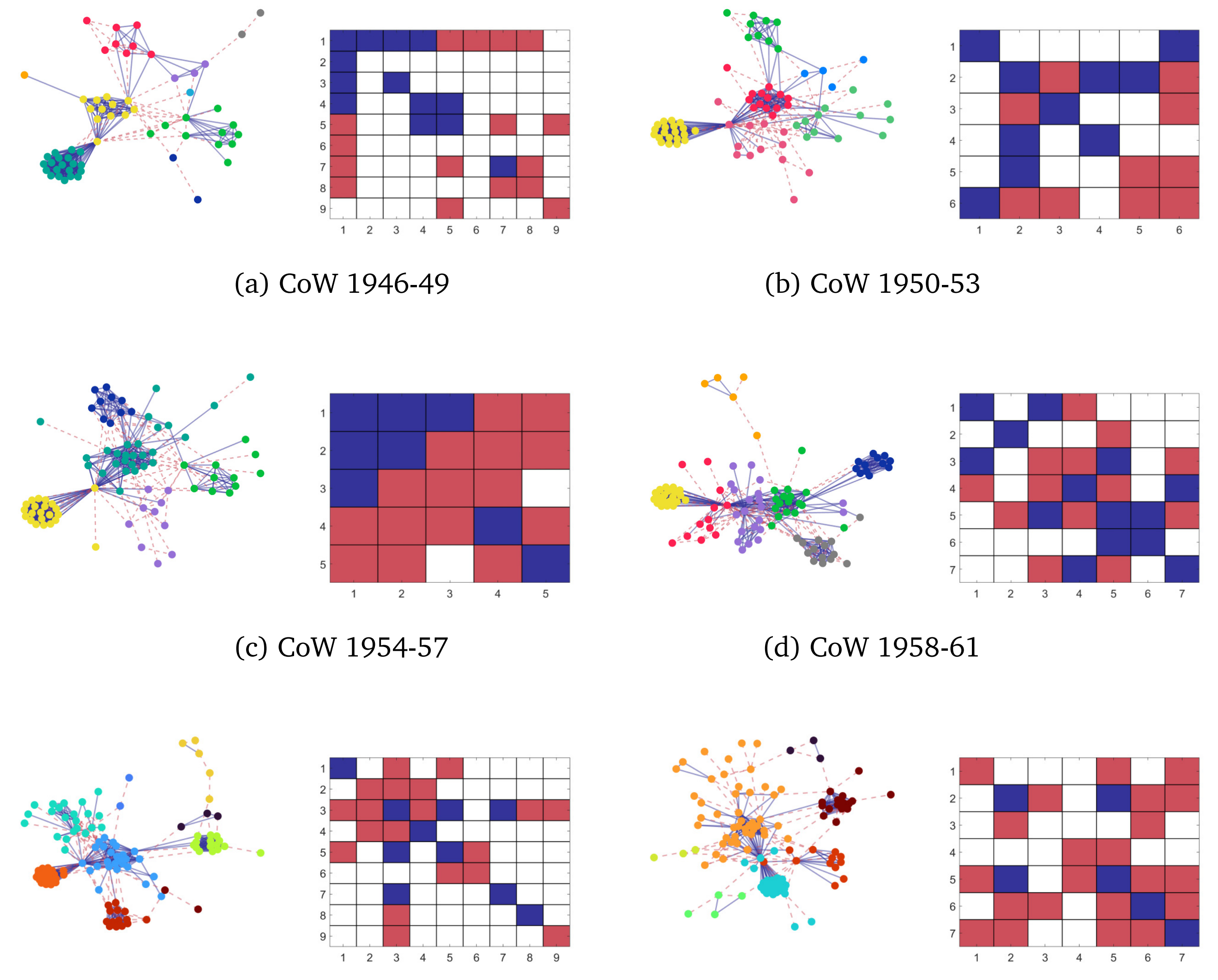


FIG. 2. Partitions recovered upon minimizing $F(\boldsymbol{\sigma})$ (left panel) and upon minimizing BIC (right panel): While minimizing $F(\boldsymbol{\sigma})$ leads to recovering a partition that is compatible with the traditional balance theory even if there is none "by design," minimizing BIC leads to recovering the homogeneous planted partition, compatible with the relaxed balance theory. Positive links are colored in blue; negative links are colored in red. Nodes of different colors belong to different groups.

Let us now apply our recipe to a number of real-world signed configurations, i.e., six snapshots of the "Correlates of Wars" (CoW) data set [18], providing a picture of the international political relationships over the years 1946–1997 and consisting of 13 snapshots of 4 years each: A positive edge between any two countries indicates an alliance, a political agreement, or the membership to the same governmental organization; conversely, a negative edge indicates that the two countries are enemies, have a political disagreement, or are part of different governmental organisations. As Fig. 3 shows, minimizing BIC leads to recover partitions that obey the statistical variant of the relaxed balance theory (a blue block is characterized by a majority of positive links; a red block is characterized by a majority of negative links); other real-world signed configurations, instead, are found to obey the statistical variant of the traditional balance theory (see Supplemental Material Sec. B [11]). All such partitions are heterogeneous. Overall, larger configurations seems to align more with the (statistical variant of the) relaxed balance theory while smaller configurations seems to align more with the (statistical variant of the) traditional balance theory.

Discussion. The present Letter proposes a statistical approach to the theory of balance, assuming that any real-world



(a) CoW 1946-49

(b) CoW 1950-53

(c) CoW 1954-57

(d) CoW 1958-61

(e) CoW 1962-65

(f) CoW 1966-69

FIG. 3. Partitions recovered upon minimizing BIC on six snapshots of the CoW data set [18], providing a picture of the international political relationships over the years 1946–1997. A generic block, indexed as rs, is colored in blue if $L^+_{rs} > L^-_{rs}$, in red if $L^+_{rs} < L^-_{rs}$, and in white if $L^+_{rs} = L^-_{rs}$: As blue blocks do not appear only on the diagonal and red blocks do not appear only off diagonal, the considered configurations obey the statistical variant of the relaxed balance theory. Nodes of different colors belong to different groups.

signed configuration is the result of a generative process, probabilistic in nature. As some unavoidable degree of statistical noise is expected to affect any observed network structure, the criterion adopted to assess the consistency with balance theory can be recast in terms of the signs of the *estimated probabilities* to observe positive and negative links, i.e., $p^+_{rs} - p^-_{rs}, \forall\, r \leqslant s$. Estimating these coefficients by minimizing BIC allows one to unambiguously assess which variant of the theory is obeyed, from a statistical perspective, by any signed configuration.

On the contrary, minimizing $F(\boldsymbol{\sigma})$ is practically equivalent to carrying out a sort of one-sided test of hypothesis, allowing one to conclude if a given partition *does not obey* the traditional balance theory (as a matter of fact, practically always) but incapable of providing a univocal classification for a generic signed configuration. Moreover, it "works" even with configurations generated by the signed random graph model, i.e., a model carrying no information about a network modular structure, hence *overfitting* (i.e., misinterpreting statistical noise as a genuine signal—see Supplemental Material Sec. B [11]).

Under this respect, maximizing the *signed modularity* $Q(\boldsymbol{\sigma})$ is of no help, being defined as

$$Q(\boldsymbol{\sigma}) = \sum_{i=1}^{N} \sum_{j(>i)} [(a^+_{ij} - p^+_{ij}) - (a^-_{ij} - p^-_{ij})]\delta_{\sigma_i,\sigma_j} = -F(\boldsymbol{\sigma}) + \langle F(\boldsymbol{\sigma})\rangle + L^+ - \langle L^+\rangle, \quad (9)$$

with the obvious meaning of the symbols (the addendum $L^+ - \langle L^+\rangle$ is just an offset not depending on the specific partition and amounting to zero for any model reproducing the total number of positive links) [26]. In other words, the signed modularity compares the empirical amount of frustration of a given signed configuration with the one predicted by a properly defined reference model: One may thus define a partition as *statistically balanced* if it satisfies the relationship $F(\boldsymbol{\sigma}) < \langle F(\boldsymbol{\sigma}) \rangle$, i.e., $Q(\boldsymbol{\sigma}) > 0$. Although reasonable, such a criterion does not differ (much) from the one embodied by the F test: More formally, it can be proven that the relationship $L^+ \gg L^-$ (often, if not always, found to hold true for real-world signed networks) favors the fragmentation of the negative cliques into singletons, hence leading to recover traditionally balanced configurations even when there is none "by design" (see Fig. 2 and Supplemental Material Sec. C [11]).

Conclusions. Although the problem of partitioning a signed network has been approached in the past (e.g., by maximizing the signed modularity [9,26,27]), the existing works have completely overlooked the issue of harmonizing the request of having balanced configurations with that of having modular configurations, in most of the cases verifying either the "degree of balance" of modular structures or the "degree of modularity" of balanced structures *a posteriori* [28,29]; generative models, instead, can accommodate both requests, thus avoiding to return contradictory results, i.e., those one generally gets when combining a purely structure-based community detection with a purely sign-based one, while laying the basis of a more comprehensive theory of balance, grounded on probability theory (see Refs. [30–32] for related results).

Recasting the idea of balance within a statistical framework allows the presence of noise in the empirical link signs to be accounted for, hence overcoming the limitations of the current deterministic theories, which are doomed to misinterpret random patterns for genuine signals of (im)balance. Our inference scheme instead allows us to unambiguously assess whether a real-world signed graph obeys the traditional notion of balance or aligns with a more relaxed variant of it. As a last point, we would like to stress that, within such a framework, the standard notion of frustration should be replaced by a fuzzy one, to be interpreted as proxying the "distance" of a given configuration from the closest, either traditionally or relaxedly, balanced one.

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