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List of publications related to the thesis

- D. Herceg, P. Sopasakis, and P. Patrinos. ”Risk-averse model predictive control.”. 36 th Benelux Meeting on Systems and Control (2017)
- Sopasakis, Pantelis, Domagoj Herceg, Panagiotis Patrinos, and Alberto Bemporad. ”Stochastic economic model predictive control for Markovian switching systems.” IFAC-PapersOnLine 50, no. 1 (2017): 524-530.
- D. Herceg, P. Sopasakis, A. Bempoard and P. Patrinos. ”Risk-averse risk-constrained optimal control.” .Book of Abstracts - 37th Benelux Meeting on Systems and Control (2018)
- Sopasakis, Pantelis, Domagoj Herceg, Alberto Bemporad, and Panagiotis Patrinos. ”Risk-averse model predictive control.” Automatica 100 (2019): 281-288.

Other publications not included in the thesis

- Herceg, D., Georgoulas, G., Sopasakis, P., Castaño, M., Patrinos, P., Bemporad, A., Niemi, J. and Nikolakopoulos, G., 2017, July. Data-driven modelling, learning and stochastic predictive control for the steel industry. In 2017 25th Mediterranean Conference on Control and Automation (MED) (pp. 1361-1366). IEEE.
- Herceg, D., Ntouskas, S., Sopasakis, P., Dokoumetzidis, A., Macheras, P., Sarimveis, H. and Patrinos, P., 2017. Modeling and administration scheduling of fractional-order pharmacokinetic systems. IFAC-PapersOnLine, 50(1), pp.9742-9747.

To my family.

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Abstract

This thesis attempts to shed some additional light on pressing questions regarding the control of uncertain systems. Special focus is given to systems with uncertain uncertainty (inexactly known distribution), numerical optimization methods to enable the use of proposed advanced optimization methods in practice and systems controlled by an economic controller where stability is not always the primary objective. Current state-of-the-art methods often neglect that underlying uncertainty in a stochastic model is, in fact, uncertain as well in the sense that its probability distribution function is unknown and can only be (inaccurately) estimated from data. Hence, theoretical guarantees obtained by such methods, e.g., mean-square stability, may not be satisfied in practice. Moreover, many control methods use convex costs as a performance index to be optimized, which may not be the most descriptive choice for real-world problems. Here, we endeavour to remedy the shortcomings of such methods. We focus on three important extension in particular, i) theoretical developments to deal with non-convex performance indices in stochastic optimal control problems and ii) novel methods to deal with "uncertainty in the uncertainty" in a rigorous and theoretically sound way and iii) numerical optimization methods to solve these problems efficiently.

Model predictive control (MPC) is an advanced control method that has found its way into many practical applications. Since its introduction and popularization in the 80's in the process industry, it has now taken a long way to automotive applications, large scale networks and robotics. MPC itself uses a mathematical model of a system to predict its possible future

trajectories. A sequence of control actions is then calculated by solving an optimization problem by minimizing a performance index of the state and input cost along predicted trajectories . When the system moves to a new state, the new state is sampled and the whole procedure is applied again. Part of its popularity stems from the fact that the MPC framework can also incorporate state and input constraints and handle multiple input output systems naturally.

Stochastic economic model predictive control is concerned with problems with non-convex costs which are readily found in real-world applications. Rather than minimizing a deviation from a prescribed (optimal/best) set-point or a tracking reference, the main objective is to optimize a given economic cost functional. The control paradigm that optimizes the process economics within the MPC formulation is usually known as economic MPC (EMPC). Several research directions have discussed the closed-loop properties of EMPC-controlled deterministic systems, however, little have uncertain systems been studied. In this thesis we propose EMPC formulations for nonlinear Markovian switching systems which guarantee recursive feasibility, asymptotic performance bounds and constrained mean square (MS) stability. For nonlinear systems we provide design guidelines based on the system linearization using only mild assumptions on the system dynamics and stage cost function.

Risk-averse model predictive control is an approach to bridge the gap between two popular control strategies dubbed stochastic and robust MPC. In robust MPC, modeling errors and disturbances are assumed to be unknown-but-bounded quantities and the performance index is minimized with respect to the worst-case realization of the uncertainty (min-max) approach). However, such worst-case events which are unlikely to occur in practice and render robust MPC severely conservative since all statistical information, typically available from past measurements, is completely ignored. On the other hand, in stochastic MPC it is assumed that the underlying uncertainty is a random vector following some probability distribution. In reality, not always can the probability distribution be accurately estimated from available data, nor does it remain constant in time. Nonetheless, theoretical guarantees of such algorithms hinge on this unrealistic assumption. Using the theory of risk measures, which originated in the field of stochastic finance, we devise a novel algorithmic and theoretical solutions to combine advantages of robust and stochastic optimal control by proposing a unifying framework that extends and contains both as special cases. In this thesis, we propose risk-averse formulations where the total cost of the MPC problem is expressed as a nested composition of conditional risk mappings. We focus on constrained nonlinear Markovian switching systems and derive Lyapunov-type risk-averse stability conditions. Moreover, for the nonlinear system we prescribe a linearization based controller design procedure and we show that linearized system locally inherits stability properties of its linear counterpart.

Finally, we propose a splitting for risk-averse problems which makes the problem a candidate for proximal algorithms. Usually, risk-averse problems are solved using stochastic dual dynamics programming approaches or generic interior point method solvers. Both of these approaches are are not adept to deal with problems of large dimension. However, we show that risk-averse problems posses a rich structure that we can exploit to devise very efficient and massively parallelisable methods to solve them.

Chapter 1

Introduction

1.1 Background and motivation

1.1.1 Model predictive control

Model predictive control has had its place in industrial control setting for decades now. First know use was under the name Dynamic matrix control used by Shell company. Proven in practice, it has also caught interest of academics research due to its successful application. Nowadays model predictive control has it's place in automotive industry with new and emerging applications in aerospace industry. Academic research has enabled new applications with ever decreasing time scale. This is usually discussed in the field as the field of embedded model predictive control.

Main research today focused on the stochastic, distributed end fast model predictive control. It has even caught attention of the finance sector (BGPB10; NBLM19). Main advantage od MPC is that its the basic idea is very simple to understand. At each time systems' state is sampled, then an optimal input is computed based on the systems predicted behavior. Predictions are made based on the mathematical model of the system which strikes a balance between being accurate enough to capture dominant dynamic behavior of a system and simple enough to be usefully from computational point of view. System input is applied and system is sampled at a new state and whole process is repeated *ad infi-*

mum. Model predictive control can also handle state and input controls gracefully as it can deal with multiple input multiple output systems.

1.1.2 Uncertainty in the uncertainty

State of the art approaches fail to account for another very important facet of algorithms that deal with uncertainty – the uncertainty of the uncertainty itself. Almost always, the underlying uncertainty, upon which we build control algorithms is tacitly assumed to be correct – yet it almost never is. In this thesis, we tackle the problem in the most principal way by accounting directly for the uncertainty in the uncertainty by using *risk measures*. The importance of measuring risk has been recognized in the field of finance, where it plays a central role. A significant breakthrough in understanding of risk was with the introduction of Value at Risk (V@R), which dates back to 80's and was popularized by J.P. Morgan.The latest one has started with the adoption of *coherent* risk measures which were introduced in a seminal paper by (ADEH99) In the above paper authors postulate four main axioms which any measure of risk must obey in order to be considered a coherent measure of risk. Risk measures which are coherent have remarkable mathematical properties which allow for design of computationally efficient algorithms. Possibly the most famous and widely used coherent risk measure is the average value-at-risk (AV@R). Authors in their celebrated paper (RU^+00) give an optimization based formulation for calculation of such value. It is worth nothing that in the rest of the thesis we shall closely follow the notation of (SDR14a) which is somewhat closer to one used in the engineering field, as opposed to (ADEH99) which use terms closer to finance. In 2014 the Basel Committee on Banking Supervision changed has suggested to replace V@R metric with AV@R for assessing market risk.

Of course these ideas are not unique to finance sector. An important concept in engineering is the so called "probability of failure". We refer the reader to an excellent discussion in (RR10), and to (MU18) for further mathematical properties. There it is argued that a much better concept would be what they call the "buffered probability of failure".

This is equivalent to the move form V@R to AV@R in finance terms. Authors argue that not accounting for the events after some fixed threshold makes things much worse when unlikely events do happen. On the other hand taking into account the values above the threshold makes losses more manageable when they do happen. Aside from more robust design, they are much easier to handle mathematically. In simple terms, for discrete distributions, problem with V@R objective amounts to a combinatorial problem, which gets prohibitive as the number of data points grows for example in big-data applications. Contrary to that, AV@R problems amount to solving a convex problem, for which reliable and efficient methods already exist (BV04).

Virtually, the only conceptual drawback to this new concept is that it is somewhat harder to back-test for, i,e., to estimate the values from past measurements, in sense that confidently estimating AV@R requires usually more data than estimating VAR. However, with the measuring capabilities in the era of big data, we expect this problem to be less pronounced than some time in the past. It is worth nothing that concept of risk-aversion is deeply rooted in human psychology as well. It is empirically shown that humans experience emotionally the same magnitude of loss roughly twice as negatively as opposed to positive feeling form the same gain (Kah11). Risk-based decision making hence is a natural candidate for many emerging human-centered technologies.

In our work, we extend the known results in the field of stochastic MPC by giving, through use of risk measures, an unifying overview of the two most popular approaches in the field. Risk measures capture both of these approaches and offer an interpolation between the two. In this way, we provide a designer of the system with a turning knob to account for the belief in the nominal uncertainty on which the control system is build. The current theory is expanded upon in two major directions. Fist, a novel stability analysis is presented for the design or risk-averse controllers, including linearization procedure for nonlinear control systems. Secondly, a modeling procedure is sketched out on how to ”untangle” convoluted risk measures for any polytopic coherent risk measure.

1.1.3 Computational challenges

Development of theoretical properties for risk-averse methods leads us to a natural and very important question – are we able to solve these problems accurately and timely? The crucial link missing, in the authors' view, are precisely numerical solvers that can do this. Theoretical developments in risk-averse control are not restricted to control problems only, even though that was our main focus. However, risk-averse optimization lends it self to every branch of science and technology where uncertainty is inherent. Moreover, risk-averse problems are by its nature large scale problems. Usually the data is estimated from a large number of scenarios, which provide a fine probability distribution. The need for large data set stems from the fact that we are mostly interested int the right tail of the distribution and as such we would like to have a good representation of very unlikely scenarios. Usually, the resulting risk-averse problems are solved by interior point solvers which offer hight accuracy, but the iterations of algorithm become very slow as the size of the problem increases until they become infeasible. Another popular approach is to use stochastic dual dynamics programming approaches which are not as efficient. Another inherent difficulty of risk-averse problems, aside from its large scale, is the fact that they are non-smooth, and the optimization problem boils down mostly to projections on epigraphs. Problem is further complicated by knowing that most firs-order methods are relatively imprecise.

The rise of GPU capable hardware and a general interest in large-scale problems has brought attention to the field of proximal algorithms once again. It is worth nothing that major theoretical fundamentals were done in the sixties. However, we leverage some recent results in proximal algorithms to arrive at robust and accurate solutions to risk-averse problems.

1.2 Scope and Contributions of the thesis

In this thesis, we endeavor the bridge some of the main shortcoming of such a thesis. Namely these are (i) extending the theory of stochastic economic model predictive control to account for underlying stochasticity (ii) proposing novel and general theory to handle uncertainty in the uncertainty and (iii) proposing a computationally efficient numerical scheme to handle risk-averse problems which are usually large scale.

Publications and presentations

Work presented in this thesis lead to following publications and presentations

- D. Herceg, P. Sopasakis, and P. Patrinos. "Risk-averse model predictive control.". 36 th Benelux Meeting on Systems and Control (2017)

- Sopasakis, Pantelis, Domagoj Herceg, Panagiotis Patrinos, and Alberto Bemporad. "Stochastic economic model predictive control for Markovian switching systems." IFAC-PapersOnLine 50, no. 1 (2017): 524-530.

- D. Herceg, P. Sopasakis, A. Bempoard and P. Patrinos. "Risk-averse risk-constrained optimal control." .Book of Abstracts - 37th Benelux Meeting on Systems and Control (2018)

- Sopasakis, Pantelis, Domagoj Herceg, Alberto Bemporad, and Panagiotis Patrinos. "Risk-averse model predictive control." Automatica 100 (2019): 281-288.

Work in progress

- Proximal methods for risk-averse problems, D. Herceg,P. Sopasakis, A. Bemporad, and P. Patrinos.

Other publications and reports

I have decided to omit the following publications from the thesis, but they were developed during my Phd stay at IMT Lucca.

- Herceg, Domagoj, George Georgoulas, Pantelis Sopasakis, Miguel Castaño, Panagiotis Patrinos, Alberto Bemporad, Jan Niemi, and George Nikolakopoulos. "Data-driven modelling, learning and stochastic predictive control for the steel industry." In 2017 25th Mediterranean Conference on Control and Automation (MED), pp. 1361-1366. IEEE, 2017.

- Herceg, Domagoj, Sotiris Ntouskas, Pantelis Sopasakis, Aris Dokoumetzidis, Panos Macheras, Haralambos Sarimveis, and Panagiotis Patrinos. "Modeling and administration scheduling of fractional-order pharmacokinetic systems." IFAC-PapersOnLine 50, no. 1 (2017): 9742-9747.

1.3 Structure

Chapter 2 - Background

This chapter servers as an introduction into the main concepts upon which we expand in the reset of the thesis. The reader is acquainted with main theoretical properties of model predictive control and mathematical tools needed for stability analysis. Secondly, risk measures are given some introductory results so the reader can better understand results presented in chapter 4. Finally, basic notions of proximal operators and algorithms are given which we use in Chapter 5.

Chapter 3 - Stochastic economic model predictive control

This chapter is concerned with extending the theory stochastic economic model predictive control. We introduce extend the notion of dissipativity, the central notion in economic MPC, to *stochastic* dissipativity. Form there we prescribe how to recover other properties or economic MPC it the stochastic setting. Moreover, we prescribe a linearization procedure for the design of non-linear stochastic economic MPC.

Chapter 4 - Risk-averse model predictive control

Stochastic control algorithms usually come in two flavors with respect to the underlying uncertainty. Designers have to decide to blindly trust the underlying uncertainty and take it at face value. Or to disregard most of the information contained in a distribution and focus on the worst case outcome. Clearly both approaches suffer from complementary issues. In this chapter, we explore novel theoretical ideas to More precisely we present, stability theory for a class of Markovian systems where we assume that the state of the system is measurable at each step. This allows for a less conservative theoretical developments than the opposite case where the state is not measures. Presented theory is valid for non-convex systems as well. We also describe a linearization procedure for the non-linear system around origin and show that locally, the non-linear system inherits all of the theoretical properties of the risk-averse MPC. Moreover, we present an algorithmic scheme on how to untangle nested risk measures in the objective function to arrive at a convex optimization problem. At the end, simulations are provided to highlight the advantages of such formulation.

Chapter 5 - Proximal methods for scenario based risk-averse problems

The crucial link missing are the algorithms that can tackle these problems with computational efficiency. In this chapter we present a parallelizable algorithmic scheme based on recently introduced SuperMann algorithm. Simulation results are provided to test the feasibility of the proposed approach.

Chapter 6 - Conclusions and future research outlook

This chapter summarizes research done in this thesis and offers outlook for future research.

Chapter 2

Background

2.1 Convexity

Here we introduce basic concept of convex theory. Convexity plays a major role in our exposition, because of its unique mathematical properties and well developed tools for solving such problems. We will mostly follow the notation of standard textbooks on this subject (BV04; Ber09; Roc70).

2.1.1 Convex sets

A set $C \subseteq \mathbb{R}^n$ is a convex set if for every pair $x \in C$,$y \in C$ it holds that

$$\alpha x + (1-\alpha)y \in C, \quad \forall \alpha \in [0,1]. \tag{2.1}$$

A hyperplane is set of the form

$$C := \{x \in \mathbb{R}^n \,|a^\top x \leq b\}, \tag{2.2}$$

for $a \in \mathbb{R}^n$ ($a \neq 0$) and $b \in \mathbb{R}$. Another important example of a convex set is a *polyhedron*. A set is called a polyhedron if it can be defined by a set of inequalities and equalities

$$C := \{x \in \mathbb{R}^n \,|\, Ax \leq b,\ Ex = d\} \tag{2.3}$$

for some matrices $A \in \mathbb{R}^{m \times n}, E \in \mathbb{R}^{p \times n}$ and vectors $b \in \mathbb{R}^m, d \in \mathbb{R}^p$.

Next, we introduce an important concept of a *cone*. We say that set $\mathcal{K} \in \mathbb{R}^n$ is a cone if and only if it is closed under nonnegative scalar multiplication, i.e. $x \in \mathcal{K} \implies \alpha x \in \mathcal{K}$ for all $\alpha \in \mathbb{R}^+$. If it also holds that $x, y \in \mathcal{K} \implies x + y \in \mathcal{K}$ than we say that cone $\mathcal{K}$ is *convex*. Furthermore, cone $\mathcal{K}$ is *pointed* if and only if $\mathcal{K} \cap -\mathcal{K} = \{0\}$. Usually we require a cone to be solid, i.e. that its interior is non empty. Here, however, we will focus on the *relative* interior of a set. By relative interior of a set we understand the interior with respect to its affine hull. The relative interior of a cone is more useful in our exposition because, as we will see later, we will deal with a subspace of a probability simplex which has a lower dimension than the original space. In the rest of the thesis we shall assume that all the cones are convex, closed and pointed. In this light, we define conic inequalities. For a cone $\mathcal{K}$ we can define an inequality

$$x \succeq_{\mathcal{K}} y$$

which is to be understood as $x - y \in \mathcal{K}$. Strict conic inequality $x \succ_{\mathcal{K}} y$ is to be understood as $x - y \in \mathbf{ri}(\mathcal{K})$. Next we list some important cones and their duals. The *nonnegative orthant* is defined by

$$\mathcal{K}_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}. \tag{2.4}$$

Second-order cone or the *Lorentz* cone is defined as

$$\mathcal{K}_{\mathrm{s}} = \{(x, t), x \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq t\}. \tag{2.5}$$

Both of these cones are self-dual. Another interesting cone which is not self dual is the *exponential cone*. Consider the set $\mathcal{K}_{\mathrm{e}}$ in $\mathbb{R}^3$

$$\mathcal{K}_{\mathrm{e}} = \left\{(x, y, z) \in \mathbb{R}^3 \mid y e^{\frac{x}{y}} \leq z, y > 0\right\} \tag{2.6}$$

The closure of $\mathcal{K}_{\mathrm{e}}$ is the exponential cone $\mathcal{K}_{\mathrm{exp}} = \mathbf{cl}\,\mathcal{K}_{\mathrm{e}}$. The exponential cone is not self-dual and its dual is given by

$$\mathcal{K}_{\mathrm{exp}}^* = \left\{(u, v, w) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} -u\,\mathbf{ln}(-\frac{-u}{w}) + u \leq w \\ u \leq 0,\ w \geq 0 \end{array} \right\}. \tag{2.7}$$

Note that for computational purposes we shall work with cones can be easily projected on. In general, we shall work with a product of cones $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n$. The dual of this product is the product of the individual duals, i.e. $\mathcal{K}^* = \mathcal{K}_1^* \times \mathcal{K}_2^* \times \cdots \times \mathcal{K}_n^*$. As a short hand notation we shall use $\mathcal{K}^n = \mathcal{K} \times \cdots \times \mathcal{K}$ for the product of n cones of the same type.

2.1.2 Convex functions

Let $C \in \mathbb{R}^n$ be a convex set. We say that a function $f : C \to \mathbb{R}$ is *convex* if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad \forall x, y \in C, \forall \alpha \in [0,1]. \tag{2.8}$$

A function is said to be *strictly convex* if the above inequality holds with strict inequality ($<$) for $\alpha \in (0,1)$ and $x \neq y$.

For the remainder of this thesis, we shall be working with the extended real-valued function which can take value in the $[-\infty, \infty]$ domain at some points. Let us introduce the shorthand notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. These functions are naturally characterized by the notion of the epigraph which we now introduce. The epigraph of a function $f : C \to \overline{\mathbb{R}}$, where $C \subset \mathbb{R}^n$, is defined to be a subset of $\mathbb{R}^{n+1}$ given by

$$\mathbf{epi}(f) = \{(x, w) \,|\, x \in C,\ w \in \mathbb{R},\ f(x) \leq w\}. \tag{2.9}$$

The effective domain of f is the set

$$\mathbf{dom}(f) = \{x \in C \,|\, f(x) < \infty\}. \tag{2.10}$$

Furthermore, a function that has at least one point such that $f(x) < \infty$ and $f(x) > -\infty$ is called *proper*. There is an important connection between convex functions and convex sets which is stated below.

Let C be a convex subset of $\mathbb{R}^n$. We say that an extended real-valued function $f : C \to \overline{\mathbb{R}}$ is convex if $\mathbf{epi}(f)$ is a convex subset of $\mathbb{R}^{n+1}$. A proper function f is called *lower semicontonous* at a vector $x \in C$ if

$$f(x) \leq \liminf_{k\to\infty} f(x_k) \tag{2.11}$$

for every sequence $\{x_k\} \subset C$ with $x_k \to x$. We say that a function is *lower semicontinous* if it is lower semincontinous at each point $x \in C$. A function $f : C \to \overline{\mathbb{R}}$ is *closed* if its epigraph $\mathbf{epi}(f)$ is a closed set. The following proposition binds these important notion together

Proposition 1 (Proposition 1.1.2,(Ber09)) *For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ the following are equivalent:*
(i) The level set $\{x \mid f(x) \leq \gamma\}$ is closed for every scalar γ.
(ii) f is lower semincontinous.
(iii) $\mathbf{epi}(f)$ is closed.

We shall impose the assumption that functions with which we work in this thesis are *proper* and *lower semincontinous* possibly without explicitly stating it. Moreover, usually we shall assume convexity as well, but we will make this explicit.

A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called β-Lipschitz continuous, with $\beta \geq 0$ if

$$\|F(x_1) - F(x_2)\| \leq \beta\|x_1 - x_2\|. \tag{2.12}$$

Next, we shall introduce notions of a subgradient and a subdifferential. A vector $g \in \mathbb{R}^n$ is a subgradient of f at $x \in \mathbf{dom}\, f$ if

$$f(y) \geq f(x) + \langle g, y - x\rangle \quad \forall y \in \mathbf{dom}\, f. \tag{2.13}$$

The *subdifferential* of f at $x \in \mathbf{dom}\, f$ is the set of all subgradients g of f at $x \in \mathbf{dom}\, f$ and denoted by $\partial f(x)$

$$\partial f(y) \geq f(x) + \langle g, y - x\rangle \quad \forall y \in \mathbf{dom}\, f. \tag{2.14}$$

A conjugate function of closed, convex, proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ if the function $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{x^\top y - f(x)\}, \quad y \in \mathbb{R}^n. \tag{2.15}$$

Conjugate function is lower semicontinous and convex even if f is not convex. We can also take the conjugate of the conjugate function and in case that f is closed, proper and convex we have that $f^{**} = f$.

2.2 Elements of monotone operator theory

In this section we introduce some basic notions of monotone operator theory. Properties introduced here are relevant for algorithmic solution of risk-averse problems in Chapter 5. Definitions here are taken form (BC11), which is a standard text on this subject. For an easy to read survery, we point the reader to (RB16).

Fixed set of an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ a sets

$$\mathbf{fix}\, T = \{x \in \mathbb{R}^n \mid x = Tx\}. \tag{2.16}$$

Let us denote the identity operator with $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$. Many optimization problems can be formulated as problems of finding a fixed point of the associated operator T. Let $C \subset \mathbb{R}^n$. We say that an operator $T : C \to \mathbb{R}^n$ is *non-expansive* if it is Lipschitz continuous with constant one, i.e.

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C. \tag{2.17}$$

If it if Lipschitz continous with constant $\beta \in [0, 1)$ then it is *contractive*. We say that an operator T is *firmly non-expansive* if

$$\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2, \forall x, y \in C. \tag{2.18}$$

An operator $T : C \to \mathbb{R}^n$ is *averaged* with constant $\alpha \in (0, 1)$ if there exist a nonexpansive operator $R : C \to \mathbb{R}^n$ such that

$$T = (1 - \alpha)\mathrm{Id} + \alpha R. \tag{2.19}$$

The importance of these properties is highlighted by the following theorems.

Theorem 2.2.1 (Krasnoselskii) *Let T be an averaged operator with a fixed point. Then, the iteration*

$$x_{k+1} = Tx_k \tag{2.20}$$

converges weakly to a fixed point of T.

Theorem 2.2.2 (Krasnoselskii-Mann) *Let T be a nonexpansive operator with a fixed point. Then, the iteration*

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k T x_k \tag{2.21}$$

converges weakly to a fixed point of T as long

$$\lambda_k > 0, \quad \sum_k \lambda_k(1 - \lambda_k) = \infty.$$

2.2.1 Proximal operator

Given a proper, convex and lower-semicontinuous function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and a positive scalar μ, its proximal operator $\mathbf{prox}_{\mu f} : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\mathbf{prox}_{\mu f}(v) := \underset{x \in \mathbb{R}^n}{\mathbf{argmin}}\{f(x) + \tfrac{1}{2\mu}\|x - v\|^2\}. \tag{2.22}$$

Because of the quadratic term inside the above expression, proximal operator is a uniquely defined mapping. Let us introduce and function and an operator which are heavily used later on. The *indicator function* of a set $C \subseteq \mathbb{R}^n$ is the extended-real valued function $\delta_C : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined as

$$\delta_C = \begin{cases} 0, & x \in C \\ \infty, & x \notin C. \end{cases} \tag{2.23}$$

Given a non-empty, closed and convex set $C \in \mathbb{R}^n$ we define a *projection* operator

$$\Pi_C(x) := \underset{y \in C}{\mathbf{argmin}}\,\|y - x\|^2. \tag{2.24}$$

Euclidean distance of $x \in \mathbb{R}^n$ from C is given as $d(x, C) = \mathbf{min}_{y \in C}\|y - x\|^2$. When function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinous $\mathbf{prox}_f$ is *firmly nonexpansive*.

Proximal mapping, can be seen as a generalized projection operator. Indeed, when function f is an indicator function of a convex set C its proximal operator is a projection.

$$\mathbf{prox}_{\delta_C(v)} = \underset{x \in C}{\mathbf{argmin}}\,\|x - v\|_2^2 = \mathbf{\Pi}_C(v) \tag{2.25}$$

In case of C being the positive orthant, i.e. $C = \{x \in \mathbb{R}^n \mid x \geq 0\}$,

$$\mathbf{\Pi}_C(x) = \mathbf{max}(0, x) = [x]_+$$

where $\mathbf{max}$ operator is understood to be elementwise.

Let C be a hyperplane $C = \{x \mid a^\top x = b\}$, the projection is

$$\mathbf{\Pi}_C(x) = x - \frac{a^\top x - b}{\|a\|^2} a.$$

If C is a halfspace, $C = \{x \mid a^\top x \leq b\}$, then

$$\mathbf{\Pi}_C(x) = x - \frac{[a^\top x - b]_+}{\|a\|^2} a$$

Another important projection is the projection onto the affine set $C = \{x \in \mathbb{R}^n \mid Ax = b\}$ for $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Projection onto C is given as

$$\mathbf{\Pi}_C(x) = \{x - A^\top (AA^\top)^{-1}(Ax - b)\}. \tag{2.26}$$

Note that $A^\top (AA^\top)^{-1}$ is the pseudo inverse of A. This is particularity useful for dynamical systems where the system dynamics can be written as the above set.

Another very useful set is the second order cone $\mathcal{K}_s$. The projection of onto second-order cone $\mathcal{K}_s$ for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ is given by the explicit formula

$$\mathbf{\Pi}_{\mathcal{K}_s}(x, t) = \begin{cases} (x, t), & \text{if} \quad \|x\|_2 \leq t \\ (0, 0), & \text{if} \quad \|x\|_2 \leq -t \\ \frac{\|x\|_2 + t}{2}\left(\frac{x}{\|x\|_2}, 1\right) & \text{otherwise} \end{cases} \tag{2.27}$$

This formula will prove crucial later on when we solve risk-averse problems with quadratic cost. Note that we will usually write $x \in \mathbb{R}^{n+1} \in \mathcal{K}_s$ to mean $((x_i)_{i=1}^n, x_n) \in \mathcal{K}_s$.

An important property of proximal operators it the so called *separable sum property*. Let $f(x, y) = g(x) + h(y)$ then

$$\mathbf{prox}_f = (\mathbf{prox}_g(x), \mathbf{prox}_h(y)) \tag{2.28}$$

Another important property is the *Extended Moreau identity* which links the proximal operator of a function f to its conjugate function f^*. This is stated as

$$v = \mathbf{prox}_{\mu f}(v) + \mu\, \mathbf{prox}_{\mu^{-1} f^*}(\mu^{-1} v) \tag{2.29}$$

This properties allows us to easily compute proximal mapping of a conjugate function if we know the proximal mapping of the original one and *vice versa*.

2.3 Measuring risk

Let $\Omega = \{\omega_i\}_{i=1}^n$ be a finite sample space equipped with the discrete σ-algebra 2^Ω and a probability measure P with $\mathrm{P}(\{\omega_i\}) = \pi_i$. Hereafter, we will assume that $\pi_i > 0$. The pair (Ω, P) is called a *probability space*. A vector $p \in \mathbb{R}^n$ is called a *probability vector* if $p_i \geq 0$ for all $i \in \mathbb{N}_{[1,n]}$ and $\sum_{i=1}^n p_i = 1$. The set of all probability vectors in $\mathbb{R}^n$ is called the *probability simplex* and is denoted by $\mathcal{D}_n$. A real-valued *random variable* over (Ω, P) is a mapping $Z : \Omega \to \mathbb{R}$ with $Z(\omega_i) = Z_i$; this can be identified by the vector $Z = (Z_1, \ldots, Z_n) \in \mathbb{R}^n$.

A *risk measure* is a mapping $\rho : \mathbb{R}^n \to \mathbb{R}$. A risk measure ρ is called *coherent* if it satisfies the following properties (SDR14a, Sec. 6.3)

A1. *Convexity*. For $Z_1, Z_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$, $\rho(\lambda Z_1 + (1-\lambda) Z_2) \leq \lambda\rho(Z_1) + (1-\lambda)\rho(Z_2)$,

A2. *Monotonicity*. For $Z_1, Z_2 \in \mathbb{R}^n$ with $Z_1 \leq Z_2$, $\rho(Z_1) \leq \rho(Z_2)$,

A3. *Translation equivariance*. For $a \in \mathbb{R}$ and $Z \in \mathbb{R}^n$, $\rho(a+Z) = a + \rho(Z)$,

A4. *Positive homogeneity*. For $\alpha \geq 0$ and $Z \in \mathbb{R}^n$, $\rho(\alpha Z) = \alpha\rho(Z)$.

Trivially, the *expectation* operator $\mathbb{E}(Z) := \sum_{i=1}^n \pi_i Z_i$ is a coherent risk measure and so is the *essential maximum* $\mathbf{essmax}(Z) := \mathbf{max}\{Z_i; \pi_i > 0\}$.

A popular risk measure is the *average value-at-risk*, often called *expected shortfall*, which is defined as

$$\text{AV@R}_\alpha(Z) = \begin{cases} \mathbf{min}_{t\in\mathbb{R}}\{t + \alpha^{-1}\mathbb{E}\left[Z-t\right]_+\}, & \alpha \in (0,1], \\ \mathbf{max}\{Z_i; \pi_i > 0\}, & \alpha = 0. \end{cases}$$

$\text{AV@R}_\alpha[Z]$ is the expected value of Z above its $1-\alpha$-quantile $Q_{1-\alpha}$, that is $\text{AV@R}_\alpha[Z] = \mathbb{E}[Z \mid Z \geq Q_{1-\alpha}]$.

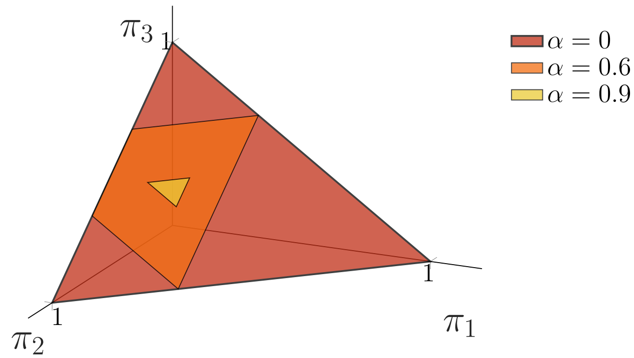

Figure 1: The ambiguity set $\mathcal{C}_\alpha$ of AV@R_α for different values of α on a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $\pi_1 = 0.2$, $\pi_2 = 0.3$ and $\pi_3 = 0.5$. $\mathcal{C}_0$ is the whole probability simplex in $\mathbb{R}^3$.

An important duality result is that all coherent risk measures can be written as (SDR14a)

$$\rho(Z) = \max_{\zeta\in\mathcal{A}} \langle \zeta, Z\rangle_{\text{P}} = \max_{\zeta\in\mathcal{A}} \sum_{i=1}^{n} \pi_i \zeta_i Z_i, \tag{2.30}$$

where $\mathcal{A}$ is a compact convex set (because ρ is convex and positively homogeneous) called the *ambiguity set* of ρ whose elements satisfy the properties $\mathbb{E}[\,\zeta\,] = 1$ (because of A3) and $\zeta_i \geq 0$ (because of A2). Being the support function of a compact convex set, ρ is a continuous mapping.

We can also define a probability distribution μ with $\mu_i = \zeta_i \pi_i$ for every $\zeta \in \mathcal{A}$. By doing so we have defined another compact convex set of probability distributions $\mathcal{A}$ and we can state (2.30) as

$$\rho(Z) = \max_{\mu \in \mathcal{C}} \sum_{i=1}^{n} \mu_i Z_i = \max_{\mu \in \mathcal{C}} \mathbb{E}_{\mu}[Z]. \tag{2.31}$$

In this way we can think of a coherent risk measure as the worst case expectation with respect to a probability distribution taken from a set of probability vectors $\mathcal{C} = \{\mu \mid \mu_i = \pi_i \zeta_i, \forall \zeta \in \mathcal{A}\}$. From now on, we shall refer to both sets $\mathcal{C}$ and $\mathcal{A}$ as ambiguity sets.

A risk measure is called *polytopic* if its ambiguity set is a polytope, i.e., it can be written as the convex hull of a finitely many elements, see (ER05)), that is $\mathcal{C} = \mathbf{conv}\{\mu^{(l)}\}_{l=1}^{\kappa}$. Then

$$\rho(Z) = \max_{l \in \mathbb{N}_{[1,\kappa]}} \sum_{j=1}^{n} \mu_j^{(l)} Z_j = \max_{l \in \mathbb{N}_{[1,\kappa]}} \mathbb{E}_{\mu^{(l)}}[Z]. \tag{2.32}$$

Examples of polytopic risk measures involve the aforementioned average value-at-risk, whose ambiguity set is

$$\mathcal{C}_{\alpha} = \left\{ \mu \in \mathbb{R}^n \middle| \sum_{i=1}^{n} \mu_i = 1, 0 \leq \mu_i \leq \frac{\pi_i}{\alpha} \right\}. \tag{2.33}$$

Note that for $\alpha = 1$, $\mathcal{C}_1 = \{\pi\}$ and $\mathrm{AV@R}_1[Z] = \mathbb{E}[Z]$. The maximal ambiguity set is attained for $\alpha = 0$ and it is $\mathcal{C}_0 = \{\mu \in \mathbb{R}^n \mid \sum_{i=1}^{n} \mu_i = 1, \mu_i \geq 0\}$. As illustrated in Fig. 1, $\mathrm{AV@R}_{\alpha}$ interpolates between the risk-neutral expectation operator ($\mathrm{AV@R}_1 = \mathbb{E}$) and the worst-case essential maximum ($\mathrm{AV@R}_0 = \mathbf{essmax}$). Another important property of $\mathrm{AV@R}_{\alpha}$ is that its relation to VAR_{α} which is essentialy $Q_{1-\alpha}$. As we will briefly discuss later in the thesis VAR_{α} can be used to model chance constraints, but this non-coherent risk measure is hard to deal with numerically. It is well know that $\mathrm{AV@R}_{\alpha} Z \geq VAR_{\alpha}[Z]$ and it its tightest convex bound.

Some other polytopic risk measures are the *mean-upper semideviation* $\rho[Z] = \mathbb{E}[Z] + c\mathbb{E}[Z - \mathbb{E}[Z]]_+$ with $c \in [0, 1]$ and, of course, the expectation and the essential maximum.

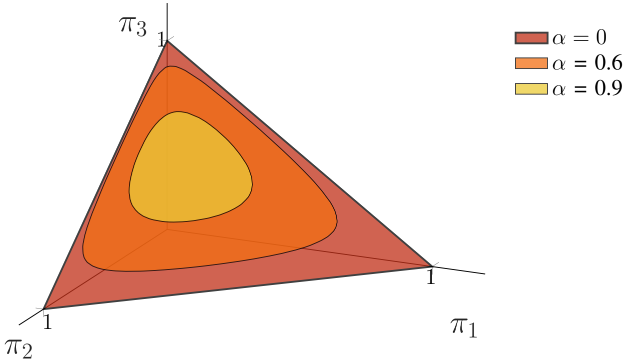

Figure 2: The ambiguity set $\mathcal{C}_\alpha$ of non-polytopic risk measure EV@R_α for different values of α on a probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $\pi_1 = 0.2$, $\pi_2 = 0.3$ and $\pi_3 = 0.5$. $\mathcal{C}_0$ is the whole probability simplex in $\mathbb{R}^3$.

Another interesting, albeit non-polytopic, risk measure is the entropic-value-at-risk (EV@R_α) (AJ12), whose ambiguity set is given by

$$\mathcal{A} = \{D_{KL}(\mu \| \pi) \leq -\ln(1-\alpha)\}. \tag{2.34}$$

Here $D_{KL}(\mu \| \pi)$ is the Kullback-Liebler divergence from μ to π, and in the case of finite discrete distributions is given by

$$D_{KL}(\mu \| \pi) = -\sum \pi_i \ln\left(\frac{\mu_i}{\pi_i}\right). \tag{2.35}$$

2.4 Elements of control theory

Firs, we will explain a notion of stability that we use in model predictive control.

2.4.1 Lyapunov stability

Stability in the control loop probably a central topic in any controller design. The most salient notion in stability is the one of Lyapunov stability. A more elaborate discussion than presented here can be found in standard textbooks (Kha02; Sas99; Son98; SL91; Las76).

Mostly we will consider discrete systems described by difference equations

$$x_{k+1} = f(x_k, u_k) \tag{2.36}$$

where $x_k \in \mathbb{R}^n$ is the system state and $u_k \in \mathbb{R}^m$ is the control input. As a special case, we consider linear systems

$$f(x_k, u_k) = Ax_k + Bu_k, \tag{2.37}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. If we define a state-feedback control law with $\kappa(x_k)$ we can re-write the system above as $x_{k+1} = f(x_k, \kappa(x_k))$. With $\phi(k, x, \mathbf{u})$ we denote the trajectory of the system at time k with initial state $x_0 = x$ and control inputs $\mathbf{u} = \{u_0, u_1, \ldots, u_{n-1}\}$

For the sake of simplicity we shall describe a system given by

$$x_{k+1} = f(x_k). \tag{2.38}$$

A point x_e is called an equilibrium point of system if (2.38) it holds that $f(x_e) = x_e$. Without loss of generality we shall assume that equilibrium state is at the origin.

Definition 1 (Positive invariant set) *Set $\mathcal{X} \in \mathbb{R}^n$ is positively invariant for the system* (2.38) *if $x \in \mathcal{X}$ implies $f(x) \in \mathcal{X}$.*

Definition (1) implies that once system enters positively invariant set it remains there.

We are ready to state stability theorems, but first we shall introduce useful class of functions. Function $h : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, $h(0) = 0$ and strictly increasing. Function $h : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function and $h(s) \to \infty$ when $s \to \infty$, i.e. it is unbounded.

Definition 2 (Stability in the sense of Lyapunov) *Equilibrium point $x_e = 0$ is stable in the sense of Lyapunov if for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $\|x(k) - x_e\| \leq \epsilon$ for all $k \geq 0$ whenever $\|x(0) - x_e\| \leq \delta_\epsilon$.*

Definition 3 (Asymptotic stability) *An equilibrium point is said to be asymptotically stable if it is stable in the sense of Lyapunov and there exists $\delta > 0$ such* *that whenever $\|x(0) - x_e\| \leq \delta$ we have that $x(k) \to x_e$ as k increases for every $x(0) \in \mathcal{X}$. It is globally asymptotically stable if it is asymptotically stable and $x(k) \to x_e$ as k increases for any $x(0) \in \mathcal{X}$.*

Stronger notion of stability is called exponential stability.

Definition 4 (Exponential stability) *An equilibrium point of* (2.38) *is said to be exponentially stable in $\mathcal{X}$ if there exist constants $\alpha > 0$ and $\gamma \in (0, 1)$ such that*

$$\|x_k - x_e\| \leq \alpha\gamma^k \|x_0 - x_e\|, \tag{2.39}$$

for all $x_0 \in \mathcal{X}$ and all $k \geq 0$.

Finally, we can introduce Lyapunov function.

Definition 5 (Lyapunov function) *A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a Lyapunov function if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive definite function α_3 such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{2.40a}$$

$$V(f(x)) - V(x) \leq -\alpha_3(\|x\|) \tag{2.40b}$$

2.4.2 Model predictive control

Model predictive control is an advanced control strategy that is widely applied in industrial control systems (QB03; FPHG15) mostly due to the fact that it can handle constraints naturally. Moreover, each control action is decided based on the minimizing the objective which may include conflicting criteria.

MPC has stared as a practical approach to handle process dynamics. It was later on that a rich theory was developed. More on general MPC can be found in textbooks (Mac02; GSDD06; BC07) and articles (BM99a; MRRS00). Note that MPC is still an active research area (May14).

Nominal MPC

We define $\ell_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ to be a penalty function at each stage k for all $k \in \mathbb{N}_{[0,N-1]}$. A special and very common case is to choose ℓ to be quadratic, i.e.

$$\ell_k(x_k, u_k) = x_k^\top Q x_k + u_k^\top R u_k$$

, with $Q \in \mathbf{S}_{+}^{n\times n}, R \in \mathbf{S}_{++}^{m\times m}$. Terminal state penalty function is denoted with $\ell_N : \mathbb{R}^n \to \mathbb{R}$, and again we will usually choose $\ell_N(x_N) = x^\top Px$. Additionally, we require each state state-control pair (x_k, u_k) to be inside a (usually convex) set Y_k. For a terminal state x_N we require it to be inside a set denoted with X^f. Terminal set and penalty function usually play an important role in stability analysis of MPC scheme and needs to be selected carefully (MRRS00). However, there also exist approaches which can avoid using the terminal penalty (LASC06; BGW14; Ala17). Finally we form the objective to be optimized at each stage as

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell_k(x_k, u_k) + \ell_N(x_N) \tag{2.41}$$

where $\mathbf{u} = \{u_0, u_1, \ldots, x_{N-1}\}$. States x_k for $k > 0$ can be reconstructed form $\mathbf{u}$ and (2.36). We state the MPC optimization problem $\mathbb{P}$ below.
MPC problem

$$\mathbb{P} : V_N^\star(x) = \min_{x,u} V_N(x, u) \tag{2.42a}$$

$$= \min_{x,u} \sum_{i=0}^{N} \ell(x_k, u_k) + \ell_N(x_N) \tag{2.42b}$$

$$x_{k+1} = Ax_k + Bu_k, \tag{2.42c}$$

$$(x_k, u_k) \in Y_k \tag{2.42d}$$

$$x_N \in X^f \tag{2.42e}$$

$$x_0 = x \tag{2.42f}$$

for $k = 0, \ldots, N-1$.

An important property we require is that model predictive control law is *recursively feasible*.

Definition 6 *The MPC problem is recursively feasible, if for all feasible initial states feasibility is guaranteed at every state along the closed-loop trajectory.*

Of course, here we assume that we have no external disturbances which may drive the system away from the nominal trajectory. It is also important to keep in mind that recursive feasibility does not imply stability.

Chapter 3

Stochastic Economic Model Predictive Control

3.1 Introduction

3.1.1 Background and motivation

Recently, a new approach to model predictive control (MPC) termed *economic model predictive control* (EMPC) has gained a lot of attention. Rather than minimizing a deviation from a prescribed (optimal/best) set-point or a tracking reference, the main objective in EMPC is to optimize a given economic cost functional (AAR12). Often, in engineering practice, the main objective is to devise control algorithms which asymptotically guarantee an economic operation of the controlled plant.

Already, a considerable body of theoretical results has been reported in the literature characterizing the asymptotic performance of EMPC. Perhaps *dissipativity* is the most salient notion in the pertinent literature which is shown to be a sufficient condition for proving optimal operation at a steady state and stability of EMPC formulations (AAR12). The same authors show that economic MPC has no worse an asymptotic average performance than the best admissible steady state operation (MAA13).

The introduction of a, possibly non-quadratic and nonconvex, economic cost into the MPC framework disqualifies the standard stability analysis used in the MPC literature. (AAR12) propose the use of a simple terminal constraint to guarantee stability of EMPC-controlled systems which is generalized by (ARA11) using terminal set constraints. (FT13) use a generalized terminal state equality constraint where the target terminal state is left as a free variable to be optimized which increases the feasibility region of EMPC. This concept was further generalized to include a terminal region constraint (MAA14). It was further shown that EMPC can achieve near-optimal operation without terminal constraints and costs for a sufficiently large prediction horizon (Grü13). Similar results exist for a system that is best operated at a periodic regime (ZGD13). It is worth noting that this wealth of results concerns only deterministic systems.

In spite of the noticeable interest for the idea of EMPC there are very few theoretical results accounting for uncertainty, which is often relevant in a real-world operation. (BJ14) propose a scenario-based EMPC formulation for fault-tolerant constrained regulation and a similar approach is pursued by (LAB^+14a). (LAB^+14b) present a multi-stage scenario-based nonlinear MPC control strategy validated on a benchmark example, but no performance guarantees or stability analysis is provided. An interesting theoretical treatment is given by (BMA14) where a tube-based EMPC formulation is proposed for constrained systems with bounded additive disturbances. Very recently (BLMA16) proposed a robust economic MPC formulation for linear systems with bounded additive uncertainty with known probability distribution.

3.1.2 Contributions

In this chapter we endeavor to cover the theoretical gap in EMPC for an important class of stochastic systems — the Markovian switching systems. We first study the properties of an MPC formulation for Markovian switching systems where optimal steady states are mode-dependent. We propose an MPC scheme which is recursively feasible and satisfies an asymptotic performance bound. Assuming that there is a common optimal steady state, we show that the MPC-controlled system is mean-square (MS) stable when a stochastic dissipativity condition is satisfied. We then formulate a variant of the MPC problem using mode-dependent terminal constraints and provide mean-square stability conditions and performance bounds. We then provide guidelines for the design of mean-square stabilizing predictive controllers for nonlinear systems imposing weak conditions on the system dynamics and the EMPC stage cost.

3.1.3 Notation and mathematical preliminaries

Let $\mathbb{R}$ and $\mathbb{R}_+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ denote the sets of real numbers, nonnegative reals, n-dimensional real vectors and n-by-m matrices. Let $\mathcal{B}_\delta$ be the ball of $\mathbb{R}$ of radius δ, that is $\mathcal{B}_\delta := \{x : \|x\| < \delta\}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *lower semicontinuous* if its epigraph, that is the set $\mathbf{epi}\, f = \{(x, \alpha) \in \mathbb{R}^{n+1} : f(x) \leq \alpha\}$, is closed. We say that $f : \mathbb{R}^n \to \mathbb{R}$ is *level-bounded* if its level sets, $\mathbf{lev}_\alpha f = \{x : f(x) \leq \alpha\}$, are bounded. We say that $f : \mathbb{R}^n \times \mathbb{R}^m \ni (x, u) \mapsto f(x, u) \in \mathbb{R}$ is *level-bounded in u locally uniformly in x* if for every $\bar{x}$ there is a neighborhood of $\bar{x}$, $V_{\bar{x}} \subseteq \mathbb{R}^n$, so that $\{(x, u) : x \in V_{\bar{x}}, f(x, u) \leq \alpha\}$ is bounded. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *β-smooth* if it is differentiable with β-Lipschitz gradient, that is $\|\nabla f(y) - \nabla f(x)\| \leq \beta\|y - x\|$ for all $x, y \in \mathbb{R}^n$; then, we have that $\|f(y) - f(x) - \nabla f(x)(y - x)\| \leq \frac{\beta}{2}\|y - x\|^2$. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is positive definite around x_0 if $f(x_0) = 0$ and $f(x) > 0$ for $x \neq x_0$. $A \succcurlyeq 0$ denotes that A is a positive semidefinite matrix and $A \succ 0$ means that A is positive definite. We denote the transpose of a matrix A by $A^\top$.

3.2 Stochastic Economic Model Predictive Control

3.2.1 System dynamics

Consider the following Markovian switching system

$$x_{k+1} = f(x_k, u_k, \theta_k), \tag{3.1}$$

driven by the random parameter θ_k which is a time-homogeneous irreducible and aperiodic Markovian process with values in a finite set $\mathcal{N} = \{1, \dots, \nu\}$ with transition matrix $P = (p_{ij}) \in \mathbb{R}^{\nu \times \nu}$ and *initial distribution* $v = (v_1, \dots, v_\nu)$ (CFM05). We assume that at time k we measure the full state x_k and the value of θ_k. Markov jump linear systems (MJLS) with additive disturbances are a special case of system (3.1) with

$$f(x, u, \theta) = A_\theta x + B_\theta u + w_\theta.$$

Let $\Omega := \prod_{k \in \mathbb{N}} (\mathbb{R}^n \times \mathbb{R}^m \times \mathcal{N})$ and $\mathcal{F}_k$ be the minimal σ-algebra over the Borel-measurable rectangles of Ω with k-dimensional base and $\mathcal{F}$ be the minimal σ-algebra over all Borel-measurable rectangles. Define the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k \in \mathbb{N}}, \mathrm{P})$ where P is the unique product probability measure according to (Ash72, Th. 2.7.2) with

$$\mathrm{P}(\theta_0 = i_0, \theta_1 = i_1, \dots, \theta_k = i_k) = v_{i_0} p_{i_0 i_1} \cdots p_{i_{k-1} i_k}$$

for any $i_0, i_1, \dots, i_k \in \mathcal{N}$ and $k \in \mathbb{N}$, where θ_k is an $\mathcal{F}_k$-adapted random variable from Ω to $\mathcal{N}$. We will use the notation $u \lhd \mathcal{F}_k$ to denote that the random variable u is $\mathcal{F}_k$-measurable.

Let $\mathbb{E}[\cdot]$ denote the expectation of a random variable with respect to P and $\mathbb{E}[\cdot | \mathcal{F}_k]$ the conditional expectation. It can be shown (TGG10) that the augmented state (x_k, θ_k) contains all the probabilistic information relevant to the evolution of the Markovian switching system for all time instants $t > k$.

Definition 7 (Cover and bet node) *For every node $i \in \mathcal{N}$, the cover of i is the set $\mathcal{C}(i) = \{j \in \mathcal{N} \mid p_{ij} > 0\}$. The bet node of an $i \in \mathcal{N}$ is a node $\mathrm{bet}(i) \in \mathcal{C}(i)$ with $p_{i\mathrm{bet}(i)} \geq p_{ij}$ for all $j \in \mathcal{C}(i)$.*

A bet of a mode $\theta_k = i$ is one of the most likely successor modes θ_{k+1}. System (3.1) is subject to the following joint state-input constraints

$$(x_k, u_k) \in Y_{\theta_k}. \tag{3.2}$$

Let $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{N} \to \mathbb{R}$ be a mode-dependent cost function.

Assumption 3.2.1 (Well-posedness) *For each $\theta \in \mathcal{N}$, $\ell(\cdot,\cdot,\theta)$ are nonnegative, lower semicontinuous and level-bounded in u locally uniformly in x, $f(\cdot,\cdot,\theta)$ are continuous and sets Y_θ are nonempty and compact. The random process $\{\theta_k\}_k$ is an irreducible and aperiodic Markov chain.*

Definition 8 (Optimal steady states) *Given a stage cost function $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}$ which satisfies Assumption 3.2.1, a pair (x_s^θ, u_s^θ) is called an optimal steady state of* (3.1) *subject to* (4.4) *with respect to ℓ if it is a minimizer of the problem*

$$\ell_s(\theta) := \min_{x,u} \{\ell(x,u,\theta) | f(x,u,\theta) = x, (x,u) \in Y_\theta\}$$

For reasons that will be better elucidated in the next section, we need to draw the following controllability assumption essentially requiring that if $x_k = x_s^i$ and $\theta_k = j$ then there is a control action $\bar{u}_s^{i,j}$ so that at time $k+1$ the state is steered to $x_{k+1} = x_s^{\mathrm{bet}(j)}$.

Assumption 3.2.2 (Weak controllability) *In addition to Assumption 3.2.1, for all $i,j \in \mathcal{N}$ there is a control law $\bar{u}_s : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ with $\bar{u}_s(x_s^i, j) = \bar{u}_s^{i,j}$ so that $(x_s^i, \bar{u}_s^{i,j}) \in Y_j$ and $f(x_s^i, \bar{u}_s^{i,j}, j) = x_s^{\mathrm{bet}(j)}$.*

3.2.2 Model predictive control

In this section we shall present a model predictive control framework for constrained Markovian switching systems with mode-dependent optimal steady-state points.

Let $u_k \lhd \mathcal{F}_k$ for $k \in \mathbb{N}_{[0,N-1]}$ and $\mathbf{u}_N = (u_0, \ldots, u_{N-1})$, and define

$$V_N(x_0, \theta_0, \mathbf{u}_N) = \mathbb{E}\left[V_f(x_N, \theta_N) + \sum_{j=0}^{N-1} \ell(x_j, u_j, \theta_j) \middle| \mathcal{F}_0 \right].$$

Here, we take $V_f = 0$ and let the state sequence satisfy (3.1).

We introduce the following stochastic economic model predictive control problem

$$\mathbb{P}(x,\theta) : V_N^\star(x,\theta) = \inf_{\mathbf{u}_N} V_N(x,\theta,\mathbf{u}_N), \tag{3.3a}$$

and for $k = 0, \ldots, N-1$, subject to

$$x_{k+1} = f(x_k, u_k, \theta_k) \tag{3.3b}$$

$$(x_k, u_k) \in Y_{\theta_k} \tag{3.3c}$$

$$(x_0, \theta_0) = (x, \theta) \tag{3.3d}$$

$$x_N = x_s^{\mathbf{bet}(\theta_{N-1})} \tag{3.3e}$$

$$u_k \lhd \mathcal{F}_k. \tag{3.3f}$$

Because of Assumption 3.2.1 and in light of (RW09, Thm. 1.17) the infimum in (3.3) is attainable and the corresponding set of minimizers is compact. Note that in the above formulation the minimization is carried out in a space of control policies $\mathbf{u} = \{u_0, \ldots, u_{N-1}\}$ where u_k are *causal* control laws — as required by (3.3f).

Let $\mathbf{u}^\star(x, \theta) = \{u_0^\star(x, \theta), \ldots, u_{N-1}^\star(x_{N-1}, \theta_{N-1})\}$ be an optimizer of (3.3). The receding horizon control law that accrues from this problem is

$$\kappa_N(x, \theta) := u_0^\star(x, \theta)$$

and the closed-loop system satisfies

$$x_{k+1} = f(x_k, \kappa_N(x_k, \theta_k), \theta_k). \tag{3.4}$$

3.2.3 Recursive feasibility

We will now prove that the MPC problem in (3.3) is recursively feasible.

Proposition 2 *Let $X_N \subseteq \mathbb{R}^n \times \mathcal{N}$ be the domain of problem $\mathbb{P}$. If Assumption 3.2.2 holds and problem $\mathbb{P}(x_k, \theta_k)$ is feasible, then problem $\mathbb{P}(x_{k+1}, \theta_{k+1})$, with $x_{k+1} = f(x_k, \kappa_N(x_k, \theta_k), \theta_k)$ and $\theta_{k+1} \in \mathcal{C}(\theta_k)$, is also feasible.*

Proof 3.2.3 *For given $(x, \theta) \in X_N$ let $\pi(x, \theta) = \{u_0^\star, \ldots, u_{N-1}^\star\}$ be an optimizer of $\mathbb{P}(x, \theta)$ and let $x^\star(x, \theta) = \{x, x_1^\star, \ldots, x_N^\star\}$ be the corresponding sequence of states. Because of* (3.3e) *we have that*

$$x_N^\star = x_s^{\mathrm{bet}(\theta_{N-1})}.$$

Now take $x^+ = f(x, u_0^\star(x, \theta), \theta)$ and $\theta^+ \in \mathcal{C}(\theta)$. We need to show that $\mathbb{P}(x^+, \theta^+)$ is feasible. Take $\tilde{\pi}^+(x^+, \theta^+) := \{u_1^\star, \ldots, u_{N-1}^\star, u\}$ and let $u = \bar{u}_s(x_N, \theta_N)$. Then, by virtue of Assumption 3.2.2, $x_{N+1}^\star = x_s^{\mathrm{bet}(\theta_N)}$, so $\tilde{\pi}^+$ will satisfy the constraints of $\mathbb{P}(x^+, \theta^+)$. □

3.2.4 Performance assessment

We will now prove that the closed-loop system has a bounded expected asymptotic average cost (Thm. 3.2.6). First, we need to give the following result:

Lemma 3.2.4 *Let Assumption 3.2.2 hold and let*

$$\ell_N(\theta_k) := \mathbb{E}\Big[\ell(x_s^{\mathrm{bet}(\theta_{N-1})}, \bar{u}_s^{\mathrm{bet}(\theta_{N-1}),\theta_N}, \theta_N) \mid \theta_0 = \theta\Big]$$

and

$$\mathcal{L}V_N^\star(x_k, \theta_k) := \mathbb{E}[V_N^\star(x_{k+1}, \theta_{k+1}) - V_N^\star(x_k, \theta_k) | \mathcal{F}_k];$$

then, the following holds for all $(x_k, \theta_k) \in X_N$

$$\mathcal{L}V_N^\star(x_k, \theta_k) \leq \ell_N(\theta_k) - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k). \tag{3.5}$$

Proof 3.2.5 *Let* $(x, \theta) \in X_N$*; then* $\tilde{\pi}^+(x_{k+1}, \theta_{k+1})$ *is feasible — but not necessarily optimal — for* $\mathbb{P}(x_{k+1}, \theta_{k+1})$*, therefore,*

$$V_N^\star(x_{k+1}, \theta_{k+1}) \leq V_N(x_{k+1}, \theta_{k+1}, \tilde{\pi}^+(x_{k+1}, \theta_{k+1})).$$

By the tower property of the conditional expectation we know that $\mathbb{E}[\mathbb{E}[\cdot \mid \mathcal{F}_{k+1}] \mid \mathcal{F}_k] = \mathbb{E}[\cdot \mid \mathcal{F}_k]$ *since* $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$*. We then have*

$$\begin{aligned}
\mathcal{L}V_N^\star(x_k, \theta_k) &\leq \mathbb{E}\left[\sum_{j=k+1}^{k+N-1} \ell(x_j, u_{j-k}^\star, \theta_j) + \ell(x_{k+N}, \bar{u}_s, \theta) - \right.\\
&\qquad \left. - \sum_{j=k}^{k+N-1} \ell(x_j, u_j, \theta_j) \mid \mathcal{F}_k\right]\\
&= \mathbb{E}\left[\ell(x_s^{\mathrm{bet}(\theta_{k+N-1})}, \bar{u}_s^{\mathrm{bet}(\theta_{k+N-1}),\theta_{k+N}}, \theta_{k+N})\right.\\
&\qquad \left. - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k) \mid \mathcal{F}_k\right]\\
&= \ell_N(\theta_k) - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k),
\end{aligned}$$

where $u_{j-k}^\star = u_{j-k}^\star(x_j, \theta_j)$ *and this completes the proof.* □

The irreducibility and aperiodicity assumptions (Assumption 3.2.1) imply the existence of a limiting probability vector $\pi = (\pi^1, \ldots, \pi^\nu) \in \mathbb{R}^\nu$ which satisfies $\pi P = \pi$ and does not depend on the initial distribution v (LPW09).

Theorem 3.2.6 (Asymptotic performance) *Let Assumption 3.2.2 hold and let $\{x_k\}_k$ be a sequence satisfying* (3.4). *Define the asymptotic average cost as the random variable*

$$J := \mathbb{E}\Bigg[\limsup_{T\to\infty} \tfrac{1}{T}\sum_{k=0}^{T-1} \ell(x_k, u_k, \theta_k)\Bigg] \tag{3.6a}$$

where $u_k = \kappa_N(x_k, \theta_k)$ and assuming $(x_0, \theta_0) \in X_N$. Then,

$$J \leq \ell_\infty := \sum_{i\in\mathcal{N}} \pi_i \ell_N(i). \tag{3.6b}$$

Proof 3.2.7 *Firstly, since $(x_0, \theta_0) \in X_N$ and $u_k = \kappa_N(x_k, \theta_k)$ we have that $(x_k, \theta_k) \in X_N$ by Proposition* (2). *By taking asymptotic averages and the expectation with respect to $\mathcal{F}_0$ on both sides of* (3.5) *we have*

$$\begin{aligned}
&\mathbb{E}\Bigg[\liminf_{T\to\infty} \tfrac{1}{T}\sum_{k=0}^{T-1} \mathcal{L}V_N^\star(x_k, \theta_k)\Bigg] \\
&\leq \mathbb{E}\Bigg[\liminf_{T\to\infty} \tfrac{1}{T}\sum_{k=0}^{T-1} \ell_N(\theta_k) - \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k)\Bigg] \\
&\leq \mathbb{E}\Bigg[\liminf_{T\to\infty} \tfrac{1}{T}\sum_{k=0}^{T-1} \ell_N(\theta_k) \\
&\qquad - \limsup_{T\to\infty} \tfrac{1}{T}\sum_{k=0}^{T-1} \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k)\Bigg]
\end{aligned}$$

and using Fatou's lemma which we can apply because of nonnegativity of ℓ

$$\begin{aligned}
&\leq \liminf_{T\to\infty} \mathbb{E}\Bigg[\tfrac{1}{T}\sum_{k=0}^{T-1} \ell_N(\theta_k)\Bigg] \\
&\qquad - \mathbb{E}\Bigg[\limsup_{T\to\infty} \tfrac{1}{T}\sum_{k=0}^{T-1} \ell(x_k, \kappa_N(x_k, \theta_k), \theta_k)\Bigg].
\end{aligned} \tag{3.7}$$

We now use the fact that $\mathbb{E}[\ell_N(\theta_k)] = \sum_{i\in\mathcal{N}} \pi_k^i \ell_N(i)$, where $\pi_k^i = \mathrm{P}[\theta_k = i]$ and since $\pi_k^i \to \pi^i$ as $k \to \infty$, we have that $\mathbb{E}[\ell_N(\theta_k)] \to \ell_\infty$ and the right hand side of (3.7) *is equal to $\ell_\infty - J$.*

Using (PSSB14b, Lemma 19) and because of the fact that ℓ are nonnegative,

$$\begin{aligned}
&\mathbb{E}\left[\liminf_{T\to\infty} \tfrac{1}{T} \sum_{k=0}^{T-1} \mathcal{L}V_N^\star(x_k, \theta_k)\right] \\
&= \mathbb{E}\left[\liminf_{T\to\infty} \tfrac{1}{T}(V_N^\star(x_T, \theta_T) - V_N^\star(x_0, \theta_0))\right] \\
&\geq \liminf_{T\to\infty} \left(-\tfrac{1}{T} V_N^\star(x_0, \theta_0)\right) = 0.
\end{aligned}$$

Combining the two results completes the proof. □

3.2.5 Mean-square stability

We will now study the conditions under which a Markovian system is mean square stable towards an equilibrium point.

Assumption 3.2.8 (Common optimal equilibrium) *There exists one common optimal stationary point (x_s, u_s) for all modes which is the solution of the optimization problem in Definition 8. Without loss of generality assume $x_s = 0$, $u_s = 0$. In other words $\ell_s(\theta)$ is independent of θ. Hereafter, we denote $\ell_s = \ell_s(\theta)$.*

Consider the following Markovian switching system

$$x_{k+1} = f(x_k, \theta_k), \tag{3.8}$$

and let $r_k = (\theta_0, \ldots, \theta_k)$ be an admissible *switching sequence* starting from θ_0. Let $\phi(k; x_0, r_k)$ be the trajectory of (3.8) with $\phi(0; x_0, r_0) = x_0$.

Definition 9 (Mean Square Stability) *We say that* (3.8) *is mean square stable if* $\mathbb{E}[\|\phi(k; x_0, r_k)\|^2] \to 0$*, as $k \to \infty$ for all x_0 and θ_0.*

We extend the notion of dissipativity to Markovian systems as follows

Definition 10 (Stochastic dissipativity) *We say that system* (3.8) *is stochastically dissipative with respect to a stochastic supply rate*

$$s : \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{N} \to \mathbb{R}$$

if there is a function

$$\lambda : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}$$

, lower semicontinuous in the first argument, so that for all $x_k \in \mathbb{R}^n$ and $\theta_k \in \mathcal{N}$

$$\mathcal{L}\lambda(x_k, \theta_k) \leq s(x_k, u_k, \theta_k). \tag{3.9}$$

where

$$\mathcal{L}\lambda(x_k, \theta_k) := \mathbb{E}[\lambda(x_{k+1}, \theta_{k+1}) - \lambda(x_k, \theta_k) \mid \mathfrak{F}_k].$$

We say that (3.1) *is strictly stochastically dissipative with respect to s if there is a convex function $\rho : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}_+$, positive definite with respect to x_s, so that the left hand side of* (3.9) *is no larger than $s(x_k, u_k, \theta_k) - \rho(x_k, \theta_k)$.*

Assumption 3.2.9 (Strict stochastic dissipativity) *Function $\lambda(x_s, \theta)$ is independent of θ and let $\lambda_s := \lambda(x_s, \theta)$. In addition to Assumption 3.2.8, system* (3.8) *is strictly stochastically dissipative with supply rate $s(x, u, \theta) = \ell(x, u, \theta) - \ell_s$.*

Let us define the *rotated stage cost* function as

$$L(x_k, u_k, \theta_k) := \ell(x_k, u_k, \theta_k) - \mathcal{L}\lambda(x_k, \theta_k). \tag{3.10}$$

We now define the *rotated cost function* $\widetilde{V}_N(x, \theta, \mathbf{u}_N)$ as follows

$$\widetilde{V}_N(x_0, \theta_0, \mathbf{u}_N) = \mathbb{E}\left[\sum_{j=0}^{N-1} L(x_j, u_j, \theta_j) \middle| \mathcal{F}_0\right]$$

using again $V_f = 0$ and we introduce the rotated MPC problem

$$\bar{\mathbb{P}}(x, \theta) : \widetilde{V}_N^\star(x, \theta) = \inf_{\mathbf{u}_N} \widetilde{V}_N(x, \theta, \mathbf{u}_N), \tag{3.11}$$

subject to (3.3b)–(3.3f).

Lemma 3.2.10 *Problem $\bar{\mathbb{P}}(x, \theta)$ is recursively feasible and it has the same set of minimizers as $\mathbb{P}(x, \theta)$. Let $\tilde{\kappa}_N$ be the receding horizon control law which accrues from $\bar{\mathbb{P}}(x, \theta)$. If Assumption 3.2.9 holds, then*

$$\mathcal{L}\widetilde{V}_N^\star(x_k, \theta_k) \leq -\rho(x_k, \theta_k), \tag{3.12}$$

where $\rho : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}_+$ is a positive definite function in the first argument with respect to x_s.

Proof 3.2.11 *Problems $\mathbb{P}$ and $\bar{\mathbb{P}}$ have the same set of constraints, therefore, they have the same feasibility domain and the recursive feasibility of $\bar{\mathbb{P}}$ follows from Prop. 2. The rotated cost function can be expanded as*

$$\begin{aligned}\widetilde{V}_N(x_k,\theta_k,\mathbf{u}_N) &= \mathbb{E}[\textstyle\sum_{j=k}^{k+N-1} L(x_j,u_j,\theta_j) \mid \mathfrak{F}_k] \\ &= \mathbb{E}[\sum_{j=k}^{k+N-1} \ell(x_j,u_j,\theta_j) - \mathcal{L}\lambda(x_k,\theta_k) \mid \mathfrak{F}_k].\end{aligned}$$

We now use the fact that

$$\begin{aligned}\mathbb{E}[\sum_{j=k}^{k+N-1} \mathcal{L}\lambda(x_k,\theta_k) \mid \mathcal{F}_k] &= \mathbb{E}[\lambda(x_{k+N-1},k+N-1) - \lambda(x_k,\theta_k) \mid \mathcal{F}_k] \\ &= \lambda_s - \lambda(x_k,\theta_k).\end{aligned}$$

Therefore,

$$\widetilde{V}_N(x_k,\theta_k,\mathbf{u}_N) = V_N(x_k,\theta_k,\mathbf{u}_N) + \lambda(x_k,\theta_k) - \lambda_s.$$

The rotated and original cost functions differ only by a constant so the two problems, $\mathbb{P}$ and $\bar{\mathbb{P}}$, share a common optimal sequence. Proceeding as in Lemma 3.2.4 the following holds

$$\mathcal{L}\tilde{V}_N^\star(x_k,\theta_k) \leq \ell_s - L(x_k,\tilde{\kappa}_N(x_k,\theta_k),\theta_k), \tag{3.15}$$

By tracing the arguments of (RAB12) we have that $L(x_k,u_k,\cdot) \geq \ell_s$. Combining (3.10) *and Assumption 3.2.9 we arrive at*

$$L(x_k,u_k,\theta_k) \geq \rho(x_k,\theta_k) + \ell_s, \tag{3.16}$$

which completes the proof. □

Next, we draw an additional assumption on $\rho(\cdot,\theta)$:

Assumption 3.2.12 (Quadratic lower bound) *There exist a positive constant γ, such that $\rho(x,i) \geq \gamma\|x - x_s\|^2$ holds for all x.*

Theorem 3.2.13 *Suppose Assumption 3.2.12 is satisfied. Then, system* (3.8) *is MSS.*

Proof 3.2.14 *All assumptions required by (PSSB14b, Theorem 24) are met and entail mean square stability.* □

3.3 Uniform Invariance and Terminal Constraints

In this section we relax the restrictive requirement $x_N = x_s^{\mathrm{bet}(\theta_{N-1})}$ and we instead replace it with a terminal constraint of the form $(x_N, \theta_N) \in X^f$ along with a terminal penalty function V_f and we derive conditions so that the controlled system is mean-square stable.

We will now make use of the following definition (PSSB14b)

Definition 11 (Uniform positive invariance) *A family of nonempty sets $C = \{C_i\}_{i\in\mathcal{N}}$ is said to be* uniformly positive invariant *(UPI) for the constrained Markovian switching system* (3.8) *if for every $x_k \in C_{\theta_k}$, $x_{k+1} \in C_{\theta_{k+1}}$.*

As before, we make an assumption that there is one stationary point ℓ_s and require, with a slight abuse of notation, that $\lambda_s = \lambda(x_s, \theta)$, $V_f(x_s) = V_f(x_s, \theta)$ for all $\theta \in \mathcal{N}$. Now we make a central assumption regarding our exposition

Assumption 3.3.1 (Terminal control law) *There exists a control law $\kappa_f : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ and a collection of sets $X^f = \{X_i^f\}_{i\in\mathcal{N}}$ so that*

i. X^f is UPI for the closed-loop system controlled by κ_f and

ii. for all $(x, \theta) \in X^f$

$$\mathcal{L}V_f(x_k, \theta_l) \leq -\ell(x_k, \kappa_f(x_k, \theta_k), \theta_k) + \ell_s. \tag{3.17}$$

We now consider the following stochastic economic model predictive control problem:

$$\mathbb{P}_T(x, \theta) : V_N^\star(x, \theta) = \inf_{\mathbf{u}_N} V_N(x, \theta, \mathbf{u}_N) \tag{3.18a}$$

and for $k = 0, \ldots, N-1$, it is subject to

$$x_{k+1} = f(x_k, u_k, \theta_k) \tag{3.18b}$$

$$(x_k, u_k) \in Y_{\theta_k} \tag{3.18c}$$

$$(x_0, \theta_0) = (x, \theta) \tag{3.18d}$$

$$x_N \in X_{\theta_k}^f \tag{3.18e}$$

$$u_k \lhd \mathfrak{F}_k. \tag{3.18f}$$

Again, the same reasoning as in Section 3.2.2 applies regarding the existence of optimal solutions. Let $\hat{\mathbf{u}}^\star(x,\theta) = \{u_0^\star(x,\theta), \dots, u_{N-1}^\star(x_{N-1}, \theta_{N-1})\}$ be an optimizer of (3.18). The receding horizon control law is given by $\hat{\kappa}_N(x,\theta) := u_0^\star(x,\theta)$.

In light of the state-input constraints (3.18c) we must require that the sets X_i^f in Assumption 3.3.1 are subsets of X_N, the feasibility domain of $\mathbb{P}_T$.

3.3.1 Recursive feasibility

Here, we will show that stochastic economic model predictive control problem (3.18) is recursively feasible.

Proposition 3 *Let $X_N \subseteq \mathbb{R}^n \times \mathcal{N}$ be the feasibility domain of $\mathbb{P}_T$ and let Assumption 3.3.1-i hold. Then, X_N is UPI for the MPC-controlled system.*

Proof 3.3.2 *For given $(x,\theta) \in X_N$ let*

$$\pi(x,\theta) = \{u_0^\star, \dots, u_{N-1}^\star\}$$

be an optimizer of $\mathbb{P}_T(x,\theta)$ and let

$$x^\star(x,\theta) = \{x, x_1^\star, \dots, x_N^\star\}$$

be the corresponding sequence of states. Because of (3.18e) *we have $x_N^\star \in X_{\theta_N}^f$. Now take $x^+ = f(x, u_0^\star(x,\theta), \theta)$, $\theta^+ \in \mathcal{C}(\theta)$ and let*

$$\tilde{\pi}^+(x^+,\theta^+) := \{u_1^\star, \dots, u_{N-1}^\star, u_f\}$$

where $u_f = \kappa_f(x_N, \theta_N)$. Then, since X^f is a UPI set, $(x_{N+1}, \theta_{N+1}) \in X^f$, so $\tilde{\pi}^+$ satisfies the constraints of $\mathbb{P}_T(x^+,\theta^+)$. □

3.3.2 Expected asymptotic average performance

We show next that the asymptotic average cost of the EMPC-controlled system with terminal constraints is no higher than the cost of the best stationary point.

Theorem 3.3.3 *Let Assumption 3.3.1 hold and let $\{x_k\}_k$ be a sequence satisfying* (3.4) *with $\hat{\kappa}_N(x_k,\theta_k)$. Then, $J \le \ell_s$.*

Proof 3.3.4 *Using the optimal solution* $\pi(x, \theta)$ *of* (3.18) *with initial conditions* (x, θ) *we construct a feasible shifted policy* $\pi^+(x^+, \theta^+)$ *as in the proof of the Prop. 3. Then* $V_N^\star(x_{k+1}, \theta_{k+1}) \leq V_N(x^+, \tilde{\pi}^+, \theta^+)$ *and*

$$\begin{aligned}\mathcal{L}V_N^\star(x_k, \theta_k) = \mathbb{E}\Bigg[& \sum_{j=k+1}^{k+N-1} \ell(x_j, u_j^\star, \theta_j) \\ & + \ell(x_{k+N}, \kappa_f(x_{k+N}, \theta_{k+N}), \theta_{k+N}) + V_f(x_{k+N+1}, \theta_{k+N+1}) \\ & - \sum_{j=k}^{k+N-1} \ell(x_j, u_j^\star, \theta_j) - V_f(x_{k+N}, \theta_{k+N}) \mid \mathfrak{F}_k \Bigg] \\ & \leq \ell_s - \ell(x, \hat{\kappa}_N(x, \theta), \theta).\end{aligned}$$

Here, we used the tower property and Assumption 3.3.1. Proceeding as in Thm. 3.2.6 we prove the assertion. □

3.3.3 Mean-square stability

In this section we will give conditions under which a Markovian system with terminal region constraint is mean-square stable towards a common equilibrium point. Once again, our main argument will be the equivalence between the original and a suitably *rotated* problem.

We define the following *rotated terminal function*

$$\tilde{V}_f(x_k, \theta_k) = V_f(x_k, \theta_k) + \lambda(x_k, \theta_k) - V_f(x_s) - \lambda_s. \tag{3.19}$$

Combining condition (3.9) (Definition 10) with the rotated stage cost we may easily derive

$$L(x_k, u_k, \theta_k) \geq \rho(x_k, \theta_k). \tag{3.20}$$

Lemma 3.3.5 *Suppose Assumption 3.3.1 holds. Then*

$$\mathcal{L}\tilde{V}_f(x_k, \theta_k) \leq -L(x_k, \kappa_f(x_k, \theta_k), \theta_k). \tag{3.21}$$

Proof 3.3.6 *By adding* $\mathcal{L}\lambda(x_k, \theta_k)$ *to both sides of* (3.17) *we get*

$$\begin{aligned}\mathcal{L}\tilde{V}_f(x_k, \theta_k) + \mathcal{L}\lambda(x_k, \theta_k) \leq & - \ell(x_k, \kappa_f(x_k, \theta_k), \theta_k) + \ell_s \\ & + \mathcal{L}\lambda(x_k, \theta_k).\end{aligned}$$

The right hand side is equal to the rotated stage cost

$$\mathbb{E}\Bigg[V_f(f(x_k,\kappa_f(x_k,\theta_k)),\theta_{k+1}) + \lambda(x_{k+1},\theta_{k+1}) \\ - V_f(x_k,\theta_k) - \lambda(x_k,\theta_k) \mid \mathcal{F}_k\Bigg] \leq -L(x_k,\kappa_f(x_k,\theta_k),\theta_k).$$

We add $V_f(x_s) + \lambda_s - V_f(x_s) - \lambda_s$ to the left hand side and, after rearranging, arrive at (3.21). ☐

Now, we introduce the *rotated stochastic economic MPC problem*

$$\bar{\mathbb{P}}_T(x,\theta) : \tilde{V}_N^\star(x,\theta) = \inf_{\mathbf{u}_N} \tilde{V}_N(x,\theta,\mathbf{u}_N) \tag{3.23}$$

subject to (3.18b)-(3.18f).

Theorem 3.3.7 *Problem $\bar{\mathbb{P}}_T(x,\theta)$ is recursively feasible and has the same set of minimizers as $\mathbb{P}_T(x,\theta)$.*

Proof 3.3.8 *Problems $\mathbb{P}_T$ and $\bar{\mathbb{P}}_T$ have the same set of constraints, therefore, they have the same feasibility domains and the recursive feasibility of $\bar{\mathbb{P}}$ follows from Prop. 3. The rotated cost is*

$$\begin{aligned}
\tilde{V}_N(x_k,\theta_k,\mathbf{u}_k) &= \mathbb{E}[\sum_{j=k}^{k+N-1} L(x_j,u_j,\theta_j) + \tilde{V}_f(x_N,u_N,\theta_N) \mid \mathfrak{F}_k] \\
&= \mathbb{E}[\sum_{j=k}^{N-1}(\ell(x_j,u_j,\theta_j) + \lambda(x_j,\theta_j) - \mathbb{E}[\lambda(x_{j+1},\theta_{j+1} - \ell_s) \mid \mathfrak{F}_j]) \\
&\quad + V_f(x_N,\theta_N) + \lambda(x_N,\theta_N) - V_f(x_s) - \lambda_s) \mid \mathfrak{F}_N] \mid \mathfrak{F}_k] \\
&= V_N(x,\mathbf{u},\theta) + \lambda(x,\theta) - N\ell_s - V_f(x_s) - \lambda_s.
\end{aligned}$$

The two cost functions, V_N and $\tilde{V}_N$ differ by feedback-invariant quantities, hence, the optimal solutions of the two problems coincide. ☐

Theorem 3.3.9 *Suppose Assumptions 3.2.12 and 3.3.1 are satisfied. Then,* (3.4) *is MSS with domain of attraction X_N.*

Proof 3.3.10 *It follows from (PSSB14b, Thm. 24).* ☐

3.3.4 Linearization-based design

In this section we demonstrate how to design a terminal cost function and give a terminal control law using local linearization around the origin. In other words, we give conditions under which Assumption 3.3.1 -ii is satisfied, given that Assumption 3.3.1 -i holds for a nonlinear system with a particular control law. In the next section we shall also demonstrate how to design an ellipsoidal set X^f such that it satisfies Assumption 3.3.1 -i.

To simplify the notation let

$$\bar{\ell}(x,\theta) = \ell(x,\kappa_f(x,\theta),\theta) - \ell(0,0,\theta), \tag{3.25}$$

for all $\theta \in \mathcal{N}$, be a shifted stage cost function. Define

$$\hat{f}_\theta(x) := f(x,\kappa_f(x,\theta),\theta), \tag{3.26}$$

where $\kappa_f(x,\theta)$ is a terminal control law that we will introduce shortly. The evolution of the controlled nonlinear system is described by

$$x_{k+1} = \hat{f}_\theta(x_k),$$

for all $\theta \in \mathcal{N}$. To proceed we need the following assumption which does not require the system dynamics or the cost function to be twice differentiable, often a demanding condition, but commonly used in the literature (RM09a).

Assumption 3.3.11 (Smoothness) *Functions $\hat{f}_\theta(x)$ are β_f^θ-smooth and $\bar{\ell}(x,\theta)$ are β_ℓ^θ-smooth for all $\theta \in \mathcal{N}$.*

Let

$$z_{k+1} = A_{\theta_k} z_k + B_{\theta_k} u_k \tag{3.27}$$

be the corresponding linearized Markovian jump linear systems (MJLS), where $A_i = \frac{\partial f_i}{\partial x}(0,0)$ and $B_i = \frac{\partial f_i}{\partial u}(0,0)$ for all $i \in \mathcal{N}$. Hereafter, we make an assumption that

Assumption 3.3.12 *The set of pairs $\{(A_i,B_i)\}_{i\in\mathcal{N}}$ is mean-square stabilizable.*

(CFM05) provide conditions for Assumption 3.3.12 to hold. We recall the following result for MJLS (PSSB14b)

Proposition 4 (MSS of MJLS) *Consider system* (3.27) *subject to* (4.4) *in closed loop with* $\kappa(x,i) = K_i x$. *Suppose there is a UPI set* X^f *and matrices* $P^f = \{P_i^f\}_{i\in\mathcal{N}}$ *so that*

$$P_i^f \succcurlyeq \Gamma_i^\top \mathcal{E}_i(P^f)\Gamma_i + Q_i^*$$

with

$$\Gamma_i := A_i + B_i K_i, \quad \mathcal{E}_i(P^f) := \sum_{j\in\mathcal{C}(i)} p_{ij} P_j^f$$

and $Q_i^* = (Q_i^*)^\top \succ 0$. *Then, the closed-loop system is MS stable in* X^f.

Next, we will design a terminal cost function $V_f(x,\theta)$ which, under certain assumptions (see Thm. 3.3.15) satisfies a desired Lyapunov-type inequality (see Assumption 3.3.1 -ii). First, we design a quadratic cost function $\ell_q(x,\theta)$ which is an upper bound on the shifted cost.

Lemma 3.3.13 *Let*

$$\ell_q(x,\theta) := \tfrac{1}{2} x^\top Q_\theta^* x + q_\theta^\top x \tag{3.28}$$

where $Q_\theta^* = (\alpha + \beta_\ell^\theta) I, q_\theta = \nabla \bar{\ell}(0,\theta)$. *Then it holds that*

$$\ell_q(x,\theta) \geq \bar{\ell}(x,\theta) + \frac{\alpha}{2}\|x\|^2$$

for any $\alpha > 0$, *for all* $\theta \in \mathcal{N}$.

Proof 3.3.14 *By Assumption 3.3.11 on* $\bar{\ell}(x,\theta)$, *we have that*

$$|\bar{\ell}(x,\theta) - q_\theta^\top x| \leq \beta_l^\theta / 2\|x\|^2.$$

Adding $\alpha/2\|x\|^2$ *to both sides the assertion follows.* □

We may now choose our terminal cost to be the following infinite sum

$$V_f(x,i) = \mathbb{E}\left[\sum_{k=0}^{\infty} \ell_q(x_k,\theta_k) \,\middle|\, \mathcal{F}_0\right], \tag{3.29}$$

for the MJLS $x_{k+1} = \Gamma_{\theta_k} x_k$, with $x_0 = x, \theta_0 = \theta$. Using the linearity of expectation we have

$$V_f(x, \theta) = \mathbb{E}\left[\sum_{k=0}^{\infty} \tfrac{1}{2} x_k^\top Q_{\theta_k}^* x_k\right] + \mathbb{E}\left[\sum_{k=0}^{\infty} q_{\theta_k}^\top x_k\right]$$

and V_f can be written in the form

$$V_f(x, i) = \tfrac{1}{2} x^\top P_i^f x + p_i^\top x, \tag{3.30}$$

where P_i^f are computed as in Prop. 4 with $=$ in lieu of $\succcurlyeq$ (CFM05, Prop. 3.20). Because of the parametrization of Q_i^* in Lemma 3.3.13, we may choose $P_i^f = P_i^\beta + \alpha P_i^I$ and require that

$$P_i^I = I + \Gamma_i^\top \mathcal{E}_i(P^I)\Gamma_i, \tag{3.31a}$$

$$P_i^\beta = \beta_\ell^i I + \Gamma_i^\top \mathcal{E}_i(P^\beta)\Gamma_i. \tag{3.31b}$$

If we chose K_i, P_i^I and P_i^β so that they satisfy (3.31), then $P_i^f = P_i^\beta + \alpha P_i^f$ satisfies $P_i^f = \Gamma_i^\top \mathcal{E}(P^f)\Gamma_i + Q_i^*$ where $Q_\theta^* = (\alpha + \beta_\ell^\theta)I$ as in Lemma (3.3.13).

For convenience we re-introduce the operator $\mathcal{L}$, but this time with a distinction between nonlinear and linear systems:

i. $\mathcal{L}V_f(x_k, \theta_k) = \mathbb{E}[V_f(\hat{f}_{\theta_k}(x_k), \theta_{k+1}) - V_f(x_k, \theta_k) \mid \mathcal{F}_k]$

ii. $\mathcal{L}V_f^{\text{lin}}(x_k, \theta_k) = \mathbb{E}[V_f\,(\Gamma_{\theta_k} x_k, \theta_{k+1}) - V_f(x_k, \theta_k) \mid \mathcal{F}_k]$.

Parameter α will be used to bound the mismatch between $\mathcal{L}V_f(x_k, \theta_k)$ and $\mathcal{L}V_f^{\text{lin}}(x_k, \theta_k)$ and a method for choosing it is presented in the proof of the next theorem.

Theorem 3.3.15 *Consider the control law $\kappa_f(x, i) = K_i x$ and let Assumptions 3.3.11 and 3.3.12 hold. Then $\mathcal{L}V_f(x, \theta) \leq -\bar{\ell}(x, \theta)$ for $x \in \mathcal{B}_\delta$ for some $\delta > 0$. If X^f satisfies Assumption 3.3.1 -i with $X_i^f \subseteq \mathcal{B}_\delta$ and Assumption 3.2.12 is satisfied, the controlled system is locally mean square stable.*

Proof 3.3.16 *Let us introduce the shorthand*

$$\Delta\mathcal{L}V_f(x_k, \theta_k) := \mathbb{E}[V_f(\hat{f}_{\theta_k}(x_k), \theta_{k+1}) - V_f\,(\Gamma_{\theta_k} x_k, \theta_{k+1}) \mid \mathcal{F}_k]. \tag{3.32}$$

By the linearity of the conditional expectation

$$\mathcal{L}V_f(x,\theta) = \mathcal{L}V_f^{lin}(x,\theta) + \Delta\mathcal{L}V_f(x,\theta).$$

By (3.29)*, the first term is*

$$\mathcal{L}V_f^{lin}(x,\theta) = -\ell_q(x,\theta).$$

The last term is

$$\Delta\mathcal{L}V_f(x,\theta) = \tfrac{1}{2}e(x,\theta)^\top \mathcal{E}_\theta(P^f)e(x,\theta) - (\Gamma_\theta x)^\top \mathcal{E}_\theta(P^f)e(x,\theta) + \mathcal{E}_\theta(p)^\top \cdot e(x,\theta),$$

where $e(x,\theta) := \hat{f}_\theta(x) - \Gamma_\theta x$ *is the linearization error. Under Assumption 3.3.11,*

$$\|e(x,\theta)\| \leq \tfrac{\beta_f^\theta}{2}\|x\|^2$$

and

$$\Delta\mathcal{L}V_f(x,\theta) \leq \tfrac{(\beta_f^\theta)^2}{8}\|\mathcal{E}_\theta(P^f)\|\|x\|^4 + \tfrac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathcal{E}_\theta(P^f)\|\|x\|^3 + \tfrac{\beta_f^\theta}{2}\|\mathcal{E}_\theta(p)\|\|x\|^2.$$

We need to show that $\Delta\mathcal{L}V_f(x,\theta)$ *is upper bounded by* $\frac{\alpha}{2}\|x\|^2$ *in a region of the origin for adequately large* α*. Recall that* $\mathcal{E}_\theta(P^f)$ *depends on* α *as follows*

$$\mathcal{E}_\theta(P^f) = \mathcal{E}_\theta(P^\beta) + \alpha\mathcal{E}_\theta(P^I). \tag{3.33}$$

Using the triangle inequality

$$\begin{aligned}\Delta\mathcal{L}V_f(x,\theta) &\leq \tfrac{(\beta_f^\theta)^2}{8}\|\mathcal{E}_\theta(P^\beta)\|\|x\|^4 \\ &+ \tfrac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathcal{E}_\theta(P^\beta)\|\|x\|^3 + \tfrac{\beta_f^\theta}{2}\|\mathcal{E}_\theta(p)\|\|x\|^2 \\ &+ \alpha\left(\tfrac{(\beta_f^\theta)^2}{8}\|\mathcal{E}_\theta(P^I)\|\|x\|^4 + \tfrac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathcal{E}_\theta(P^I)\|\|x\|^3\right)\end{aligned} \tag{3.34}$$

For the right hand side of the last inequality to be upper bounded by $\frac{\alpha}{2}\|x\|^2$ *it suffices to take* $x \in \mathcal{B}_\delta$ *with* $\delta > 0$ *and*

$$\max_{\theta\in\mathcal{N}} \tfrac{(\beta_f^\theta)^2}{8}\|\mathcal{E}_\theta(P^I)\|\delta^2 + \tfrac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathcal{E}_\theta(P^I)\|\delta < 1, \tag{3.35}$$

and α *so that*

$$\alpha \geq \max_{\theta\in\mathcal{N}} \frac{\frac{(\beta_f^\theta)^2}{8}\|\mathcal{E}_\theta(P^\beta)\|\delta^2 + \frac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathcal{E}_\theta(P^\beta)\|\delta + \frac{\beta_f^\theta}{2}\|\mathcal{E}_\theta(p)\|}{1 - \frac{(\beta_f^\theta)^2}{8}\|\mathcal{E}_\theta(P^I)\|\delta^2 + \frac{\beta_f^\theta}{2}\|\Gamma_\theta\|\|\mathcal{E}_\theta(P^I)\|\delta}. \tag{3.36}$$

Now for $x \in \mathcal{B}_\delta$ and α as above we have

$$\Delta\mathcal{L}V_f(x,\theta) \leq \tfrac{\alpha}{2}\|x\|^2,$$

and since

$$\mathcal{L}V_f(x,\theta) = -\ell_q(x,\theta) + \Delta\mathcal{L}V_f(x,\theta)$$

we have

$$\mathcal{L}V_f(x,\theta) \leq -\ell_q(x,\theta) + \tfrac{\alpha}{2}\|x\|^2,$$

and employing Lemma 3.3.13 we obtain $\mathcal{L}V_f(x,\theta) \leq -\bar{\ell}(x,\theta)$. If Assumption 3.2.12 holds all assumptions of Thm. 3.3.9 are fulfilled and the controlled system is locally mean square stable. □

Let us repeat some crucial steps for the design of the controller. Our main goal is to find such a region where we can approximate the nonlinear system sufficiently accurately with its linearized version. We can design a terminal control law and terminal region for the linearized system, which is computationally much more practical. Then we need to ensure that the nonlinear system inherits the required MSS properties by carefully choosing the system parameters and a terminal region. This is done by suitably defining "mock"stage cost as in (3.28) which is always above the shifted cost of the nonlinear system. We can now calculate P_i^f for the linear system which can be further decomposed as in (3.31) . Having chosen K_i, P_i^I and P_i^β, there exist $\delta > 0$, properly small so that it satisfies (3.35). Then α is taken to be an upper bound of (3.36). Using the fact that the linearization error between linear and nonlinear system is bounded we arrive at a bound (3.32) which is precisely $\alpha/2\|x\|^2$, the same factor that bounds the difference between the "mock" cost and the shifted cost. Since the controlled law is stabilizing for the linear system and linearization error vanishes quadratically with respect to the distance of the origin, we conclude that the nonlinear system inside a X^f set will be stable as well. It remains to see how we can compute a UPI set X^f which lies inside $\mathcal{B}_\delta$.

3.3.5 Computation of X^f

We demonstrate a possible way of finding X^f such that the requirements of Thm. 3.3.15 are satisfied. Take $X^f = \{X_i^f\}_{i\in\mathcal{N}}$ to be ellipsoidal of the form $X_i^f = \{x : x^\top P_i x \le 1\}$. By Assumption 3.3.11, there exist constants $\gamma_i > 0$, $i \in \mathcal{N}$, such that

$$x_{k+1} = A_{\theta_k} x_k + B_{\theta_k} \kappa_f(x_k, \theta_k) + d_{k,\theta_k}, \tag{3.37}$$

with $\|d_{k,i}\|^2 \le \gamma_i x_k^\top P_i^f x_k$ where $d_{k,i} = e(x_k, i)$ is the linearization error. For X^f to be UPI for the κ_f-controlled system it must satisfy

$$\max_{j \in \mathcal{C}(i)} \{x_{k+1}^\top P_j x_{k+1}\} \le x_k^\top P_i x_k,\ \forall i \in \mathcal{N}$$

$$\Leftrightarrow \begin{bmatrix} x_k \\ d_{k,i} \end{bmatrix}^\top \begin{bmatrix} P_i - \Gamma_i^\top P_j \Gamma_i & -\Gamma_i^\top P_j \\ -P_j \Gamma_i & -P_j \end{bmatrix} \begin{bmatrix} x_k \\ d_{k,i} \end{bmatrix} \ge 0, \tag{3.38a}$$

for all $j \in \mathcal{C}(i)$ and $i \in \mathcal{N}$ whenever $d_{k,i}^\top d_{k,i} \le \gamma_i x_k^\top P_i^f x_k$, or, for $i \in \mathcal{N}$

$$\begin{bmatrix} x_k \\ d_{k,i} \end{bmatrix}^\top \begin{bmatrix} \gamma_i P_i^f & \\ & -I \end{bmatrix} \begin{bmatrix} x_k \\ d_{k,i} \end{bmatrix} \ge 0. \tag{3.38b}$$

Using the S-lemma, (3.38b) implies (3.38a) so long as

$$\begin{bmatrix} P_i - \Gamma_i^\top P_j^f \Gamma_i & -\Gamma_i^\top P_j^f \\ -P_j \Gamma_i & -P_j^f \end{bmatrix} - \tau \begin{bmatrix} \gamma_i P_i^f & \\ & -I \end{bmatrix} \succcurlyeq 0 \tag{3.39}$$

for some $\tau \ge 0$ and for all $i \in \mathcal{N}$ and $j \in \mathcal{C}(i)$.

By rearranging the terms in the two matrices, equation (3.39) can be equivalently written as

$$\begin{bmatrix} \tau\gamma_i P_i^f + \Gamma_i^\top P_j^f \Gamma_i & \Gamma_i^\top P_j^f \\ * & P_j^f \end{bmatrix} \preccurlyeq \begin{bmatrix} P_i^f & \\ & \tau I \end{bmatrix}. \tag{3.40}$$

The left hand side of (3.40) is equal to

$$\begin{bmatrix} P_i^f & \\ P_j^f \Gamma_i & P_j^f \end{bmatrix}^\top \begin{bmatrix} \frac{1}{\gamma_i \tau} P_i^f & \\ & P_j^f \end{bmatrix}^{-1} \begin{bmatrix} P_i^f & \\ P_j^f \Gamma_i & P_j^f \end{bmatrix}$$

Using the Schur complement we get

$$\begin{bmatrix} P_i^f & 0 & P_i^f & \Gamma_i^\top P_j^f \\ 0 & \tau I & 0 & P_j^f \\ * & * & \frac{1}{\gamma_i \tau} P_i^f & 0 \\ * & * & 0 & P_j^f \end{bmatrix} \succcurlyeq 0. \tag{3.41}$$

Introducing the variables $P_i^f = Z_i^{-1}$ and $K_i = Y_i Z_i^{-1}$, this is equivalent to the matrix inequality

$$\begin{bmatrix} Z_i & 0 & \tau Z_i & Z_i A_i^\top + Y_i^\top B_i^\top \\ 0 & \tau I & 0 & I \\ * & * & \tau\gamma_i^{-1} Z_i & 0 \\ * & * & 0 & Z_j \end{bmatrix} \succcurlyeq 0. \tag{3.42}$$

As required by Thm. 3.3.15, X_i^f must be in $\mathcal{B}_\delta$. This is equivalently written as

$$\begin{bmatrix} \delta I & P_i \\ P_i & I \end{bmatrix} \succcurlyeq 0. \tag{3.43}$$

We then choose P_i^f so as to satisfy (3.42) and (3.43) for all $i \in \mathcal{N}$ and $j \in \mathcal{C}(j)$. Note that (3.42) is a bilinear matrix inequality (BMI) with unknowns Z_i, Y_i and τ, but the bilinearity is only because of the term τZ_i. Although BMIs are more difficult to solve compared to LMIs, in this case since τ is a scalar, (3.42) can be solved with a simple line search method with respect to τ.

3.4 Conclusions

This chapter offers a theoretical framework for the control of Markovian switching systems using EMPC. We first studied a formulation with mode-dependent optimal steady states and terminal equality constraints for which we provided an upper bound on the expected asymptotic average cost (Thm. 3.2.6). We then studied an EMPC formulation with mode-dependent terminal region constraints and we provided design guidelines based on the system linearization assumptioning that the system dynamics and the stage cost function are β-smooth which are rather weak assumptions (Thm. 3.3.15).

Chapter 4

Risk-averse model predictive control

4.1 Motivation

There exist two main ways to deal with uncertainty in model predictive control (MPC), namely, the *robust* and the *stochastic* approaches. In *robust MPC*, modeling uncertainties or disturbances are modeled as unknown-but-bounded quantities and the performance index is minimized with respect to the worst-case realization of the uncertainty (min-max approach) (RM09b; BM99b). However, such worst-case events which are unlikely to occur in practice and render robust MPC severely conservative since all statistical information, typically available from past measurements, is completely ignored.

On the other hand, in *stochastic MPC* we assume that the underlying uncertainty is a random vector following some probability distribution (Mes16) and we minimize the expectation of a performance index; such formulations are naturally significantly less conservative. In stochastic MPC, the driving random process is often taken to be normally and independently identically distributed (HCC^{+}12) or it is assumed that it is a finite Markov process (PSSB14a) and in *scenario-based MPC*, filtered probability distributions are estimated from data (BB09;

BB12; HSB$^+$15). However, not always can we accurately estimate a probability distribution from available data, nor does it remain constant in time. Stochastic MPC will guarantee mean-square stability of the closed-loop system only with respect to the nominal probability distribution, therefore, errors in the estimation of that distribution may lead to bad performance or even instability.

Using the theory of *risk measures* (SDR14a; PR07), which sprung from the field of stochastic finance, we seek to transcend the limitations of robust and stochastic optimal control by proposing a unifying framework that extends and contains both as special cases. Worst-case approach to MPC is equivalent to taking the expectation of the random cost over all possible distributions and singling out the least favorable one. This interpretation offers a clear connection to stochastic approach where we consider expectation over only one probability distribution. The natural question that arises is whether we can interpolate between the two approaches by considering a subset of distributions in a computationally efficient manner. As we will see later, risk measures offer an elegant way to do so.

Roughly speaking, risk measures quantify the importance and effect of the right tail of a distribution of losses, that is, the impact of the occurrence of *extreme events*. As such they offer a mathematically elegant tool to tackle problems where we seek to avoid *high effect low probability* (HELP) events and can be readily used in various applications. For example, the authors in (GHK12) compute routing policies for shortest path problem on a graph with uncertain arch lengths. They show that evaluating uncertain paths by using a risk measure offers protection against high-length paths at the expense of somewhat higher average path lengths. The authors in (MDZ17) present a distributed risk-averse reinforcement learning approach to planning the exploration of the uncertain environments by a team of sensors. In this way the resulting policy seeks to avoid high-risk and low-reward events.

The analysis and design of risk-averse MPC controllers was recently identified as a contemporary challenge in stochastic MPC (Mes16). Risk-averse formulations are of great interest for various applications such as optimal bidding (KSVB15), unit commitment problems (MA15) and manufacturing systems (AJM13) without, however, being accompanied by proper theoretical stability guarantees or rigorous design guidelines.

4.1.1 Background and contributions

The first steps to risk-averse formulations can be traced back to linear-exponential-quadratic Gaussian control (Dun13) and the study of stochastic control problems under inexact knowledge of the underlying probability distribution which is often termed *distributionally robust* (ZSM17; GS10). There have been proposed distributionally robust control methodologies for linear systems with probabilistic constraints assuming knowledge of some moments of the distribution (VP15; VPKGM16). The same problem was also recently addressed for Markov decision processes with uncertain transition probabilities (YX16).

Recently, risk-averse MPC formulations for Markov jump linear systems (MJLS) were studied (CSMP17; CP14). In (CP14) the authors formulate an MPC optimization problem employing a coherent risk measure of an uncertain cost as an objective function. Furthermore, they give conditions under which the MPC algorithm is stabilizing, albeit for a system with no state-input constraints. Their approach is extended in (CSMP17) assuming ellipsoidal state-input constraints. Here, we further improve on the state of the art by studying nonlinear systems and proposing a computationally favorable formulation for risk-averse optimization problems which leads to low computation times.

In this thesis we study risk-averse model predictive control formulations for nonlinear Markovian switching systems under — generally nonconvex — joint state-input constraints. We formulate multistage risk-averse optimal control problems using conditional risk measures and draw parallels between dynamic programming and system theoretic properties to derive Lyapunov-type risk-averse stability conditions. When the system is a Markov jump linear system (MJLS) with polytopic constraints, we provide a tractable procedure for the design of stabilizing risk-averse controllers. Moreover, we propose a linearization-based controller design procedure for a class of nonlinear systems with smooth dynamics and Lipschitz-continuous gradient.

In the optimization and operations research communities, the solution of multistage risk-averse optimal control problems has been considered prohibitive as only slow cutting-plane methods are currently used (AR15; CPR12; BASS16). In a 2017 paper, Rockafellar proposed an algorithmic scheme for solving multistage problems using a non-composite (not nested) risk measure recognizing the difficulty of solving problems with nested risk measures (Roc17). Indeed, the difficulty lies in that the cost function of the optimization problem is written as a series of compositions of typically nonsmooth operators. Albeit convex, risk-averse multistage problems with nested risk measures have been difficult to deconvolve so as to solve them efficiently. In Section 4.5 we present a computationally tractable approach for the solution of multistage risk-averse problems by casting them as a simple quadratically constrained quadratic programs. This formulation renders risk-averse MPC suitable for embedded applications.

Last, we provide simulation examples to showcase the properties and advantages or risk-averse control. Using a cyber-attack scenario, we show that a conventional stochastic MPC design may fail to provide mean-square stability if the transition probabilities are inexactly known while the proposed method does stabilize the system in the mean-square sense. We evaluate risk-averse controllers with different levels of risk aversion on Samuelson's constrained Markovian macroeconomic model and demonstrate the feasibility of the proposed scheme. We apply the proposed methodology to a nonlinear constrained Markovian switching system where we design the terminal cost function and the terminal region using the system linearization at the origin.

4.1.2 Notation

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be the set of extended-real numbers, $\mathbb{N}_{[k_1,k_2]}$ the integers in $[k_1, k_2]$, for $z \in \mathbb{R}^n$ let $[z]_+ = \mathbf{max}\{0, z\}$ (where the max is taken element-wise). We denote by 1_n the vector in $\mathbb{R}^n$ with all coordinates equal to 1. We denote the sets of n-by-n symmetric positive definite (semidefinite) matrices as $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$). For two n-by-n symmetric matrices M_1, M_2, $M_1 \succcurlyeq M_2$ means that $M_1 - M_2 \in \mathcal{S}^n_{+}$. We denote the transpose of a matrix A by $A^\top$ and the identity matrix by I. For a $g : \mathbb{R}^n \to \mathbb{R}^m$, its *Jacobian matrix* is the mapping $Jg : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ defined as $Jg(x) = ({}^{\partial g_i(x)}/_{\partial x_j})_{i,j}$, provided that the partial derivatives exist. For $\epsilon \geq 0$ we define $\mathcal{B}_\epsilon = \{x \mid \|x\| \leq \epsilon\}$. For a set $C \subseteq \mathbb{R}^n$, we define its *indicator function* as $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = \infty$ otherwise. The *domain* of an extended-real-valued function $f{:}\mathbb{R}^n{\to}\overline{\mathbb{R}}$ is $\mathbf{dom}\, f = \{x \in \mathbb{R}^n \mid f(x){<}\infty\}$. An extended-real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *proper* if its domain is nonempty; it is called lower semi-continuous (lsc) if its lower level sets are closed. An $\ell : \mathbb{R}^n \times \mathbb{R}^m \ni (x, u) \mapsto \ell(x, u) \in \overline{\mathbb{R}}$ is called *level bounded in* u *locally uniformly in* x if for each $x_0 \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, there is a neighborhood U_{x_0} of x_0 along with a bounded set $B \subseteq \mathbb{R}^m$ such that $\{u \mid \ell(x, u) \leq \alpha\} \subseteq B$ for all $x_0 \in U_{x_0}$. The *effective domain* of a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined as $\mathbf{dom}\, F = \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$. For a nonempty set E and a finite set $\mathcal{N}$ we define $\mathbf{fcns}(E, \mathcal{N}) = \{V : E \times \mathcal{N} \to \overline{\mathbb{R}} \mid V(x, i) \geq 0, V(0, i) = 0, \text{ for all } x \in E, i \in \mathcal{N}\}$.

4.2 Risk-averse optimal control

4.2.1 Measuring risk

Let $\mathcal{N} = \mathbb{N}_{[1,n]}$ be a discrete sample space. A probability measure thereon can be identified by a probability vector $p \in \mathbb{R}^n$ with $\sum_{i=1}^n p_i = 1$, $p_i \geq 0$ for $i \in \mathcal{N}$. Let $Z : \mathcal{N} \to \mathbb{R}$ be a real-valued random variable on $\mathcal{N}$ which represents a random cost for $i \in \mathcal{N}$ let $Z_i = Z(i)$. The vector $(Z_i)_{i\in\mathcal{N}}$ identifies the random variable Z.

The *expectation* of a random variable Z with respect to the probability vector p is defined as

$$\mathbb{E}_p[Z] \equiv \mathbb{E}_p[Z(i); i] = \sum_{i\in\mathcal{N}} p_i Z_i. \tag{4.1}$$

The notation $\mathbb{E}_p[Z; i]$ is to emphasize that the expectation is taken with respect to i.

Definition 12 *A risk measure on $\mathbb{R}^n$ is a mapping $\rho : \mathbb{R}^n \to \mathbb{R}$. It is called coherent if it satisfies the following properties (SDR14a, Sec. 6.3) for $Z, Z' \in \mathbb{R}^n$, $\alpha \geq 0$, $\lambda \in [0, 1]$*

A1. Convexity. $\rho(\lambda Z + (1-\lambda)Z') \leq \lambda\rho(Z) + (1-\lambda)\rho(Z')$,

A2. Monotonicity. $\rho(Z) \leq \rho(Z')$ *whenever* $Z \leq Z'$,

A3. Translation equivariance. $\rho(c1_n + Z) = c + \rho(Z)$,

A4. Positive homogeneity. $\rho(\alpha Z) = \alpha\rho(Z)$.

Trivially, the *expectation* is a coherent risk measure and so is the *essential maximum* $\mathbf{essmax}[Z] := \mathbf{max}\{Z_i \mid p_i > 0, i \in \mathcal{N}\}$. A popular risk measure is the *average value-at-risk*, also known as *conditional value-at-risk* or *expected shortfall*, which is defined as

$$\mathrm{AV@R}_\alpha[Z] = \begin{cases} \min\limits_{t\in\mathbb{R}} \{t + \alpha^{-1}\mathbb{E}_p\left[Z - t\right]_+\}, & \alpha \in (0, 1] \\ \mathbf{essmax}(Z), & \alpha = 0. \end{cases}$$

As a result of assumptions A1–A4, coherent risk measures can be written in the following dual form (SDR14a, Thm. 6.5)

$$\rho[Z] = \max_{\mu\in\mathcal{A}(p)} \mathbb{E}_\mu[Z], \tag{4.2}$$

where $\mathcal{A}(p) \subseteq \mathbb{R}^n$ is a compact convex set of probability vectors containing p which we shall call the *ambiguity set* of ρ. We may think of a coherent risk measure as the worst-case expectation with respect to a probability distribution taken from a set of probability vectors. We call ρ a *polytopic* risk measure if $\mathcal{A}(p)$ is a polytope, i.e., it can be described by $\rho(Z) = \mathbf{max}\{\mu^\top Z \mid 1_n^\top \mu = 1, F(p)\mu \leq b(p)\}$ for some $F(p) \in \mathbb{R}^{q \times n}$ and $b(p) \in \mathbb{R}^q$. The expectation, the essential maximum and $\mathrm{AV@R}_\alpha$ are polytopic risk measures. The ambiguity set of $\mathrm{AV@R}_\alpha$ for $\alpha \in [0,1]$ is the polytope $\mathcal{A}_\alpha(p) = \{\mu \in \mathbb{R}^n \mid \sum_{i=1}^n \mu_i = 1, \mu_i \geq 0, \alpha\mu_i \leq p_i\}$. The ambiguity set $\mathcal{A}_0(p)$ is the whole probability simplex. Apparently $\mathrm{AV@R}_\alpha$ is a polytopic risk measure. $\mathrm{AV@R}_\alpha$ interpolates between the risk-neutral expectation operator ($\mathrm{AV@R}_1 = \mathbb{E}_p$, with $\mathcal{A}_0(p) = \{p\}$) and the worst-case essential maximum ($\mathrm{AV@R}_0 = \mathbf{essmax}$).

4.2.2 Markovian switching systems

In this chapter we consider Markovian switching systems

$$x_{k+1} = f(x_k, u_k, i_k), \tag{4.3}$$

driven by the random parameter i_k which is a time-homogeneous irreducible and aperiodic Markovian process with values in a finite set $\mathcal{N} = \mathbb{N}_{[1,n]}$ with transition matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$ (CFM05), that is

$$\mathrm{P}[i_{k+1} = j \mid i_k = i] = p_{ij}.$$

We call the states of this Markov chain, the *modes* of (4.3). We denote the *cover* of each mode by $\mathcal{C}(i) := \{j \in \mathcal{N} \mid p_{ij} > 0\}$ and assume that at time k we measure the full state x_k and the value of i_k. As the probabilistic information available up to time k is fully described by the pair (x_k, i_k), the control actions u_k may be decided by a causal control law

$$u_k = \kappa_k(x_k, i_k).$$

This formulation aligns with that of the classic textbook (CFM05), Often, one is able to design an estimator $\hat{i}_k$ for i_k and study the resulting closed-loop stability properties in light of the probability distribution of measurement accuracy (dVB99; CFM05). There exist formulations where i_k is not known at time k and the control law is a function of x_k only (CSMP17; BB12). Moreover, i_k is assumed to be an independent identically distributed process.

Each $f(\cdot,\cdot,i) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, $i \in \mathcal{N}$, is assumed to be continuous and satisfy $f(0,0,i) = 0$. MJLS are a special case of (4.3) with

$$f(x,u,i) = A_i x + B_i u, \quad i \in \mathcal{N}.$$

System (4.3) is subject to the joint state-input constraints

$$(x_k, u_k) \in Y_{i_k}, \tag{4.4}$$

and we shall assume that for all $i \in \mathcal{N}$, Y_i are nonempty, closed sets containing the origin in their interiors.

4.2.3 Markov risk measure

Consider the space of pairs (i, j) in $\Omega := \mathcal{N} \times \mathcal{N}$ equipped with the σ-algebra $\mathcal{F} = 2^{\Omega}$ and the probability measure $\mathrm{P}[\{(i, j)\}] = p_{ij}$. The conditional probability conditioned by the knowledge of i can be identified with the probability vector P_i — the i-th row of P. For a random variable $Z : \Omega \to \mathbb{R}$, the conditional expectation of Z conditioned by i, denoted as $\mathbb{E}_i[Z; j]$, is a random variable on $\mathcal{N}$, that is $\mathcal{N} \ni i \mapsto \mathbb{E}_i[Z; j] \in \mathbb{R}$, with

$$\mathbb{E}_i[Z; j] := \mathbb{E}_{P_i}[Z; j] = \sum_{j \in \mathcal{N}} p_{ij} Z(i, j). \tag{4.5}$$

We may extend this definition to define conditional variants of risk measures. Following (4.5), we give the following definition

Definition 13 (Markov risk measure) *Given a coherent risk measure ρ with ambiguity set $\mathcal{A}$ and a probability transition matrix P of a Markov chain, we define the Markov risk measure $\rho_i\,[Z; j]$ as*

$$\rho_i\,[Z; j] = \max_{\mu \in \mathcal{A}(P_i)} \mathbb{E}_{\mu}[Z; j] = \max_{\mu \in \mathcal{A}(P_i)} \textstyle\sum_{j \in \mathcal{N}} \mu_j Z(i, j) \tag{4.6}$$

for all random variables $Z : \Omega \to \mathbb{R}$.

This definition falls into the general framework of (Rus10). This way, with every i we associate the coherent risk measure $\rho_i\,[Z; j]$. As with the expectation, the notation $\rho_i\,[Z; j]$ is to emphasize that the risk is computed with respect to j.

4.2.4 Risk-averse optimal control and dynamic programming

Conditional risk mappings enable us to formulate risk-averse finite-horizon optimal control problems. Consider a *stage cost* function

$$\ell \in \mathbf{fcns}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_u}, \mathcal{N})$$

and a *terminal cost*

$$\ell_N \in \mathbf{fcns}(\mathbb{R}^{n_x}, \mathcal{N}).$$

Functions ℓ are extended-real-valued, therefore, they can encode constraints such as (4.4) by taking $\mathbf{dom}\,\ell(\cdot,\cdot,i) = Y_i$, $i \in \mathcal{N}$. Likewise, ℓ_N can encode constraints on the terminal state of the form $x_N \in X^f_{i_N}$ by taking $\mathbf{dom}\,\ell_N(\cdot,i) = X^f_i$, $i \in \mathcal{N}$, where X^f_i contain the origin in their interiors. We may now introduce the following finite-horizon risk-averse optimal control problem

$$\begin{aligned}\operatorname*{\mathbf{minimize}}_{u_0}\ \ell(x_0,u_0,i_0) + \rho_{i_0}\Bigg[&\inf_{u_1} \ell(x_1,u_1,i_1)\\ &+ \rho_{i_1}\Big[\inf_{u_2} \ell(x_2,u_2,i_2) + \cdots\\ &+ \rho_{i_{N-1}}[\ell_N(x_N,i_N);i_N]\cdots;i_2\Big];i_1\Bigg],\end{aligned} \tag{4.7}$$

where $x_{k+1} = f(x_k,u_k,i_k)$, for all $k \in \mathbb{N}_{[0,N-1]}$. As it will become evident in what follows, each one of the infima at stage k in (4.7) is parametric in x_k and i_k, that is, the minimization takes place over causal control laws $u_0,\ldots,u_{N-1}$.

Note that under assumptions A1 and A2, we may interchange the Markov risk measures with the infima (SDR14a, Prop. 6.60) leading to risk-averse multistage formulations discussed in (SDR14a, Sec. 6.8.4).

Problem (4.7) can be described by a dynamic programming (DP) recursion. Inspired by (SDR14a, Sec. 6.8), for a $V \in \mathbf{fcns}(\mathbb{R}^{n_x}, \mathcal{N})$ we define the DP operator $\mathbf{T} : \mathbf{fcns}(\mathbb{R}^{n_x}, \mathcal{N}) \to \mathbf{fcns}(\mathbb{R}^{n_x}, \mathcal{N})$ so that

$$\begin{aligned}(\mathbf{T}V)(x,i) &= \inf_u \{\ell(x,u,i) + \rho_i\,[V(f(x,u,i),j);j]\}\\ &= \inf_u \ell(x,u,i) + \max_{\mu\in\mathcal{A}(P_i)} \sum_{j\in\mathcal{N}} \mu_j V(f(x,u,i),j).\end{aligned}$$

Let $(\mathbf{S}V)(x,i)$ be the corresponding set of minimizers for the optimization problem involved in the definition of $(\mathbf{T}V)(x,i)$. This defines the following DP recursion

$$V_{k+1}^\star = \mathbf{T}V_k^\star, \tag{4.8a}$$

$$\mathcal{U}_{k+1}^\star = \mathbf{S}V_k^\star, \tag{4.8b}$$

for $k \in \mathbb{N}_{[0,N-1]}$ with $V_0^\star(x,i) := \ell_N(x,i)$, $i \in \mathcal{N}$. For $C = \{C_i\}_{i\in\mathcal{N}}$ with $C_i \subseteq \mathbb{R}^{n_x}$, we define the mode-dependent *predecessor operator* $R(C) = \{R_i(C)\}_{i\in\mathcal{N}}$ with

$$R_i(C) = \{x \in \mathbb{R}^{n_x} \mid \exists u \in \mathbb{R}^{n_u}, (x,u) \in Y_i, f(x,u,i) \in \bigcap_{j\in\mathcal{C}(i)} C_j.\} \tag{4.9}$$

Next, we present some fundamental properties of the DP operator $\mathbf{T}$.

Proposition 5 *If $\ell_N(\,\cdot\,,i)$ are proper, lsc and $\ell(\cdot,\cdot,i)$ are proper, lsc and level bounded in u locally uniformly in x for all $i \in \mathcal{N}$, then for all $i \in \mathcal{N}$:*

(i) $\mathbf{T}V \in \mathbf{fcns}(\mathbb{R}^{n_x}, \mathcal{N})$ *for* $V \in \mathbf{fcns}(\mathbb{R}^{n_x}, \mathcal{N})$*,*

(ii) $V_k^\star(\,\cdot\,,i)$ *are lsc,*

(iii) $\mathbf{dom}\, V_k^\star(\,\cdot\,,i) = \mathbf{dom}\, \mathcal{U}_k^\star(\,\cdot\,,i) \neq \emptyset$*,*

(iv) $\mathcal{U}_k^\star$ *is compact-valued,*

(v) $\mathbf{dom}(V_{k+1}^\star) = R(\mathbf{dom}(V_k^\star))$*.*

Proof 4.2.1 *The proof goes along the lines of (PSSB14a, Thm. 11a) using (RW11, Prop. 1.17, Prop. 1.26(a)).*

We may easily verify the monotonicity property

$$\mathbf{T}V \leq \mathbf{T}V', \tag{4.10}$$

for all V, V' with $V \leq V'$following (Ber12). An observation that will prove useful in what follows is that if $\mathbf{T}\ell_N \leq \ell_N$, then $V_{k+1}^\star \leq V_k^\star$. The above risk-averse optimal control problem leads naturally to the statement of a risk-averse MPC problem where control actions are computed by a control law $\kappa_N^\star(x,i) \in \mathcal{U}_N^\star(x,i)$. In Section 4.3 we state an appropriate risk-based notion of stability and provide conditions on ℓ_N for the MPC-controlled system $x_{k+1} = f(x_k, \kappa_N^\star(x_k,i_k), i_k)$ to be stable.

4.3 Risk-averse stability

Consider the following Markovian switching system which is controlled by some control law $u_k = \kappa(x_k, i_k)$

$$x_{k+1} = f^\kappa(x_k, i_k) := f(x_k, \kappa(x_k, i_k), i_k), \tag{4.11}$$

subject to the constraints $(x_k, i_k) \in X^\kappa := \{(x, i) \mid (x, \kappa(x, i)) \in Y_i\}$. For convenience, we introduce the notation $X_i^\kappa = \{x \mid (x, \kappa(x, i)) \in Y_i\}$, for $i \in \mathcal{N}$. Let $i_{[k]} = (i_0, i_1, \ldots, i_k)$ denote an admissible path of length k of the Markov chain $\{i_t\}_{t\in\mathbb{N}}$, that is, $i_{t+1} \in \mathcal{C}(i_t)$ for $t \in \mathbb{N}_{[0,k-1]}$. For a given initial state x_0, the solution of (4.11) at time k is denoted as $\phi(k, x_0, i_{[k-1]})$.

In order to be able to define risk-based notions of stability, we must first introduce an appropriate notion of invariance for Markovian switching systems (PSSB14a).

Definition 14 (Uniform invariance) *Let $X = \{X_i\}_{i\in\mathcal{N}}$ be a collection of nonempty closed subsets of $\mathbb{R}^{n_x}$ and $X_i \subseteq X_i^\kappa$. X is called uniformly invariant (UI) for* (4.11) *subject to constraints $x \in X_i^\kappa$ if $f^\kappa(x, i) \in \bigcap_{j\in\mathcal{C}(i)} X_j$, whenever $x \in X_i$ for all $i \in \mathcal{N}$.*

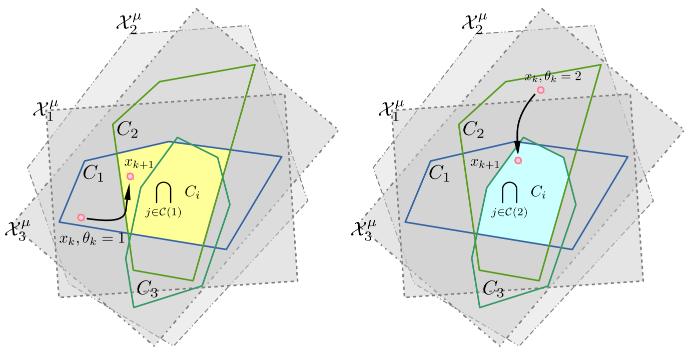

Figure 3: Illustration of the concept of uniform invariance for a Markovian system with $\nu = 3$ modes and $\mathcal{C}(1) = \{1, 2\}$ and $\mathcal{C}(2) = \{1, 2, 3\}$. (Left) If $\theta_k = 1$, x_{k+1} must be contained in $C_1 \cap C_2$, (Right) If $\theta_k = 2$, x_{k+1} must be in $C(1) \cap C(2) \cap C(3)$.

The concept of uniform invariance is illustrated in Fig. 3.

For the controlled system (4.11), the predecessor operator is now defined as $R_i(C) = \{x \in X^\kappa \mid f^\kappa(x,i) \in \bigcap_{j \in \mathcal{C}(i)} C_j\}$. We have that C is UI if and only if $C_i \subseteq R_i(C)$ for all $i \in \mathcal{N}$ (PSSB14a).

Given a coherent risk measure ρ and a random variable $\psi(i_0, i_1, \ldots, i_k)$, let $\bar{\rho}_1[\psi] = \rho_{i_0}\left[\psi(i_0, i_1, \ldots, i_k); i_1\right]$ and recursively define

$$\bar{\rho}_k = \bar{\rho}_{k-1} \circ \rho_{i_{k-1}}\left[\,\cdot\,; i_k\right]$$

, that is

$$\bar{\rho}_k[\psi] = \rho_{i_0}\left[\rho_{i_1}\left[\cdots \rho_{i_{k-1}}\left[\psi(i_0, i_1, \ldots, i_k); i_k\right] \cdots ; i_2\right]; i_1\right]$$

as explained in (SDR14a, Sec. 6.8.2). We may now give the following stability notion (CP14).

Definition 15 (Risk-square exponential stability) *We say that the origin is risk-square exponentially stable (RSES) for system* (4.11) *over a set $X = \{X_i\}_{i \in \mathcal{N}}$ if X is UI and for $x_0 \in X_{i_0}$*

$$\bar{\rho}_{k-1}\left[\|\phi(k, x_0, i_{[k-1]})\|^2\right] \leq \lambda \beta^k \|x_0\|^2,$$

for all $k \in \mathbb{N}$, for some $\beta \in [0,1)$, $\lambda \geq 0$.

RSES entails that the origin is exponentially mean-square stable for system (4.11) not only for the nominal probability distribution, but also for those probability distributions in the ambiguity set of the risk measure. In the unconstrained case, RSES corresponds to the notion of uniform global risk-sensitive exponential stabilty which is defined using the notion of dynamic risk measures (CP14). If the underlying risk measure is the expectation operator, then RSES reduces to mean-square exponential stability, whereas, if it is the essential supremum operator, it yields the definition of robust exponential stability. Additionally, since all coherent risk measures are lower bounded by the expectation, RSES is a stronger notion of stability compared to mean-square stability. The following lemma provides Lyapunov-type stability conditions for RSES.

Lemma 4.3.1 (RSES conditions) *Suppose there is a $V \in$ **fcns**$(\mathbb{R}^{n_x}, \mathcal{N})$, proper, lsc function such that* *(i)* $\mathbf{dom}\, V$ *is a UI set*

(ii) $\rho_i \left[V(f^\kappa(x,i),j);j\right] - V(x,i) \leq -c\|x\|^2$*, for some* $c > 0$ *for all* $(x,i) \in \mathbf{dom}\, V$.

Then, $\bar{\rho}_{k-1}\left[\sum_{t=0}^{k} \|\phi(t,x_0,i_{[t-1]})\|^2\right]$*, is uniformly bounded in* k *for* $(x_0,i_0) \in \mathbf{dom}\, V$*. If, additionally,*

(iii) *for all* $(x,i) \in \mathbf{dom}\, V$*,* $\alpha_1\|x\|^2 \leq V(x,i) \leq \alpha_2\|x\|^2$*, for some* $\alpha_1, \alpha_2 > 0$*,*

then, the origin is RSES for system (4.11) *over* $\mathbf{dom}\, V$.

Proof 4.3.2 *Proof of Lemma 4.3.1. Define* $V_k := V(x_k, i_k)$ *and, for fixed* $x_0 \in \mathbf{dom}\, V_N^\star(\cdot, i_0)$ *let* $x_t := \phi(t, x_0, i_{[t-1]})$*. We have*

$$\begin{aligned}
\bar{\rho}_k\Big[V_k - V_0 + c\sum_{t=0}^{k-1}\|x_t\|^2\Big] &= \bar{\rho}_k\Big[\sum_{t=0}^{k-1} V_{t+1} - V_t + c\|x_t\|^2\Big] \\
&\leq \sum_{t=0}^{k-1}\bar{\rho}_k\left[V_{t+1} - V_t + c\|x_t\|^2\right] \\
&= \sum_{t=0}^{k-1}\bar{\rho}_{t+1}\left[V_{t+1} - V_t + c\|x_t\|^2\right],
\end{aligned} \tag{4.12}$$

where the inequality is because of the subadditivity of ρ *(A1 and A4) and the last equality is because* $V_{t+1} - V_t + c\|x_t\|^2$ *is independent of* $i_{t+2}, \ldots, i_k$*. In light of Cond. (ii) and given that*

$$\begin{aligned}
&\bar{\rho}_{t+1}[V_{t+1} - V_t + c\|x_t\|^2] = \rho_{i_0} \circ \ldots \circ \rho_{i_t}\left[V_{t+1} - V_t + c\|x_t\|^2; i_{t+1}\right] \\
&= \rho_{i_0} \circ \ldots \circ \rho_{i_t}\left[V(f^\kappa(x_t,i_t), i_{t+1}) - V(x_t,i_t) + c\|x_t\|^2; i_{t+1}\right] \leq 0,
\end{aligned}$$

and because of (4.12) *and property A2 we have that*

$$\bar{\rho}_k\left[-V_0 + c\sum_{t=0}^{k-1}\|x_t\|^2\right] \leq \bar{\rho}_k\left[V_k - V_0 + c\sum_{t=0}^{k-1}\|x_t\|^2\right] \leq 0.$$

Using properties A3 and A4 gives

$$\bar{\rho}_k\left[\sum_{t=0}^{k-1}\|x_t\|^2\right] \leq {}^{V_0}/_{c}$$

which proves the first part of Lemma 4.3.1.

By Cond. (ii),

$$\rho_{i_k}\left[V_{k+1} - V_k; i_{k+1}\right] \leq -c\|x_k\|^2 \leq -c\alpha_2^{-1} V_k \leq -\eta V_k$$

for some $\eta \in (0, 1)$*, so*

$$\rho_{i_k}\left[V_{k+1}; i_{k+1}\right] \leq \beta V_k$$

with $\beta := 1 - \eta \in (0, 1)$*. We have* $\rho_{i_0}\left[V_1; i_1\right] \leq \beta V_0$ *and* $\rho_{i_1}\left[V_2; i_2\right] \leq \beta V_1$*, so* $\rho_{i_0}\left[\rho_{i_1}\left[V_2; i_2\right]; i_1\right] \leq \beta \rho_{i_0}\left[V_1; i_1\right] \leq \beta^2 V_0$*. Then,* $\bar{\rho}_2[V_2] \leq \beta^2 V_0$ *and recursively*

$$\bar{\rho}_k\left[V_k\right] \leq \beta^k V_0. \tag{4.14}$$

By the left hand side of Cond. (iii), $\|x_k\|^2 \leq {}^{1}/_{\alpha_1} V_k$ *and applying* $\bar{\rho}_k$ *and using* (4.14) *and, subsequently the right hand side of Cond. (iii),*

$$\bar{\rho}_k(\|x_k\|^2) \leq \bar{\rho}_k({}^{V_k}/_{\alpha_1}) \leq \tfrac{1}{\alpha_1}\bar{\rho}_k(V_k) \leq \tfrac{1}{\alpha_1}\beta^k V_0 \leq \tfrac{\alpha_2}{\alpha_1}\beta^k \|x_0\|^2.$$

□

The uniform boundedness condition in Lemma 4.3.1 is reminiscent of the notion of stochastic stability in (CFM05, Sec. 3.3.1). In fact, if the risk measure in Lemma 4.3.1 is the expectation operator, then the uniform boundedness condition is equivalent to mean-square stability (CFM05, Thm. 3.9(6)).

We call a function $V \in \mathbf{fcns}(\mathbb{R}^{n_x}, \mathcal{N})$ which satisfies all requirements of Lemma 4.3.1, a (mode-dependent) *risk-averse Lyapunov function*. We may now state conditions on the stage cost ℓ and the terminal cost ℓ_N which entail RSES for the risk-averse MPC-controlled system.

4.4 Risk-averse MPC

Theorem 4.4.1 (RSES of MPC) *Suppose that*

(i) $c\|x\|^2 \leq \ell(x, u, i)$ *for some* $c > 0$ *for all* $(x, u) \in Y_i$*,* $i \in \mathcal{N}$

(ii) $\ell_N(x, i) \leq d\|x\|^2$*, for some* $d > 0$ *for all* $x \in X_i^f$*,*

(iii) X_i^f *contain the origin in their interiors*

(iv) $V_N^\star$ *is locally bounded over its domain, that is, for every compact set* $\bar{X} \subseteq \mathbf{dom}\, V_N^\star$*, there is an* $M \geq 0$ *so that* $V_N^\star(x,i) \leq M$ *for all* $(x,i) \in \bar{X}$ *and*

$$\mathbf{T}\ell_N \leq \ell_N. \tag{4.15}$$

Then, the origin is RSES for the risk-averse MPC-controlled system $x_{k+1} = f(x_k, \kappa_N^\star(x_k, i_k), i_k)$ *over all compact uniformly invariant subsets of* $\mathbf{dom}\, V_N^\star$*.*

Proof 4.4.2 *Proof of Theorem 4.4.1. Let* $\bar{X} \subseteq \mathbf{dom}\, V_N^\star$ *be a compact UI set. By* (4.8)*,*

$$V_N^\star(x,i) = \rho_i\left[V_{N-1}^\star(f^{\kappa_N^\star}(x,i),j);j\right] + \ell(x, \kappa_N^\star(x,i), i).$$

Then, for $(x,i) \in \bar{X}$*,*

$$\begin{aligned}
&\rho_i\left[V_N^\star(f^{\kappa_N^\star}(x,i),j);j\right] - V_N^\star(x,i)\\
={}& \rho_i\left[V_N^\star(f^{\kappa_N^\star}(x,i),j);j\right] - \ell(x,\kappa_N^\star(x,i),i)\\
&- \rho_i\left[V_{N-1}^\star(f^{\kappa_N^\star}(x,i),j);j\right]\\
\leq{}& -\ell(x,\kappa_N^\star(x,i),i) \leq -c\|x\|^2.
\end{aligned}$$

The first inequality is because $V_N^\star \leq V_{N-1}^\star$ *and property A2. We have that*

$$V_N^\star(x,i) \leq \ell_N(x,i) \leq d\|x\|^2$$

for all $x \in X_i^f$*. Because of Cond. (iii), we may find* $\epsilon > 0$ *such that* $\mathcal{B}_\epsilon \subseteq X_i^f$*, for* $i \in \mathcal{N}$*. By Cond. (iv), there is an* $M > d\epsilon^2$*. Then, for all* $x \in \bar{X}_i \setminus X_i^f$*,*

$$\frac{M}{\epsilon^2}\|x\|^2 \geq M \geq V_N^\star(x,i).$$

Because of Cond. (i) and the definition of $\mathbf{T}$*, we have that* $V_t^\star(x,i) \geq c\|x\|^2$ *for all* $(x,i) \in \mathbf{dom}\, V_t^\star$ *for* $t \in \mathbb{N}_{[1,N]}$*. The proof is complete since* $V = V_N^\star + \delta_{\bar{X}}$ *satisfies all conditions of Lemma 4.3.1.* □

In Thm. 4.4.1 we show that $V_N^\star$ is a mode-dependent risk-averse Lyapunov function over compact uniformly invariant subsets of $\mathbf{dom}\, V_N^\star$. We shall use this result in the following sections to design risk-averse stabilizing MPC controllers for MJLS as well as nonlinear Markovian switching systems. Note that Condition (iv) in Thm. 4.4.1 holds if the following assumption is satisfied (see (RM09b, Prop. 2.15))

Assumption 4.4.3 (Local boundedness of $V_N^\star$) *For all $i \in \mathcal{N}$, functions $\ell(\cdot,\cdot,i)$ and $\ell_N(\cdot,i)$ are continuous on their domains, and the sets $U_i(x) := \{u \in \mathbb{R}^{n_u} \mid (x,u) \in Y_i\}$ are compact and bounded uniformly in x.*

Additionally, because of the monotonicity property of $\mathbf{T}$ and since $\mathbf{T}\ell_N \leq \ell_N$, condition (4.15) implies $V_{k+1}^\star \leq V_k^\star$, thus $\mathbf{dom}(V_k^\star) \subseteq \mathbf{dom}(V_{k+1}^\star) = R(\mathbf{dom}\, V_k^\star)$ (Prop. 5), thus $\mathbf{dom}\, V_k^\star$ is UI.

4.4.1 Risk-averse MPC design for MJLS

Here we provide RSES conditions and design guidelines for risk-averse MPC of MJLS (CFM05), that is $f(x,u,i) = A_i x + B_i u$, using a quadratic stage cost $\ell(x,u,i) = x^\top Q_i x + u^\top R_i u + \delta_{Y_i}(x,u)$, with $Q_i \in \mathcal{S}_{+}^{n_x}$, $R_i \in \mathcal{S}_{++}^{n_u}$ and Y_i are polytopes with the origin in their interiors. The terminal cost function is taken to be $\ell_N(x,i) = x^\top P_i^f x + \delta_{X_i^f}(x)$ with $P_i^f \in \mathcal{S}_{++}^{n_x}$ and X_i^f. We shall derive conditions on P_i^f and X_i^f so that the stabilizing conditions of Thm. 4.4.1 are satisfied.

Condition $\mathbf{T}\ell_N \leq \ell_N$ is equivalent to

$$\min_u \{x^\top Q_i x + u^\top R_i u + \rho_i \left[x^{+\top} P_j^f x^+; j\right]\} \leq x^\top P_i^f x, \tag{4.16a}$$

$$\mathbf{dom}(\mathbf{T}\ell_N) \supseteq \mathbf{dom}\,\ell_N \Leftrightarrow R(X^f) \supseteq X^f, \tag{4.16b}$$

where $x^+ = f(x,u,i)$ and the minimization in (4.16a) is over the space of admissible causal control laws $u = \kappa(x,i)$ so that $(x,i) \in X^\kappa$. An upper bound to the left hand side of (4.16a) is obtained by parametrizing

$$u = K_i x.$$

We introduce the shorthand notation

$$\bar{A}_i = A_i + B_i K_i$$

and

$$\bar{Q}_i = Q_i + K_i^\top R_i K_i$$

, for $i \in \mathcal{N}$. We use the fact that the stabilizing condition $\mathbf{T}\ell_N \leq \ell_N$ is satisfied if $\mathbf{T}_\kappa \ell_N \leq \ell_N$.

Condition (4.16b) means that X^f is a UI set for the system $x_{k+1} = (A_{i_k} + B_{i_k} K_{i_k}) x_k$ under the prescribed constraints. Such a set can be determined by the fixed-point iteration $\mathcal{O}_{k+1} = R(\mathcal{O}_k)$ with $\mathcal{O}_0 = \{(x,i) \mid (x, K_i x) \in Y_i\}$. If this iteration converges in a finite number of iterations — a sufficient condition for which is given in (PSSB14a, Lem. 21) — to a set $\mathcal{O}_\infty$, this is a *polytopic* UI set.

Assuming that ρ is a polytopic Markov risk measure with ambiguity set $\mathcal{A}(P_i) = \mathbf{conv}\{\mu_i^{(l)}\}_{l \in \mathbb{N}_{[1,s_i]}}$ and using its dual representation, condition (4.16a) becomes $\bar{Q}_i + \sum_{j \in \mathcal{C}(i)} \mu_{ij}^{(l)} \, (\bar{A}_i^\top P_j^f \bar{A}_i) \preccurlyeq P_i^f$ for all $i \in \mathcal{N}$ and

$l \in \mathbb{N}_{[1,s_i]}$. This condition can be cast as a linear matrix inequality (LMI) by a change of variables

$$(P_i^f)^{-1} = M_i, \quad K_i = Y_i M_i^{-1}, \quad F_i^l = \left[\sqrt{\mu_{i1}^{(l)}} I \dots \sqrt{\mu_{in}^{(l)}} I\right]$$

and $M = \mathbf{blkdiag}(M_1, \dots, M_n)$:

$$\begin{bmatrix} M_i & (A_i M_i + B_i Y_i)^\top F_i^l & M_i Q_i^{1/2} & Y_i^\top R_i^{1/2} \\ * & M & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} \succcurlyeq 0, \tag{4.17}$$

for all $i \in \mathcal{N}$ and $l \in \mathbb{N}_{[1,s_i]}$. The left hand side of (4.17) is a symmetric matrix, therefore, we show only its upper block triangular part and replaced the lower block triangular part by asterisks ($*$) to simplify the notation. Solving this LMI for $M_i \in \mathcal{S}_+^{n_x}$ and $Y_i \in \mathbb{R}^{n_u \times n_x}$ yields the linear gains K_i and the cost matrices P_i^f. Note that LMI (4.17) has to be solved only once and offline to determine matrices P_i^f.

4.4.2 Risk-averse MPC design for nonlinear Markovian switching systems

For nonlinear systems, an obvious choice for the terminal cost function would be $\ell_N(x,i) = \delta_{\{0\}}(x)$ — meaning, $X_i^f = \{0\}$ for $i \in \mathcal{N}$ — but that would lead to a very conservative design. Here we exploit the system linearization at the origin to determine a terminal cost function and terminal constraints which render the origin RSES for the MPC-controlled system. We first draw the following assumption for the nonlinear dynamics:

Assumption 4.4.4 *For each $i \in \mathcal{N}$, $f(\cdot,\cdot,i)$ is differentiable with L_i-Lipschitz Jacobian matrix.*

We use a parametric controller of the form $\kappa(x,i) = K_i x$ and define the associated closed-loop function $f^\kappa(x,i) = f(x, K_i x, i)$, $i \in \mathcal{N}$. Function $f^\kappa(\,\cdot\,,\,\cdot\,,i)$ can be written as a composition of $f(\,\cdot\,,\,\cdot\,,i)$ with the linear mapping $W_i : (x,u) \mapsto (x, K_i x)$, therefore, its Jacobian matrix will be Lipschitz-continuous with Lipschitz constant $L_i\|W_i\|^2$ which is bounded above by

$$\beta_i := L_i(1 + \|K_i\|^2). \tag{4.18}$$

The linearization of the nonlinear system at the origin is an MJLS

$$x_{k+1} = \hat{f}(x_k, u_k, i_k) := A_{i_k} x_k + B_{i_k} u_k$$

with A_{i_k} and B_{i_k} given by the Jacobian matrices, with respect to x and u respectively, of f at the origin. That is, $A_i = J_x f(0,0,i)$, $B_i = J_u f(0,0,i)$. For notational convenience, we define the following quantities

$$\begin{aligned}
\hat{f}^\kappa(x,i) &:= (A_i + B_i K_i)x = \bar{A}_i x,\\
\mathcal{L}\ell_N(x,i) &:= \rho_i\left[\ell_N(f^\kappa(x,i),j);j\right] - \ell_N(x,i),\\
\mathcal{L}'\ell_N(x,i) &:= \rho_i\left[\ell_N(\hat{f}^\kappa(x,i),j);j\right] - \ell_N(x,i),\\
\Delta(x,i) &:= \mathcal{L}\ell_N(x,i) - \mathcal{L}'\ell_N(x,i).
\end{aligned}$$

The objective is to design the terminal cost and terminal constraints for the risk-averse MPC problem using $\mathcal{L}'\ell_N$ to yield an LMI. While our design will be based on the linearized dynamics, we need to account for the linearization error. To this end, we shall derive a quadratic upper bound for $|\Delta(x_k, i_k)|$ over X^f.

Theorem 4.4.5 *Suppose that Assumptions 4.4.3 and 4.4.4 hold, the terminal cost has the form* $\ell_N(x, i) = x^\top P_i^f x + \delta_{X_i^f}(x)$ *with* $P_i^f \in \mathcal{S}_{++}^{n_x}$ *and for some* $\bar{Q}_i \in \mathcal{S}_{+}^{n_x}$

$$\mathcal{L}'\ell_N(x, i) \leq -x^\top(\bar{Q}_i + m_i I)x, \tag{4.19}$$

for $i \in \mathcal{N}$*,* $m_i > 0$*. Suppose* ℓ *and* X^f *satisfy the requirements of Thm. 4.4.1 and* $X_i^f \subseteq \mathcal{B}_{\delta_i}$ *for some* $\delta_i > 0$*,* $i \in \mathcal{N}$*,*

$$\sigma_i := \max_{j \in \mathcal{C}(i)} \|P_j^f\| \left(\tfrac{\beta_i^2 \delta_i^2}{4} + \beta_i \|\bar{A}_i\| \delta_i\right) < m_i, \tag{4.20}$$

and

$$\ell(x, K_i x, i) \leq x^\top(\bar{Q}_i + (m_i - \sigma_i)I)x$$

for $x \in X_i^f$*. If* X^f *is a UI set for* (4.11)*, then the origin is RSES for the MPC-controlled system*

$$x_{k+1} = f(x_k, \kappa_N^\star(x_k, i_k), i_k)$$

over the compact UI subsets of $\mathbf{dom}\, V_N^\star$*.*

Proof 4.4.6 *Proof of Theorem 4.4.5. Define*

$$e(x, i) = f^\kappa(x, i) - \hat{f}^\kappa(x, i).$$

By Assumption 4.4.4 and since $f^\kappa(0, i) = 0$ *for all* $i \in \mathcal{N}$*,*

$$\|e(x, i)\| \leq {}^{\beta_i}\!/_2 \|x\|^2.$$

It is

$$\Delta(x, i) = \rho_i \left[f^\kappa(x, i)^\top P_j^f f^\kappa(x, i); j\right] - \rho_i \left[x^\top \bar{A}_i^\top P_j^f \bar{A}_i x; j\right].$$

Since $\rho_i[\,\cdot\,]$ *is convex and monotone, it is nonexpansive with respect to the infinity norm (SDR14a, p. 302), thus for* $x \in X_i^f$

$$\begin{aligned}
|\Delta(x, i)| &\leq \max_{j \in \mathcal{C}(i)} |f^\kappa(x, i)^\top P_j^f f^\kappa(x, i) - x^\top \bar{A}_i^\top P_j^f \bar{A}_i x| \\
&= \max_{j \in \mathcal{C}(i)} |e(x, i)^\top P_j^f e(x, i) + 2x^\top \bar{A}_i^\top P_j^f e(x, i)| \\
&\leq \max_{j \in \mathcal{C}(i)} \|P_j^f\| \left(\tfrac{\beta_i^2}{4}\|x\|^4 + \beta_i \|\bar{A}_i\| \|x\|^3\right) \leq \sigma_i \|x\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}\ell_N(x,i) &= \mathcal{L}'\ell_N(x,i) + \Delta(x,i)\\ &\leq -x^\top(\bar{Q}_i + (m_i - \sigma_i))x \leq -\ell(x,\kappa(x,i),i),\end{aligned}$$

for all $x \in X_i^f$ and since X^f is UI, $\mathbf{T}\ell_N \leq \ell_N$. The assertion follows from Thm. 4.4.1. □

According to Thm. 4.4.5, one first needs to select $m_i > 0$ for each $i \in \mathcal{N}$ such that (4.19) holds true. In the common case where

$$\ell(x,u,i) = x^\top Q_i x + u^\top R_i u + \delta_{Y_i}(x,u),$$

with $Q_i \in \mathcal{S}_+^{n_x}$, $R_i \in \mathcal{S}_{++}^{n_u}$, (4.19) is an LMI of the form (4.17) with $Q_i + m_i I$ in place of Q_i solving which we obtain matrices K_i and P_i^f and determine the constants β_i and find $\delta_i > 0$ so that (4.20) holds. In that case, $\ell(x, K_i x, i) \leq x^\top(\bar{Q}_i + (m_i - \sigma_i)I)x$ is immediately satisfied with $\bar{Q}_i = Q_i + K_i^\top R_i K_i$. In order to determine a UI set X^f for the nonlinear system $x_{k+1} = f^\kappa(x_k, i_k)$ we may cast the nonlinear system as a linear one with bounded additive disturbance $x_{k+1} = \bar{A}_{i_k} x_k + e(x_k, i_k)$ — indeed, as we show in the proof of Thm. 4.4.5, $\|e(x,i)\| \leq \beta_i/2 \|x\|^2$. We may follow the approach of (SC15) in order to determine a polytopic robustly invariant set.

4.5 Computationally tractable formulation of risk-averse optimal control problems

Starting from an initial state x_0 and initial mode i_0 and computing control actions according to a causal control law u_k, the future states of the Markovian system, up to some future time N, span a *scenario tree* — a tree-like structure such as the one shown in Fig. 4. Note that the state at a node ι, the input and mode leading to that node are denoted as x^ι, u^ι and i^ι respectively.

The possible realizations of the system state at time k define the *nodes* of the tree. The set of all nodes at stage k defines the set Ω_k. The set of nodes in Ω_{k+1} which are reachable from a node $\iota \in \Omega_k$ is called the set of *children* of ι and is denoted by $\mathbf{ch}(\iota)$ which is a subset of Ω_{k+1}. The space $\mathbf{ch}(\iota)$ becomes a probability space with $\mathrm{P}[\{\eta\}] = p_{i^\iota i^\eta}$ for $\eta \in \mathbf{ch}(\iota)$. As illustrated in Fig. 4, the system dynamics on the scenario tree is described by $x^\eta = f(x^\iota, u^\eta, i^\eta)$, for $\eta \in \mathbf{ch}(\iota)$ and $x^0 = x_0$, $i^1 = i_0$.

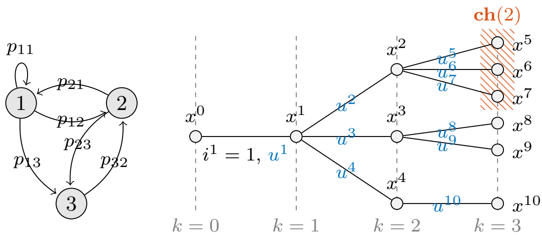

Figure 4: (Left) A Markov chain with three modes and the corresponding transition probabilities, (Right) The corresponding tree with $i_0 = 1$.

On the scenario tree, we define a process Φ as follows: for $\iota \in \Omega_N$ we define $\Phi^\iota := \rho_{i^\iota}\left[\ell_N(x^\iota, i^\eta); \eta\right] = \mathbf{max}_{\mu^\iota \in \mathcal{A}(P_{i^\iota})} \sum_{\eta \in \mathbf{ch}(\iota)} \mu^\iota_\eta \ell_N(x^\iota, i^\eta)$. Moreover, $\ell_N(x, i) = \mathbf{inf}_{\ell_N(x,i) \leq \tau} \tau$. When the underlying risk measure is polytopic with $\mathcal{A}(p) = \{\mu \in \mathbb{R}^n \mid \sum_{i=1}^n \mu_i = 1, F(p)\mu \leq b(p)\}$ with

$b(p) \in \mathbb{R}^q$, then

$$\begin{aligned}
\Phi^\iota &= \max_{\mu^\iota \in \mathcal{A}(P_{i^\iota})} \inf_{\substack{\ell_N(x^\iota, i^\eta) \leq \tau^\iota_\eta, \\ l \in \mathbf{ch}(\iota)}} \sum_{\eta \in \mathbf{ch}(\iota)} \mu^\iota_\eta \tau^\iota_\eta \\
&= \inf_{\substack{\ell_N(x^\iota, i^\eta) \leq \tau^\iota_\eta, \\ l \in \mathbf{ch}(\iota)}} \max_{\mu^\iota \in \mathcal{A}(P_{i^\iota})} \sum_{\eta \in \mathbf{ch}(\iota)} \mu^\iota_\eta \tau^\iota_\eta \\
&= \inf_{\substack{\tau^\iota, y^\iota \geq 0, \lambda^\iota \in \mathbb{R}, \\ \ell_N(x^\iota, i^\eta) \leq \tau^\iota_\eta, l \in \mathbf{ch}(\iota) \\ \tau^\iota = F(P_{i^\iota})^\top y^\iota + \lambda^\iota 1_q}} b(P_{i^\iota})^\top y^\iota + \lambda^\iota,
\end{aligned}$$

where in the first equation we interchanged $\max$ with $\inf$ using (? , Prop. 2.6.4) using the fact that the level sets of the mapping

$$\tau^\iota \mapsto \max_{\mu^\iota \in \mathcal{A}(P_{i^\iota})} \sum_{\eta \in \mathbf{ch}(\iota)} \mu^\iota_\eta \tau^\iota_\eta$$

are bounded because $\mathcal{A}(P_{i^\iota})$ is compact. The last equality is because of LP duality. Traversing indices k from $N-1$ back to 1, we define

$$\Phi^\iota := \rho_{i^\iota} \left[\ell(x^\iota, u^\eta, i^\eta) + \Phi^\eta; \eta \right],$$

which boils down to

$$\Phi^\iota = \inf_{\substack{\tau^\iota, y^\iota \geq 0, \lambda^\iota \in \mathbb{R} \\ \ell(x^\iota, u^\eta, i^\eta) + \Phi^\eta \leq \tau^\iota_\eta, l \in \mathbf{ch}(\iota) \\ \tau^\iota = F(P_{i^\iota})^\top y^\iota + \lambda^\iota 1_q}} b(P_{i^\iota})^\top y^\iota + \lambda^\iota,$$

for $\iota \in \Omega_k$. This formulation allows us to deconvolve the nested Markov risk measures. Indeed, $V^\star_N(x_0, i_0)$ is the optimal value of the following minimization problem

$$\begin{aligned}
\operatorname*{\mathbf{minimize}}_{x,u,y \geq 0, \lambda, \tau} \quad & \ell(x_0, u^1, i_0) + b(P_{i^1})^\top y^1 + \lambda^1 \\
\textbf{subject to} \quad & \ell_N(x^\iota, i^\eta) \leq \tau^\iota_\eta, \eta \in \mathbf{ch}(\iota), \iota \in \Omega_N, \\
& \tau^\iota = F(P_{i^\iota})^\top y^\iota + \lambda^\iota 1_q, \\
& \ell(x^\iota, u^\eta, i^\eta) + b(P_{i^\eta})^\top y^\eta + \lambda^\eta \leq \tau^\iota_\eta, \\
& x^\eta = f(x^\iota, u^\eta, i^\eta), \\
& \eta \in \mathbf{ch}(\iota), \iota \in \Omega_k, k \in \mathbb{N}_{[0,N]}.
\end{aligned}$$

Note that this formulation does not require the enumeration of the vertices of $\mathcal{A}(p)$ which, for instance, in the case of $\mathrm{AV@R}_\alpha$ increases exponentially with the number of modes.

The above optimization problem is solved at every time instant with x_0, i_0 being the current state and mode of the system. Solving this problem yields the optimal control actions $u^{\iota\star}$ at each node of the scenario tree. The first value, $u^{1\star}$, defines the risk-averse MPC controller $\kappa_N^\star(x,i) = u^{1\star}(x,i)$. Note that in the particular case of an MJLS where stage-wise and terminal costs are quadratic and the constraints are polyhedral and/or ellipsoidal, we obtain a QCQP which can be solved very efficiently online as we show in Section 4.6. The above reformulation can be applied to risk measures whose ambiguity set is described by a set of conic inequalities (using conic duality) such as the entropic value-at-risk.

4.6 Numerical examples

In this chapter we will present numerical simulations that highlight the advantages and properties of risk-averse MPC approach. Risk-averse control comes with somewhat higher computational footprint than stochastic or worst-case approach but the advantages are higher robustness of the overall approach than the stochastic approach and lower conservativeness than worst-case approach.

4.6.1 Samuelson's economic model

In this section we apply risk-averse model predictive control to a well-studied MJLS: Samuelson's multiplier-accelerator macroeconomic model (BS75). The system has three operating modes described with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -4.3 & 4.5 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 1 \\ 5.3 & -5.2 \end{bmatrix},$$

and

$$B_1 = B_2 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and mode-dependent polyhedral constraints with

$$Y_1 = [-10, 10]^3, Y_2 = [-8, 8]^2 \times [-10, 10], Y_3 = [-12, 12]^2 \times [-10, 10].$$

Stage costs are parametrized with

$$Q_1 = \begin{bmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{bmatrix},\ Q_2 = \begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix},\ Q_3 = \begin{bmatrix} 5 & -4.5 \\ -4.5 & 5 \end{bmatrix},$$

and R_1=2.6, R_2=1.165, R_3=1.111. The estimated transition matrix Π and the actual transition matrix Π' are

$$\Pi = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{bmatrix},\ \Pi' = \begin{bmatrix} 0.34 & 0.34 & 0.32 \\ 0.3 & 0.3 & 0.4 \\ 0.52 & 0.2 & 0.28 \end{bmatrix}.$$

Risk-averse MPC controllers for different value of α were designed as discussed in Section 4.4.1.

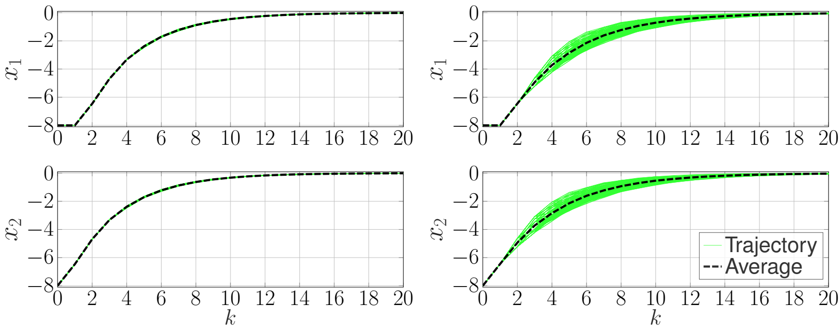

Figure 5: (Left) Trajectories of the closed-loop system with risk-averse MPC for $N = 6$ with $\alpha = 0.1$ and (Right) $\alpha = 0.5$. The thin green lines correspond to 1000 random simulations.

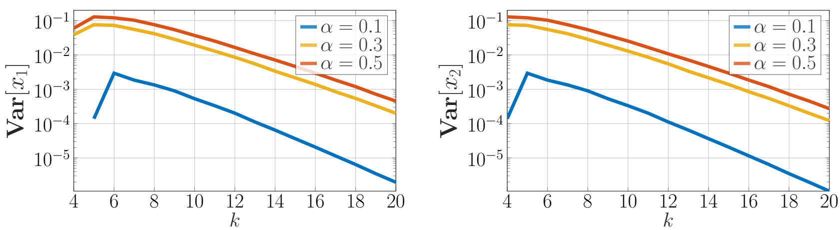

Figure 6: Estimated variance of x_k over 10^4 runs. (Left) first coordinate of the system state, (Right) second coordinate.

As we may observe in Fig. 5, lower values of α incur a lower risk behavior. This is also show in Fig. 6. In order to further assess the quality of the closed-loop behavior of the controlled system with different values of α, we compute

$$J^{\alpha} = \tfrac{1}{N_s} \sum_{k=0}^{N_s-1} \ell(x_k, u_k, \theta_k)$$

over a simulation horizon $N_s = 50$ for 10^4 random runs and we present the histogram of J^{α} in Fig. 7. Although lower values of α lead to a safer operation which can withstand higher risk, it comes at a higher operation cost. Average and maximum computation times for solving the risk-averse optimal control problem are given in Table 1 for $\alpha = 0.1$ (similar

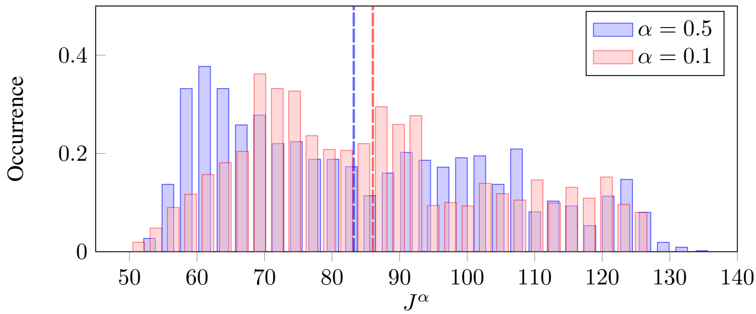

Figure 7: Histogram of J^{α} generated with 10^4 runs for $\alpha = 0.1$ and $\alpha = 0.5$. The dashed vertical lines indicate the average of J^{α}: For $\alpha = 0.1$, the average cost is 86.0, while for $\alpha = 0.5$ it is 83.2.

Table 1: Runtime for different problem sizes for $\alpha = 0.1$.

N	scenarios	mean [s]	max [s].
5	81	0.047	0.052
6	243	0.078	0.084
7	729	0.25	0.29
8	2187	0.89	0.93
9	6561	2.81	2.89

results are obtained for different values of α).

4.6.2 Risk-averse problem in (CSMP17)

In this example we apply the reformulation presented in Section 4.5 on the problem taken from (CSMP17, Sec. X.A). The system is again an MJLS with $\nu = 6$ modes and ellipsoidal state-input constraints. Even though the formulation of the MPC problem is different, as briefly discussed in Section 4.2.2, our reformulation of the optimization problem as a QCQP can be readily applied. Note that scenario tree used in (CSMP17) is the one we present in the next chapter.

Note that here we assume that we can not measure the disturbance at

Table 2: Runtime for different problem sizes on a toy problem taken from (CSMP17) for $\alpha = 0.75$. Average and maximum runtimes are taken over 100 runs for $N = 3, 4, 5$ and over 10 runs for $N = 6$.

N	scenarios	mean [s]	max [s].
3	216	0.02	0.08
4	1296	0.16	0.39
5	7776	1.06	1.65
6	46656	8.47	13.31

time k, hence system dynamics is given by

$$x_{k+1} = A(w_k)x_k + B(w_k)u_k \tag{4.21}$$

for $w_k \in \mathbb{N}_{[1,6]}$, i.e. the system will evolve according to one of six different dynamics, each one happening with probaility of $p_i = 1/6$.

We start from the initial point $x_0 = (2.5, 2.5)$ as sugggested in the orignal paper and apply the MPC control law for randomly generated switching paths. Statistics regarding the time to compute a control action running the risk-averse MPC are reported in Table 2.

Both Sameulson's economic model example and this one suggest that the runtime scales roughly linearly with the number of scenarios. Additionally, these runtimes are lower compared to those reported in (CSMP17).

All simulations were performed in MATLAB using YALMIP (Löf04) as a modeling language with the MOSEK solver (MOS16) and were executed on an Intel i5-6200U CPU at 2.30GHz with 8GB RAM running Ubuntu 16.04.

4.6.3 Resilience to actuator cyber-attacks

The purpose of this example is to demonstrate the effect of inexact knowledge of the probability distribution on the stability properties of the controlled system. Suppose that an attacker tries to alter the normal mode of operation of a system by disconnecting an actuator. We may model this as a Markovian system with two modes: $i = 1$ corresponds to normal operation and $i = 2$ corresponds to a successful attack. Suppose we have obtained the following approximate transition matrix from measurements

$$\Pi = \begin{bmatrix} 0.97 & 0.03 \\ 0.03 & 0.97 \end{bmatrix},$$

and the system dynamics is linear and described by

$$A_1 = A_2 = \begin{bmatrix} -0.4 & 0.3 \\ 5 & 0.1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

According to the above transition matrix, the attacker has probability 3% to deactivate an actuator and upon a successful attack, the system has 3% probability to recover. Suppose, however, that the attacker has 3.5% probability to gain access to an actuator, that is, the actual — and unknown — probability distribution is described by

$$\Pi' = \begin{bmatrix} 0.965 & 0.035 \\ 0.03 & 0.97 \end{bmatrix}.$$

For $Q_1 = Q_2 = R_1 = R_2 = I$, the gain matrices

$$K_1 = \begin{bmatrix} 0.28 & 1.7 \\ -5.2 & 0.29 \end{bmatrix}, K_2 = \begin{bmatrix} 0.4 & -0.3 \\ 0 & 0 \end{bmatrix},$$

and the matrices

$$P_1 = \begin{bmatrix} 22.5568 & 0.5563 \\ 0.5563 & 1.957 \end{bmatrix}, P_2 = \begin{bmatrix} 127.7554 & 7.7174 \\ 7.7174 & 4.2507 \end{bmatrix},$$

satisfy the stability Lyapunov condition (4.17) for $\alpha = 1$, therefore, stochastic MPC stabilizes the controlled system in the mean-square sense provided that the probability transition matrix is equal to Π. However, stochastic MPC in practice will fail to lead to a mean square stable closed loop as shown in Fig. 8.

Next, we design a risk-averse model predictive controller with $\alpha = 0.9$ which encompasses the transition matrix Π'. The risk-averse controller stabilizes the system in the mean-square sense as shown in Fig. 8 (Right). Using the matrix inequality (4.17) we compute the gains

$$K_1 = \begin{bmatrix} -0.4515 & -0.3112 \\ -3.9806 & -0.0654 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.4949 & -0.1910 \\ 0 & 0 \end{bmatrix}$$

and the terminal penalty matrices

$$P_1 = \begin{bmatrix} 25.5573 & 0.4948 \\ 0.4948 & 1.9502 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 184.27 & 11.1038 \\ 11.1038 & 4.6103 \end{bmatrix}.$$

This example provides clear motivation for risk-averse control as it demonstrates potential vulnerabilities of risk-neutral stochastic MPC formulations and it mitigates the lack of exact probabilistic information.

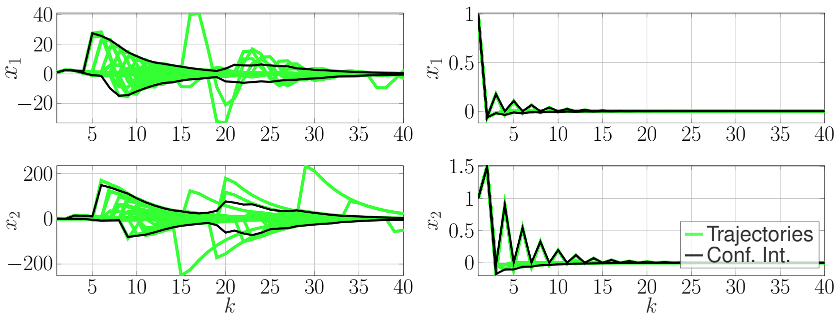

Figure 8: (Left) Closed-loop simulations with the SMPC controller starting from the initial point $x_0 = [1, 1]'$ and $\theta_0 = 1$ — 10^4 random runs. (Right) Closed-loop simulations with the risk-averse MPC controller with $\alpha = 0.9$. The black lines denote the bounds of the 99.9% confidence interval.

4.6.4 Nonlinear system

Here we demonstrate the design of stabilizing risk-averse MPC controllers for a nonlinear system. We consider the following nonlinear Markovian switching system with three modes:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A_{i_k} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + c_{i_k} \begin{bmatrix} 1 - e^{y_k} \\ 1 - e^{x_k} \end{bmatrix} + B_{i_k} u_k. \tag{4.22}$$

The system matrices are

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.1 & -0.6 \\ 0.6 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1.6 \\ 0.6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and parameters $c_1{=}0.2$, $c_2{=}-0.1$, $c_3{=}-0.3$. Stage-wise cost matrices are $Q_i = I$ and $R_i = 100 \cdot i$ for $i \in \{1, 2, 3\}$. The nominal and actual transition matrices are given by

$$P = \begin{bmatrix} 0.4 & 0.0 & 0.6 \\ 0.6 & 0.0 & 0.4 \\ 0.4 & 0.6 & 0.0 \end{bmatrix}, \quad P' = \begin{bmatrix} 0.33 & 0.0 & 0.67 \\ 0.56 & 0.0 & 0.44 \\ 0.33 & 0.67 & 0.0 \end{bmatrix}.$$

The nonlinear system is constrained to be inside the box

$$Y_1 = [-2.5, 2.5]^2 \times [-0.5, 0.5]$$

for all three modes. Using $m = 0.5$ we compute the controller design parameters of Thm. 4.4.5 which are shown in Table 3. We take the terminal sets to be ellipsoidal $X_i^f = \{x^\top P_i^f x \leq r_i\}$. Finally, we simulate the system for different values of parameter α of $\mathrm{AV@R}_\alpha$ after we formulate the problem as described in Section 4.2.4, with initial condition $x_0 = (2, -2)$ and $i_0 = 1$. Resulting system trajectories are reported in Fig. 13. The proposed methodology successfully stabilizes the nonlinear system in the presence of uncertainty in the Markov transition matrix.

A similar effect is observed when inspecting the distribution of $\ell(x_k, u_k, i_k)$ for three MPC controllers. MPC controllers with higher α (closer to stochastic MPC) allow for higher costs, albeit with low probability. On the other

Table 3: Controller design parameters

i	β_i	δ_i $\alpha = 1.0$	δ_i $\alpha = 0.9$	δ_i $\alpha = 0.5$
1	0.4421	0.2407	0.1783	0.1563
2	0.2210	0.3775	0.4121	0.3556
3	0.6631	0.1668	0.1130	0.0973

hand, the risk-averse controller with $\alpha = 0.5$ (closer to minimax MPC) tends to produce cost distributions with shorter right tails. Interestingly, the point x_0 is not feasible for the worst case controller ($\alpha = 0$). The cost distributions are shown in Fig. 10.

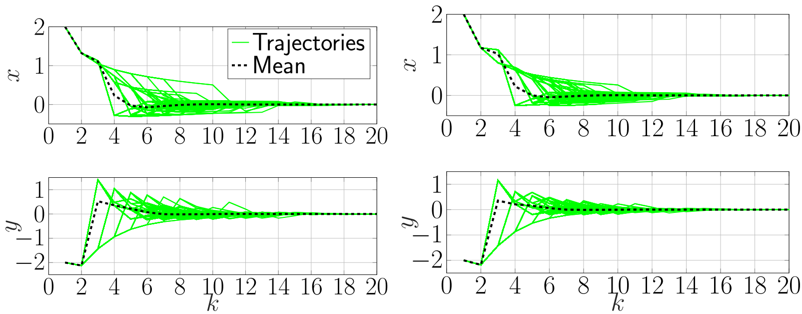

Figure 9: Trajectories of the closed-loop system with risk-averse MPC for $N = 6$ with (Left) $\alpha = 0.9$ and (Right) $\alpha = 0.5$. The green lines correspond to 1000 random simulations.

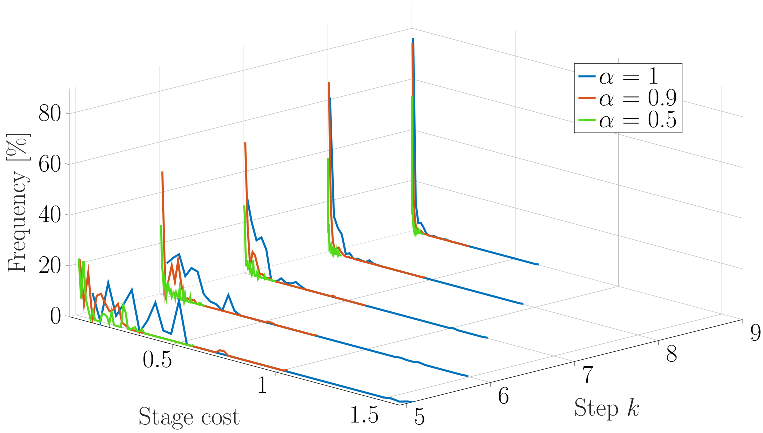

Figure 10: Distribution of $\ell(x_k, u_k, i_k)$ estimated using 1000 randomly generated switching sequences. The cost of trajectories corresponding to higher α values are more spread out compared to $\alpha = 0.5$ and have a noticeably longer right tail.

Chapter 5

Proximal methods for risk-averse problems

5.1 Introduction

Risk-averse formulations problems pose numerous advantages over stochastic and robust approaches as described in the previous chapter. However, it comes at a higher computational cost with regards to stochastic and worst-case approaches. In this chapter we seek to offer a computational algorithm by employing recent advances in proximal algorithms (PB14). Here, we shall discuss *risk-averse risk-constrained* optimal control problems which are risk-averse problems with additional risk-constraints which were introduced in (SSP19).

A major topic in stochastic control are *probabilistic constraints*. Often, in stochastic control it may be desirable to impose constraints which need to be satisfied with given probability and not for all possible realizations of uncertainty which may be too conservative in many applications. However, probabilistic constraints are hard to deal with numerically and more often than not we need to solve them with integer programming methods (She14) which do not scale well with size of the problem. More importantly, it may happen that disturbances with small probability lead to catastrophic failures of the overall system. Using the dual representation of risk-measures, we can see risk constrains as ambiguous expectation constraints (BTBB10; NS06) and we can use them to impose ”robust” version of chance constraints (Nem12; CG14). Maybe even more well known example is to use AV@R risk measure to approximate probabilistic constraints, which are equivalent to V@R constraints. An additional benefit is that using risk constraints accounts for the magnitude of constraint violation, something that pure chance constraints are blind to.

Risk-averse optimal control are typically solved using stochastic dual dynamics programming approaches (see (Sha11)). Note that the reformulation trick presented in the previous chapter allows us to use more efficient out-of-the-box methods and software such as Gurobi (GO19), Mosek (MOS16) and other. Most of these solvers use interior-point methods (NT08) , which usually do not scale well with the size of the problem and are hard to warm start (YW02). However, these problems posses a rich structure that we can exploit to devise very efficient and massively parallelisable methods to solve them. Due to these advantages, proximal algorithms have caught the attention of researches in the area of control as well (SSS$^+$16; OSB13; STSP17b).However, GPU computation is still mostly underused in optimal control problems. Some recent successful attempts in stochastic MPC are for example (SSBP15; SSBP17)

Furthermore, in this chapter we only assume that uncertainty in the model is described by a scenario tree. We make no further assumptions on the tree structure or the distribution. This approach can easily deal with the systems whose dynamics depends on the realization of uncertainty as well as systems with additive disturbances. Moreover, using the scenario tree makes the proposed approach attractive for data-driven methods as well. Indeed, such scenario trees can be readily constructed from data. In what follows, we shall restrict our interest to *coherent* risk measures (SDR14a) because of desirable mathematical properties they posses.

5.1.1 Notation

Let $\mathbb{N}_{[k_1,k_2]}$ denote the integers in $[k_1, k_2]$. For $z \in \mathbb{R}^n$ let $[z]_+ = \mathbf{max}\{0, z\}$, where the max is taken element-wise. We denote the transpose of a matrix A by $A^\top$. The dual cone $\mathcal{K}^*$ of a closed convex cone $\mathcal{K} \subseteq \mathbb{R}^n$ is the set $\mathcal{K}^* = \{y \in \mathbb{R}^n \mid y^\top x \geq 0, \forall x \in \mathcal{K}\}$. The relative interior of $\mathcal{K}$ is denoted by $\mathbf{ri}(\mathcal{K})$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *lower semicontinuous* (lsc) if its lower level sets, $\{x \mid f(x) \leq \alpha\}$, are closed and it is called *level bounded* if its lower level sets are bounded.

5.1.2 Measuring risk

Let $\Omega = \{\omega_i\}_{i=1}^n$ be a finite sample space equipped with the discrete σ-algebra 2^Ω and a probability measure P with $\mathrm{P}(\{\omega_i\}) = \pi_i$. Hereafter, we will assume that $\pi_i > 0$. The pair (Ω, P) is called a *probability space*. A vector $\pi \in \mathbb{R}^n$ is called a *probability vector* if $\pi_i \geq 0$ for all $i \in \mathbb{N}_{[1,n]}$ and $\sum_{i=1}^n p_i = 1$. The set of all probability vectors in $\mathbb{R}^n$ is called the *probability simplex* and is denoted by $\mathcal{D}_n$. A real-valued *random variable* over (Ω, P) is a mapping $Z : \Omega \to \mathbb{R}$ with $Z(\omega_i) = Z_i$; this can be identified by the vector $Z = (Z_1, \ldots, Z_n) \in \mathbb{R}^n$.

Risk measures are mappings $\rho : \mathbb{R}^n \to \mathbb{R}$. We call a risk measure coherent if it satisfies Definition (12) given in the previous chapter.

5.2 Problem formulation

In this chapter we introduce risk-averse risk optimal control formulation taken form (SSP19). We will strive to use the same notation as the authors in (SSP19) through this chapter.

5.2.1 System dynamics and scenario trees

We consider the following discrete-time dynamical system

$$x_{k+1} = f(x_k, u_k, w_k), \tag{5.1}$$

with state variable $x_k \in \mathbb{R}^{n_x}$, input $u_k \in \mathbb{R}^{n_u}$ and a disturbance $w_k \in \mathbb{R}^{n_w}$ which is a random process. From a known initial state x_0 the system states x_k for $k \in \mathbb{N}_{[1,N]}$ evolve according to (5.1) as illustrated in Figure 4 which shows a structure known as a *scenario tree*. Note that scenario trees can be generated from data (PP15; HR09) making them attractive for data-drivn approaches.

We index the *nodes* of the with an index i with $i = 0$ being the *root node* which corresponds to the initial state x_0. The nodes at subsequent stages for $k \in \mathbb{N}_{[0,N]}$ are denoted by $\mathbf{nodes}(k)$. Starting from the root node, each node i is visited with probability $\pi^i > 0$. The root node is visited with probability equal to one - $\pi^0 = 1$. This makes $\mathbf{nodes}(k)$ a probability space with probability vector $\pi_k = (\pi^i)_{i \in \mathbf{nodes}(t)}$.

The unique *ancestor* of a node $i \in \mathbf{nodes}(t) \setminus \{0\}$ is denoted by $\mathbf{anc}(i)$ and the set of *children* of $i \in \mathbf{nodes}(k)$ for $k \in \mathbb{N}_{[0,N-1]}$ is $\mathbf{ch}(i) \subseteq \mathbf{nodes}(k+1)$; this becomes a probability space with probability vector

$$\pi^{[i]} = \frac{1}{\pi^i}(\pi^{i_+})_{i_+ \in \mathbf{ch}(i)}$$

.

As shown in Fig. 4, every node i of the tree is associated with a state value x^i and all non-leaf nodes i are assigned an input u^i. Every pair of nodes i and $i_+ \in \mathbf{ch}(i)$ is connected by an edge which is associated with a disturbance w^{i_+}. The finite-horizon evolution of (5.1) on the scenario tree is given by the equation

$$x^{i_+} = f(x^i, u^i, w^{i_+}) \tag{5.2}$$

for all $i \in \mathbf{nodes}(k)$, $k \in \mathbb{N}_{[0,N-1]}$ and $i_+ \in \mathbf{ch}(i)$. Note that each control action u^i is *causal* i.e., control actions u_k are only allowed to depend on information that is available up to time k. All nodes of the tree at the last stage N are called *leaf nodes*. Moreover, every leaf node $i \in \mathbf{nodes}(N)$ identifies a *scenario*, that is, a sequence

$$\mathbf{scn}(i) = \{0, \ldots, \mathbf{anc}(\mathbf{anc}(i)), \mathbf{anc}(i), i\},$$

which can be seen as a path from inital node to one of the leaf nodes. Clearly, there are as many scenarios as there are leaf nodes.

5.2.2 Measuring risk on scenario trees

In this section we introduce the notion of conditional risk mappings which is essential in measuring the risk of a random cost which evolves in time across the nodes of a scenario tree (SDR14b, Sec. 6.8.1).

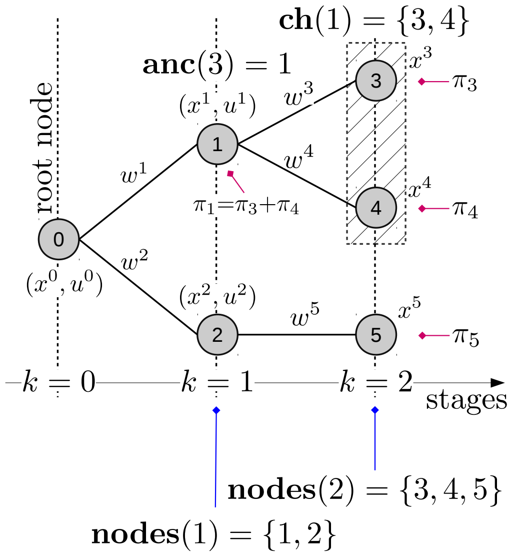

Figure 11: Structure of a general (non Markovian) scenario tree.

For $k \in \mathbb{N}_{[0,N-1]}$, let $\ell_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}$ be a stage cost function and $\ell_N : \mathbb{R}^{n_x} \to \mathbb{R}$ be the terminal cost function.

Every node $i \in \mathbf{nodes}(k+1)$, $k \in \mathbb{N}_{[0,N-1]}$ is associated with a cost value

$$Z^i = \ell_k(x^{\mathbf{anc}(i)}, u^{\mathbf{anc}(i)}, w^i).$$

For each $k \in \mathbb{N}_{[0,N-1]}$ we define a random variable $Z_k = (Z^i)_{i \in \mathbf{nodes}(k+1)}$ on the probability space $\mathbf{nodes}(k+1)$. For example, the cost at $k = 0$ is the random variable

$$Z_0 = (Z^i)_{i \in \mathbf{nodes}(1)} = (\ell_0(x^0, u^0, w^i))_{i \in \mathbf{nodes}(1)}.$$

At stage N the terminal cost is the random variable

$$Z_N = (\ell_N(x^i))_{i \in \mathbf{nodes}(N)}.$$

By defining $Z^{[i]} := (Z^{i_+})_{i_+ \in \mathbf{ch}(i)}$, $i \in \mathbf{nodes}(k)$, we partition the variable $Z_k = (Z^{[i]})_{i \in \mathbf{nodes}(t)}$ into groups of nodes which share a common ancestor.

Let $\rho^i : \mathbb{R}^{|\mathbf{ch}(i)|} \to \mathbb{R}$ be risk measures on the probability space $\mathbf{ch}(i)$. For every stage $k \in \mathbb{N}_{[0,N-1]}$ we may define a *conditional risk mapping* at stage k, $\rho_{|k} : \mathbb{R}^{|\mathbf{nodes}(k+1)|} \to \mathbb{R}^{|\mathbf{nodes}(k)|}$, as follows

$$\rho_{|k}[Z_k] = (\rho^i[Z^{[i]}])_{i \in \mathbf{nodes}(k)}. \tag{5.3}$$

Conditional can be represented using a dual representation akin to that in Eq. (**??**). Provided that all ρ^i are coherent risk measures, (5.3) yields

$$\rho_{|k}[Z_k] = \Big(\max_{\mu^i \in \mathcal{A}^i(\pi^{[i]})} \mathbb{E}^{\mu^i}[Z^{[i]}] \Big)_{i \in \mathbf{nodes}(k)}, \tag{5.4}$$

where $\mathcal{A}^i$ is the ambiguity set of ρ^i. Conditional risk mappings are used to measure the risk of a multistage stochastic process $(Z_0, \ldots, Z_k)$ of random costs, which evolves on a scenario tree. Given a sequence $(\rho_{|0}, \ldots, \rho_{|k})$ of conditional risk mappings, we define

$$\varrho_k(Z_0, \ldots, Z_k) = \rho_{|0}\big[Z_0 + \rho_{|1}[\cdots + \rho_{|k}[Z_k]]\big] \tag{5.5}$$

which is called a *nested multistage risk* measure. We define the *composite risk measure* at stage t as

$$\bar{\rho}_k[Z_k] = \varrho_k(0, \ldots, 0, Z_k). \tag{5.6}$$

If all ρ^i are coherent risk measures, then $\bar{\rho}_k$ is a coherent risk measure on $\mathbf{nodes}(k)$ (SDR14b). The composite risk measure $\bar{\rho}_t[Z_t]$ can be interpreted as the worst case expectation of Z_t accounting for the ambiguity at all intermediate stages, which is represented by $\mathcal{A}(\pi^{[i]})$, for $i \in \mathbf{nodes}(k')$, $k' \in \mathbb{N}_{[0,k]}$.

5.3 Risk-constrained risk-averse optimal control

Risk-averse risk-constrained optimal control problem (RARCOCP)with horizon N is defined via the following multistage nested formulation (SDR14b, Sec. 6.8.1)

$$\begin{aligned} V^\star = \inf_{u_0} \rho_{|0}\Bigg[\ell_0(x_0, u_0, w_0) + \inf_{u_1} \rho_{|1}\Bigg[\ell_1(x_1, u_1, w_1) \\ + \ldots + \inf_{u_{N-1}} \rho_{|N-1}\big[\ell_{N-1}(x_{N-1}, u_{N-1}, w_{N-1}) + \quad \ell_N(x_N)\big] \cdots \Bigg]\Bigg] \end{aligned} \tag{5.7a}$$

subject to

$$x_{k+1} = f(x_k, u_k, w_k), \tag{5.7b}$$

$$r_k[\phi_{j,k}(x_k, u_k, w_k)] \leq 0, j \in \mathbb{N}_{[1,q_k]}, \tag{5.7c}$$

$$r_N[\phi_{j,N}(x_N)] \leq 0, j \in \mathbb{N}_{[1,q_N]} \tag{5.7d}$$

for all $k \in \mathbb{N}_{[0,N-1]}$. Constraints (5.7c) are risk constraints involving risk measures r_k on the probability spaces $\mathbf{nodes}(k+1)$ and r_N is a risk measure on $\mathbf{nodes}(N)$. Their role is discussed in Section 5.3.1. The infima in (5.7) are taken with respect to causal control functions u_k.

The above nested formulation amounts to minimizing the nested multistage cost $\varrho_{N-1}(\ell_0(x_0, u_0, w_0), \ldots, \ell_N(x_N))$ subject to the system dynamics and additional constraints (SDR14b, Sec. 6.8). In the above, we could easily substitute risk measure with expectation operator or the max operator, which gives insite into how risk-averse problems generalize risk-neutral and minimax formulations. Moreover, the above formulation enables the stability analysis of associated model predictive control formulations (CP14; SHBP19).

5.3.1 Risk constraints

Now we discuss how to model risk constraints. At each stage $k \in \mathbb{N}_{[0,N-1]}$, let us define q_k functions $\phi_{j,k} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}$, $j \in \mathbb{N}_{[1,q_k]}$. At stage N, we also define q_N functions $\phi_{j,N} : \mathbb{R}^{n_x} \to \mathbb{R}$, $j \in \mathbb{N}_{[1,q_N]}$. Reciting (RU13), our objective is to impose that "$\phi_{j,k}$ are adequately ≤ 0," for $k \in \mathbb{N}_{[0,N]}$, in a probabilistic sense.

Let $G_{j,k} = \phi_{j,k}(x_k, u_k, w_k)$ be a real-valued random quantity defined at stage $k \in \mathbb{N}_{[0,N-1]}$ and $G_{j,N} = \phi_{j,N}(x_N)$. Similar to the definition of Z^i in Sec. 5.2.2, at every stage $k \in \mathbb{N}_{[0,N-1]}$ and node $i \in \mathbf{nodes}(t+1)$, we assign values $G_{j,t} = ((G_j^i)_{i \in \mathbf{nodes}(k+1)}$ for every $j \in \mathbb{N}_{[1,q_k]}$. Analogously, we define $G_{j,N}$ for $j \in \mathbb{N}_{[1,q_N]}$

Here, we describe two risk constraint formulations on scenario trees, namely, (i) stage-wise risk constraints, (ii) multistage nested risk constraints.

Stage-wise risk constraints

Stage-wise constraints are imposed at every stage $k \in \mathbb{N}_{[0,N-1]}$ as follows

$$r_k[G_{j,t}] \leq 0, \tag{5.8}$$

for $j \in \mathbb{N}_{[1,q_k]}$, where $r_k : \mathbb{R}^{|\mathbf{nodes}(t+1)|} \to \mathbb{R}$ are risk measures and $G_{j,k} \in \mathbb{R}^{|\mathbf{nodes}(k+1)|}$. At $k = N$, similarly, we impose $r_{N-1}[G_{j,N}] \leq 0$ for $j \in \mathbb{N}_{[1,q_N]}$. But, such risk-based constraints do not account for how the probability distribution at stage k is generated in time; indeed, the dependence on previous stages in (5.8) is disregarded.

Multistage nested risk constraints

On the other hands, multistage nested risk constraints impose nested risk constraints at each stage $k \in \mathbb{N}_{[1,N-1]}$. *Multistage nested risk constraints* are of the form

$$\bar{r}_t[G_{j,t}] = r_{|0}\big[r_{|1}\big[\cdots r_{|t}[G_{j,t}]\big]\big] \leq 0 \tag{5.9}$$

The two types of constraints are more discussed in (SSP19), here we are more interested in how to deconvove them.

5.4 Tractable reformulations

5.4.1 Conic representation of risk measures

The ambiguity set of a coherent risk measure can be written using conic inequalities, i.e., there exist matrices E, F and a vector b, such that

$$\rho[Z] = \max_{\mu \in \mathbb{R}^n, \nu \in \mathbb{R}^r} \{\mu^\top Z \mid E\mu + F\nu \preccurlyeq_{\mathcal{K}} b\}, \tag{5.10}$$

where $\mathcal{K}$ is a closed, convex cone and ν is an auxiliary variable. All widely used coherent risk measures can be written in this form. Tacitly, we have assumed that all admissible μ in (5.10) are probability vectors and the ambiguity set of ρ is the following subset of $\mathcal{D}_n$

$$\mathcal{A} = \{\mu \in \mathbb{R}^n \mid \exists \nu \in \mathbb{R}^r : E\mu + F\nu \preccurlyeq_{\mathcal{K}} b\}.$$

For example, AV@R_α is written as in (5.10) with $r = 0$ and $E = \left[\begin{smallmatrix} I & -I & 1_n \end{smallmatrix}\right]^\top$, $b = \left[\begin{smallmatrix} 1/\alpha \pi^\top & 0 & 1 \end{smallmatrix}\right]^\top$ and $\mathcal{K} = \mathbb{R}^{2n}_{\geq 0} \times \{0\}$. EV@R_α can also be written in the above form. Let $\mathcal{K}^{\text{e}} = \mathbf{cl}\{(x, y, z) \in \mathbb{R}^3 \mid ye^{x/y} \leq z, y > 0\}$ be the exponential cone. By virtue of the equivalence $x \, \mathbf{ln}(x/y) \leq t \Leftrightarrow (-t, x, y) \in \mathcal{K}^{\text{e}}$, the ambiguity set $\mathcal{A}^{\text{evar}}_\alpha(\pi)$ is

$$\mathcal{A}^{\text{evar}}_\alpha(\pi) = \left\{ \mu \in \mathcal{D}_n \,\middle|\, \begin{array}{l} \exists \nu \in \mathbb{R}^n : \sum_{i=1}^n \nu_i \leq \; - \mathbf{ln}\,\alpha, \\ (-\nu_i, \mu_i, \pi_i) \in \mathcal{K}^{\text{exp}}, i \in \mathbb{N}_{[1,n]} \end{array} \right\},$$

Provided that strong duality holds — which is the case if there exist μ^* and ν^* so that $b - E\mu^* + F\nu^* \in \mathbf{ri}(\mathcal{K})$ (BTN01, Thm. 1.4.2) — the risk measure in (5.10) can be written as

$$\rho[Z] = \min_y \{y^\top b \mid E^\top y = Z, F^\top y = 0, y \succcurlyeq_{\mathcal{K}^*} 0\}. \tag{5.11}$$

We shall use this representation of risk measures to rewrite optimal control problems involving risks.

5.4.2 Decomposition of nested formulation

Finally we are ready to state the tractable reformulation of problem (5.7). But first, we state the following result from (SSP19).

Theorem 5.4.1 (Risk-infimum interchangeability) *Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a convex risk measure and $g : \mathbb{R}^m \ni x \mapsto (g_1(x), \ldots, g_n(x)) \in \mathbb{R}^n$ where $g_i : \mathbb{R}^m \to \mathbb{R}$ is an lsc, level-bounded function over a closed set $\emptyset \neq X \subseteq \mathbb{R}^m$. Let $\mathbf{inf}_{x\in X}\, g(x) := (\mathbf{inf}_{x\in X}\, g_1(x), \ldots, \mathbf{inf}_{x\in X}\, g_n(x))$. Then*

$$\rho\Big[\inf_{x\in X} g(x)\Big] = \inf_{x\in X} \rho[g(x)] \tag{5.12a}$$

$$\operatorname*{\mathbf{argmin}}_{x\in X} g(x) \subseteq \operatorname*{\mathbf{argmin}}_{x\in X} \rho[g(x)]. \tag{5.12b}$$

Furthermore, if ρ is strictly monotone *or $\rho \circ g : \mathbb{R}^m \to \mathbb{R}$ is strictly convex over X, then*

$$\operatorname*{\mathbf{argmin}}_{x\in X} g(x) = \operatorname*{\mathbf{argmin}}_{x\in X} \rho[g(x)]. \tag{5.13}$$

The *epigraph* of a risk measure $\rho : \mathbb{R}^n \to \mathbb{R}$ is the set $\mathbf{epi}\,\rho = \{(Y, \gamma) \in \mathbb{R}^{n+1} \mid \rho[Y] \leq \gamma\}$. When ρ is a coherent risk measure given by (5.11), its epigraph is the set

$$\mathbf{epi}\,\rho = \left\{(Y,\gamma) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \exists y \succcurlyeq_{\mathcal{K}^*} 0, E^\top y = Y, \\ F^\top y = 0, y^\top b \leq \gamma \end{array} \right\}. \tag{5.14}$$

Then, for example, stage-wise risk constraints (5.8) are equivalent to

$$r_k\Big(\inf_{G_{j,k} \leq \eta_{j,k+1}} \eta_{j,k+1}\Big) \leq 0$$

for a random variable $\eta_{k+1} \in \mathbb{R}^{|\,\mathbf{nodes}(k+1)|}$. Using Thm. 5.4.1, we have that the risk constraints (5.8) are equivalent to the existence of $\eta_{j,k+1}$ such that $G_{j,k} \leq \eta_{j,k+1}$ and $(\eta_{k+1}, 0) \in \mathbf{epi}\, r_k$.

The epigraph of a conditional risk mapping is

$$\mathbf{epi}\,\rho_{|k} = \{(Y_{k+1}, Y_k) \in \mathbb{R}^{|\,\mathbf{nodes}(k+1)|+|\,\mathbf{nodes}(k)|} \mid \rho_{|k}[Y_{k+1}] \leq Y_k\}$$

which is the Cartesian product of the epigraphs of its constituent risk measures

$$\mathbf{epi}\,\rho_{|t} = \textstyle\prod_{i\in\mathbf{nodes}(t)} \mathbf{epi}\,\rho^i.$$

Proposition 6 (Nested risk epigraph) *Let $(\rho_{|0}, \ldots, \rho_{|k})$ be a sequence of coherent conditional risk mappings. Let $\bar{\rho}_k$ be the corresponding nested risk measure. Its epigraph is*

$$\mathbf{epi}\,\bar{\rho}_k = \left\{ \begin{array}{l} (Y_{k+1}, Y_0) \in \mathbb{R}^{|\,\mathbf{nodes}(k+1)|+1} \mid \exists (Y_j)_{j\in\mathbb{N}_{[1,k]}}, \\ Y_j \in \mathbb{R}^{\mathbf{nodes}(j)}, (Y_{k+1}, Y_j) \in \mathbf{epi}\,\rho_{|k},\ j \in \mathbb{N}_{[0,k]} \end{array} \right\}$$

Using Prop. 6, we may write (5.9) in the form

$$(G_{j,t}, 0) \in \mathbf{epi}\, \bar{r}_t.$$

Risk constraints, both stage-wise and nested, can be cast as conic constraints. Finally, we arrive at a tractable reformulation of the original problem which is given as

$$\mathbf{minimize}\; s^0 \tag{5.15a}$$

$$\mathbf{subject\ to}\; x^0 = x \text{ and } x^{i_+} = f(x^i, u^i, w^{i_+}), \tag{5.15b}$$

$$y^i \succcurlyeq_{(\mathcal{K}^i)^*} 0, (E^i)^\top y^i = \tau^{[i]} + s^{[i]} \tag{5.15c}$$

$$(F^i)^\top y^i = 0, (y^i)^\top b^i \leq s^i \tag{5.15d}$$

$$\ell_k(x^i, u^i, w^{i_+}) \leq \tau^{i_+}, \ell_N(x^{i'}) \leq s^{i'}, \tag{5.15e}$$

for $i' \in \mathbf{nodes}(N)$, $i \in \mathbf{nodes}(k)$, $k \in \mathbb{N}_{[0,N-1]}$ and $i_+ \in \mathbf{ch}(i)$. In (5.15c) we denote $\tau^{[i]} = (\tau^{i_+})_{i_+ \in \mathbf{ch}(i)}$.

Furthermore, if we suppose that the problem is subject to stage-wise risk constraints of the form (5.8) at stage k with a conic risk measure r_k described by the tuple $(\bar{E}_k, \bar{F}_k, \bar{b}_k, \bar{\mathcal{K}}_k)$. For notational convenience, index j is dropped.

$$\bar{y}_k \succcurlyeq_{\bar{\mathcal{K}}_k^*} 0,\; \bar{E}_k^\top \bar{y}_k = \eta_k,\; \bar{F}_k^\top \bar{y}_k = 0, \tag{5.15f}$$

$$\bar{y}_k^\top \bar{b}_k \leq 0,\; \phi_k(x^i, u^i, w^{i_+}) \leq \eta_k^{i_+}. \tag{5.15g}$$

for $i \in \mathbf{nodes}(k)$, $i_+ \in \mathbf{ch}(i)$. We have here introduced the additional variables $\eta_k \in \mathbb{R}^{|\mathbf{nodes}(k+1)|}$ and $\bar{y}_k$.

Similarly, if we suppose that the problem is subject to multistage nested risk constraints at stage t of the form (5.9) where the multistage risk is given by conic risk measures r^i described by the tuples $(\tilde{E}^i, \tilde{F}^i, \tilde{b}^i, \tilde{\mathcal{K}}^i)$. Then, (5.9) leads to the following constraints

$$\tilde{y}_k^i \succcurlyeq_{(\tilde{\mathcal{K}}^i)^*} 0,\; (\tilde{E}^i)^\top \tilde{y}_k^i = \xi_k^{[i]},\; (\tilde{F}^i)^\top \tilde{y}_k^i = 0, \tag{5.15h}$$

$$\phi_k(x^{i'}, u^{i'}, w^{i'_+}) \leq \xi_k^{i'_+},\; (\tilde{b}^i)^\top \tilde{y}_k^i \leq \xi_k^i,\; \xi_k^0 = 0, \tag{5.15i}$$

for $i' \in \mathbf{nodes}(k)$, $i'_+ \in \mathbf{ch}(i')$, $i \in \mathbf{nodes}(k')$, $k' \in \mathbb{N}_{[1,k]}$.

In all cases, the number of decision variables and constraints increases linearly with the total number of nodes. When the system dynamics is linear (or affine) and functions ℓ_k and ϕ_k are convex in x and u, then (5.15) is a convex conic problem which can be solved very efficiently with solvers such as MOSEK (MOS16), SuperSCS (SMP19) and more.

Problems (5.7) and (5.15) are equivalent in the sense that the optimal values of the objective function at the solution are the same. If all involved risk measures are strictly monotone, then the respective sets of minimizers are equal.

5.5 Conic reformulation

Here, we shall define a splitting which we find to be suitable for proximal algorithms. In order to do so, we shall require that all proximal operations in the resulting algorithm have easy-to-compute solutions. First, let us simplify the general risk-averse problem presented in (5.15) We shall assume that the system dynamics is linear, i.e.

$$x^{i_+} = A(w^{i_+})x^i + B(w^{i_+})u^i,$$

or for notational simplicity

$$x^j = A_j x^i + B_j u^i,$$

where $j \in \mathbf{ch}(i)$. Moreover, we assume that stage-wise costs and terminal costs are quadratic, i.e.

$$\ell_k(x^i, u^i, w^j) = (x^i)^\top Q_j x^i + (u^i)^\top R_j u^i, \tag{5.16a}$$

$$\ell_N(x^{i'}) = (x^{i'})^\top P x^{i'}, \tag{5.16b}$$

with $Q_j \in \mathbf{S}_+, R_j \in \mathbf{S}_{++}, P \in \mathbf{S}_{++}$ where $j \in \mathbf{ch}(i)$ for all $i \in \mathbf{nodes}(k)$, $k \in \mathbb{N}_{[0,N-1]}$ and $i' \in \mathbf{nodes}(N)$.

Note that we can have linear functions, infinity norms and other prox friendly convex functions as cost functions. Here we present the case for quadratic costs because they are prevalent in MPC formulations and we need to to employ a trick to transform them into a suitable form.

Moreover, for the sake of simpler presentation, we shall drop the F term in (5.10) which is related to more exotic risk measures such as EV@R .

Next, we define mappings F_i, G and H to facilitate the description of risk-averse problem (5.15) in conic formulation. These mappings will serve the purpose of transforming the risk-averse problem into an equivalent variant which is prox friendly. Let us start by noting that at each stage k and each node $i \in \mathbf{nodes}(k)$, $k \in \mathbb{N}_{[1,N-1]}$ we have a quadratic cost given by (5.16a). Every quadratic constraint in (5.15e) with positive-(semi)definite weighting matrix can be written as a second order cone.

Let $Q \in \mathbf{S}_+^{n_x}, R \in \mathbf{S}_+^{n_u}, x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, \tau \in \mathbb{R}$, then

$$x^\top Qx + u^\top Ru \leq \tau$$

can be written in its equivalent second-order cone form as

$$x^\top Qx + u^\top Ru \leq \tau \iff \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \\ \tau - \frac{1}{2} \\ \tau + \frac{1}{2} \end{bmatrix} \in \mathcal{K}_s. \tag{5.17}$$

This can be easily verified by applying the definition of a second-order cone and to the right-hand side of the above equivalence and squaring it. Moreover it also holds that

$$\begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \\ \tau - \frac{1}{2} \\ \tau + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} Q^{1/2} & 0 & 0 \\ 0 & R^{1/2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ u \\ \tau \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \tag{5.18}$$

The right hand side of the above equations serves as an affine mapping of decision variables which transforms the original constraint into its conic form. Having the above in mind, we can define mappings F_i and G which encode quadratic constraints. Mappings $F_i(x^i, u^i, \tau^{[i]})$ are applied at every node $i \in \mathbf{nodes}(k), \forall k \in \mathbb{N}_{[1,N-1]}$ to encode quadratic constraints . Mapping G is a mapping used at leaf nodes which does not depend on an index because we have assumed unique terminal weight function. Moreover, we define an additional linear mapping $H_i(y^i, s^i)$ to encode linear inequalities. The mappings are given below as

$$F_i(x^i, u^i, \tau^{[i]}) := \begin{bmatrix} Q_1^{1/2} & 0 & 0 & 0 & \dots & 0 \\ 0 & R_1^{1/2} & 0 & & & \\ 0 & 0 & \frac{1}{2} & & & \\ 0 & 0 & \frac{1}{2} & & & \\ Q_2^{1/2} & 0 & 0 & & & \\ 0 & R_2^{1/2} & 0 & & & \\ 0 & 0 & 0 & \frac{1}{2} & & \\ 0 & 0 & 0 & \frac{1}{2} & & \\ \vdots & & & & & \\ Q_{|\mathbf{ch}(i)|}^{1/2} & 0 & 0 & & & \\ 0 & R_{|\mathbf{ch}(i)|}^{1/2} & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x^i \\ u^i \\ \tau^{[i]} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ \vdots \\ 0 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \tag{5.19}$$

$$G(x, s) := \begin{bmatrix} P^{1/2} & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \tag{5.20}$$

$$H_i(y^i, s^i) := \begin{bmatrix} I & 0 \\ -(b^i)^\top & 1 \end{bmatrix} \begin{bmatrix} y^i \\ s^i \end{bmatrix}. \tag{5.21}$$

At last we are able to write the problem in equivalent conic form as

$$\begin{aligned} \textbf{minimize} \quad & s^0 && (5.22a) \\ \textbf{subject to} \quad & x^0 - x = 0, \quad x^j - A_j x^i - B_j u^i = 0 && (5.22b) \\ & (E^i)^\top y^i - s^{[i]} - \tau^{[i]} = 0 && (5.22c) \\ & H_i(y^i, s^i) \in \mathcal{K}_+ && (5.22d) \\ & F_i(x^i, u^i, \tau^{[i]}) \in \Pi_{j \in \mathbf{ch}(i)} \mathcal{K}_s && (5.22e) \\ & G(x^{i'}, s^{i'}) \in \mathcal{K}_s && (5.22f) \end{aligned}$$

for $j \in \mathbf{ch}(i)$, $i \in \mathbf{nodes}(k)$, for all $k \in \mathbb{N}_{[1,N-1]}$ and $i' \in \mathbf{nodes}(N)$. In the above we use the notation $[i]$ to denote a vector which is a concatenation of variables (vectors) for every $j \in \mathbf{ch}(i)$.

5.5.1 Problem splitting

Next we split the problem into the following form

$$\underset{x \in \mathbb{R}^n}{\textbf{minimize}} \quad f(x) + g(Lx) \tag{5.23}$$

where $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator which combines F_i, G, H in a suitable manner to encode constraints (5.22d)-(5.22f). Here we use x to mean a general decision variable. Dimension n equals to sum of dimension of all devision variable of the problem and m depends on the number of constraints. Assuming correct permutation of variables, we state L as

$$L = \operatorname{blkdiag}(F_0, F_1, \cdots, F_{(N_N - N_L - 1)}, \underbrace{G, G, \ldots, G}_{\times N_L}, \overbrace{H, \ldots, H}^{\times (N_N - N_L)}) \tag{5.24a}$$

where N_N is the total number of nodes and N_L is the total number of leaf nodes.

For notational convenience we define concatenation of vectors

$$\mathbf{x} = \{x^i\}_{i \in \mathbf{nodes}(k)}, \quad \forall k \in \mathbb{N}_{[0,N]}, \tag{5.25a}$$

$$\mathbf{u} = \{u^i\}_{i \in \mathbf{nodes}(k)}, \quad \forall k \in \mathbb{N}_{[0,N-1]}, \tag{5.25b}$$

$$\mathbf{y} = \{y^i\}_{i \in \mathbf{nodes}(k)}, \quad \forall k \in \mathbb{N}_{[0,N-1]}, \tag{5.25c}$$

$$\tau = \{\tau^i\}_{i \in \mathbf{nodes}(k)}, \quad \forall k \in \mathbb{N}_{[1,N]}, \tag{5.25d}$$

$$\mathbf{s} = \{s^i\}_{i \in \mathbf{nodes}(k)}, \quad \forall k \in \mathbb{N}_{[1,N]}. \tag{5.25e}$$

Note that in the above we have left s^0 out of $\mathbf{s}$. We also define additional sets

$$\mathcal{S}_1 = \left\{ (\mathbf{x}, \mathbf{u}) \,\middle|\, \begin{array}{l} x^0 = x,\ x^j = A_j x^i + B_j u^i, \\ j \in \mathbf{ch}(i), i \in \mathbf{nodes}(i), \forall k \in \mathbb{N}_{[1,N-1]} \end{array} \right\}, \tag{5.26}$$

$$\mathcal{S}_2 = \left\{ (\mathbf{y}, \tau, \mathbf{s}) \,\middle|\, \begin{array}{l} (E^i)^\top y^i - s^{[i]} - \tau^{[i]} = 0 \\ i \in \mathbf{nodes}(k), \forall k \in \mathbb{N}_{[1,N-1]} \end{array} \right\}, \tag{5.27}$$

which encode constraint (5.22b) and (5.22c).

Function f is now given as

$$f(x) = s^0 + \delta_{\mathcal{S}_1}(\mathbf{x}, \mathbf{u}) + \delta_{\mathcal{S}_2}(\mathbf{y}, \tau, \mathbf{s}) \tag{5.28a}$$

Function $f(x)$ is separable in all three terms given above, hence we can compute proximal operators separately using the separable sum property. Function g encodes all other constraints using the operator L given in (5.24a)

$$g(v) = \delta_{\mathcal{K}}(v + \mathbf{b}) \tag{5.29}$$

where $\mathcal{K}$ is defined appropriately on dual variables and $v = Lx$. For the risk-avers optimal problem above, $\mathcal{K}$ is a product of many individual cones and is given as

$$\mathcal{K} = \underbrace{\Pi_{i\in\mathbf{nodes}(1,k)}^{j\in\mathbf{ch}(i)}\mathcal{K}_s}_{\text{non-leaf quadratic penalties}} \times \underbrace{\Pi^{z\in\mathbf{nodes}(N)}\mathcal{K}_s}_{\text{terminal penalties}} \times \underbrace{\Pi^{j\in\mathbf{nodes}(k)}\mathcal{K}_+}_{\text{inequality constraints}}.$$

5.5.2 Projection onto linear dynamics via Dynamic programming

Projecting onto system dynamics admits a closed form expression given with (2.26), however the dimensions of the optimal control problem pose an obstacle. Even thought the matrix describing the system dynamics is usually very sparse, its pseudo inverse is not. Mere task of calculating the pseudo inverse becomes computationally very demanding, but storing it on a GPU could be infeasible for larger problems. Here, however, we shall take a different approach and employ dynamic programming solution to calculate the projection onto system dynamics.

Full problem is defined as

$$\begin{aligned}
\underset{\mathbf{x},\mathbf{u}}{\textbf{minimize}} \quad & \sum_j \frac{1}{2}\|x^j - z^j\|^2 + \sum_i \frac{1}{2}\|u^i - v^i\|^2 && (5.30a)\\
\textbf{subject to} \quad & x^j = A_j x^i + B_j u^i, && (5.30b)\\
& \forall j \in \mathbf{ch}(i), \forall i \in \mathbf{nodes}(k), \forall k \in \mathbb{N}_{[1,N-1]}, && (5.30c)
\end{aligned}$$

where vectors z and u are data vectors, and x and u are decision variables. Cost function at each node is given as the distance of given point to state-control pair. As we will see shortly, it will be more useful to write the problem as a quadratic expression and node-wise. Terminal cost for all leaf nodes, $i \in \mathbf{nodes}(N)$ is given as

$$J_i^\star(x^i) = \frac{1}{2}(x^i)' H_k x^i + h_i' x_i + \frac{1}{2} c_i, \tag{5.31}$$

with $H_i = I, h_i = -z_i, c_i = z_i' z_i$, i.e.

$$J_k^\star(x^k) = \frac{1}{2}\|x_k - z_k\|^2.$$

For all other nodes $i \in \mathbf{nodes}(k)$ for $k \in \mathbb{N}_{[1,N-1]}$, we find the optimal cost at a node as a solution of the optimization problem

$$J_i^\star(x^i) = \min_{u^i}\{\ell(x^i, u^i, z^i, v^i) + \sum_{j \in \mathbf{ch}(i)} J_j^\star(A_j x^i + B_j u^i)\} \tag{5.32}$$

$\forall i \in \mathbf{nodes}(1), k \in \mathbb{N}_{[1,N-1]}$, where

$$\ell(x^i, u^i, z^i, v^i) = \frac{1}{2}\|x^i - z^i\|^2 + \frac{1}{2}\|u^i - v^i\|^2$$

is the stage cost. Optimal cost at each stage will, again, be given as a quadratic function with parameters H_i, h_i and c_i which need to be computed. We can write

$$\begin{aligned} J_i^\star(x^i) = \min_{u^i} \Bigg(& \frac{1}{2}(x^i)' x^i - z_i' x^i + \frac{1}{2} z_i' z_i \\ & + \frac{1}{2}(u^i)' u^i - v_i' u^i + \frac{1}{2} v_i' v_i + \sum_{j \in \mathbf{ch}(i)} J_j^\star(x^j) \Bigg) \end{aligned} \tag{5.33}$$

where substituting $x^j \leftarrow A_j x^i + B_j u^i$ gives for each $J_j^\star(x^j)$

$$\begin{aligned} J_j^\star(A_j x^i + B_j u^i) = & \frac{1}{2}(A_j x^i + B_j u^i)' H_j (A_j x^i + B_j u^i) \\ & + h_j'(A_j x^i + B_j u^i) + \frac{1}{2} c_j. \end{aligned} \tag{5.34}$$

Differentiating with respect to u^i gives conditions for u^i to be an optimal control action at a given node.

$$\frac{\partial J_i}{\partial u^i} = u^i - v^i + \sum_{j \in \mathbf{ch}(i)} \big(B_j' H_j (A_j x^i + B_j u^i) + B_j' h_j \big) = 0, \tag{5.35}$$

which gives for each control action u^i

$$\overbrace{(I + \sum_{j \in \mathbf{ch}(i)} B_j H_j B_j)}^{\hat{B}_i} u^i = v^i - \sum_{j \in \mathbf{ch}(i)} B_j' h_j - \sum_{j \in \mathbf{ch}(i)} B_j' H_j A_j x^i \tag{5.36}$$

or in a condensed form

$$u^i = f_i + K_i x^i \tag{5.37}$$

where

$$f_i = \hat{B}_i^{-1} \left(v_i - \sum_{j \in \mathbf{ch}(i)} B_j' h_j \right) \tag{5.38a}$$

$$K_i = \hat{B}_i^{-1} \left(\sum_{j \in \mathbf{ch}(i)} -B_j' H_j A_j \right). \tag{5.38b}$$

Now, after introducing $A_{ij} = A_j + B_j K_i$, we can write

$$J_i^\star(x) = \ell_i(x^i, z^i, v^i) + \sum_{j \in \mathbf{ch}(i)} J_j \left(A_{ij} x^i + B_j f_i \right) \tag{5.39}$$

which expands to

$$\begin{aligned} J_i^\star = {} & \frac{1}{2}(x^i)' x^i - z_i' x^i + \frac{1}{2} z_i' z_i + \frac{1}{2}(f_i + K_i x^i)'(f_i + K_i x^i) \\ & - v_i'(f_i + K_i x^i)' + \frac{1}{2} v_i' v_i + \sum_{j \in \mathbf{ch}(i)} J_j^\star (A_{ij} x^i + B_j f_i). \end{aligned} \tag{5.40}$$

and after collecting the terms with x^i

$$\begin{aligned} J_i^\star = {} & \frac{1}{2}(x^i)'(I + K_i' K_i) x^i + (-z_i' + f_i' K_i - v_i' K_i) x^i \\ & + \frac{1}{2}(z_i' z_i + v_i' v_i + f_i' f_i - v_i' f_i) + \sum_{j \in \mathbf{ch}(i)} J_j^\star (A_{ij} x^i + B_j f_i). \end{aligned} \tag{5.41}$$

Every $J_j^\star(A_j x^i + B_j f_i)$ can be expanded as

$$
\begin{aligned}
J_j^\star = {} & \frac{1}{2}(x^i)'(A_{ij}')H_j A_{ij} x^i + (h_j' A_{ij} + f_i' B_j' H_j A_{ij})x^i \\
& + \frac{1}{2}(c_j + f_i' B_j B_j f_i + 2h_j' B_j f_i).
\end{aligned} \tag{5.42}
$$

Finally, optimal cost at non-leaf node i is given as

$$
J_i^\star(x^i) = \frac{1}{2}(x^i)' \left(\overbrace{I + K_i' K_i + \sum_{j \in \mathbf{ch}(i)} A_{ij}' H_j A_{ij}}^{H_i} \right) x^i + \tag{5.43a}
$$

$$
(x^i)' \left(\overbrace{-z_i + K_i' f_i - K_i' v_i + \sum_{j \in \mathbf{ch}(i)} (A_{ij}' h_j + A_{ij}' H_j' B_j f_i)}^{h_i} \right). \tag{5.43b}
$$

Above we have obtained an expression for optimal cost at all the non-leaf nodes. The recursion continues until we reach the initial state. To sum up, we initialize the recursion with

$$
H_i \leftarrow I \tag{5.44a}
$$

$$
h_i \leftarrow -z_i, \tag{5.44b}
$$

for all $i \in \mathbf{nodes}(N)$, i.e. leaf nodes. For all other nodes $i \in \mathbf{nodes}(k)$, $\forall k \in \mathbb{N}_{[1,N-1]}$ we have

$$
H_i \leftarrow I + K_i' K_i + \sum_{j \in \mathbf{ch}(i)} A_{ij}' H_j A_{ij} \tag{5.45a}
$$

$$
h_i \leftarrow -z_i + K_i'(f_i - v_i) + \sum_{j \in \mathbf{ch}(i)} (A_{ij}' h_j + A_{ij}' H_j' B_j f_i)). \tag{5.46a}
$$

or

$$
h_i \leftarrow -z_i - K_i' v_i + (K_i' + \sum_{j \in \mathbf{ch}(i)} A_{ij}' H_j' B_j) f_i + \sum_{j \in \mathbf{ch}(i)} (A_{ij}' h_j). \tag{5.47a}
$$

Examining the term

$$(K_i' + \sum_{j \in \mathbf{ch}(i)} A_{ij}' H_j' B_j)$$

we can see that after substituting $A_{ij} = A_j + B_j K_i$ gives

$$K_i'(I + \sum_{j \in \mathbf{ch}(i)} B_j' H_j B_j) + \sum_{j \in \mathbf{ch}(i)} A_{ij} H_j' B_j$$

which cancels out after plugging in the definition of K_i. Hence, the final equation is

$$h_i \leftarrow -z_i - K_i' v_i + \sum_{j \in \mathbf{ch}(i)} (A_{ij}' h_j). \tag{5.48a}$$

We can carry this recursive process until the we reach the initial node $i = 0$. Then we can calculate state and control actions going forward from the initial state x^0 by following

$$u^i = f_i + K_i x^i \tag{5.49a}$$

$$x^j = A_j x^i + B_j u^i, \quad \forall j \in \mathbf{ch}(i) \tag{5.49b}$$

where

$$f_i = \hat{B}_i^{-1} \left(v_i - \sum_{j \in \mathbf{ch}(i)} B_j' h_j \right) \tag{5.50a}$$

$$K_i = \hat{B}_i^{-1} \left(\sum_{j \in \mathbf{ch}(i)} -B_j' H_j A_j \right) \tag{5.50b}$$

This steps are summed up in Algorithm 1.

We can further notice that some expression in the original algorithm can be precomputed off-line. For example $\hat{B}_i = I + \sum_{j \in \mathbf{ch}(i)} B_j' H_j B_j$ depends on known quantities (when considering a backwards step of the recursion). Below we split the original algorithm in two parts, off-line and on-line one. Once we calculate matrices in the off-line part, they are uploaded in the system memory. During the execution of the algorithm, only the on-line part is needed which now includes small size matrix-matrix, matrix-vector multiplications and does not include computation

Algorithm 1 Projection algorithm (naive)

Input: z, v, x_0

$\{$Backward iteration$\}$

for $k \in \mathbf{nodes}(N)$ **do**

$H_k \leftarrow I, h_k \leftarrow -z_k$

end for

for $k = N-1 : -1 : 1$ **do**

for $i \in \mathbf{nodes}(k)$ **do**

$\hat{B}_i \leftarrow I + \sum_{j\in\mathbf{ch}(i)} B_j' H_j B_j$

$K_i \leftarrow -\hat{B}_i^{-1}(\sum_{j\in\mathbf{ch}(i)} B_j' H_j A_j)$

$f_i \leftarrow \hat{B}_i^{-1}(v_i - \sum_{j\in\mathbf{ch}(i)} B_j' h_j)$

$H_i \leftarrow I + K_i' K_i + \sum_{j\in\mathbf{ch}(i)}(A_{ij}' H_j A_{ij})$

$h_i \leftarrow -z_i - K_i' v_i + \sum_{j\in\mathbf{ch}(i)}(A_{ij}' h_j)$

end for

end for

$\{$Forward iteration$\}$

for $k = 1 : N-1$ **do**

for $i = \mathbf{nodes}(k)$ **do**

$u^i = f_i + K_i x^i$

for $j \in \mathbf{ch}(i)$ **do**

$x^j = A_j x^i + B_j u^i$

end for

end for

end for

Algorithm 2 Projection algorithm precompute (off-line)

Input: $A_i, B_i \quad \forall i \in \mathbf{nodes}(1, N-1)$

$H_k \leftarrow I, \quad \forall i \in \mathbf{nodes}(N)$

for $k = N-1 : -1 : 1$ **do**

for $i \in \mathbf{nodes}(k)$ **do**

$\hat{B}_i^{-1} \leftarrow (I + \sum_{j\in\mathbf{ch}(i)} B_j' H_j B_j)^{-1}$

$K_i \leftarrow -\hat{B}_i^{-1}(\sum_{j\in\mathbf{ch}(i)} B_j' H_j A_j)$

$A_{ij} \leftarrow A_j + B_j K_i, \quad \forall j \in \mathbf{ch}(i)$

$H_i \leftarrow I + K_i' K_i + \sum_{j\in\mathbf{ch}(i)}(A_{ij}' H_j A_{ij})$

end for

end for

Algorithm 3 Projection algorithm (on-line)

Input: $A_{ij}, B_j, K_i, \hat{B}_i^{-1} \quad \forall i \in \mathbf{nodes}(1, N-1), \forall j \in \mathbf{ch}(i)$

{Backward iteration}
for $k = N-1 : -1 : 1$ **do**
  for $i \in \mathbf{nodes}(k)$ **do**
    $f_i \leftarrow \hat{B}_i^{-1}(v_i - \sum_{j \in \mathbf{ch}(i)} B_j' h_j)$
    $h_i \leftarrow -z_i - K_i' v_i + \sum_{j \in \mathbf{ch}(i)} A_{ij}' h_j$
  end for
end for
{Forward iteration}
for $k = 1 : N-1$ **do**
  for all $i \in \mathbf{nodes}(k)$ **do**
    $u^i = K_i x^i + f_i$
    for all $j \in \mathbf{ch}(i)$ **do**
      $x^j = A_j x^i + B_j u^i = A_{ij} x^i + B_j f_i$
      $x^j = A_{ij} x^i + B_j f_i$
    end for
  end for
end for

of matrix inverses. These steps are detailed in Algorithms 2 and 3 respectively.

5.6 Proximal algorithms

In this section we will examine algorithmic options to solve the problem outlined in (5.22). Note that problem is (almost) completely non-smooth. The only smooth term is in the objective and is a scalar. Hence, we turn our attention to proximal algorithms as they naturally work with non-smooth functions. Moreover, we will also require all the projections to be "easy" and we shall strive to avoid solving optimization problems to compute them as much as possible.

We consider problems of the form

$$\underset{x \in \mathbb{R}^n}{\textbf{minimize}} \quad f(x) + g(Lx) \tag{5.51}$$

Let X and Y be finite dimensional vector spaces, $L : X \to Y$ a linear operator , $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ are closed, proper, convex functions which are also *prox friendly*, i.e. their proximal operators are easy to compute.

Linearized ADMM (Douglas-Rachford)

The linearized ADMM(PB14) now takes the form of

$$x^{k+1} = \mathbf{prox}_{\mu f}\left(x^k - \frac{\mu}{\lambda} L^\top (Lx^k - z^k + u^k)\right) \tag{5.52a}$$

$$z^{k+1} = \mathbf{\Pi}_{\mathcal{K}}\left(Lx^{k+1} + u^k + b\right) - b \tag{5.52b}$$

$$u^{k+1} = u^k + Lx^{k+1} - z^{k+1} \tag{5.52c}$$

Computation of $\mathbf{prox}_{\mu f}(v)$ is a projection onto a zero cone in all but one variable whose **prox** operator is easily computable and is the objective of the overall optimization problem. Cone $\mathcal{K}$ in the above equations is the product of all cones in the optimzationa problem.

5.6.1 Chambolle-Pock method

Here, we will introduce an algorithm originally proposed by Chambolle and Pock (CP11). In the literature, it is also know as the Primal-Dual Hybrid Gradient (PDHG) method (ZC08; Roc76), but here we shall refer to it as the the Chamboll-Pock method. The Chamboll-Pock method is an instance of the *preconditioned proximal point method* (PPPM) with a particular preconditioner that allows for the computation of the proximal operator proximals of f and g^* (the conjugate of g). The Chambolle-Pock method does not require either f or g to be smooth or strictly/strongly convex. Even if the function f is smooth, the Lipschitz constant of its gradient can be very large necessitating very small step sizes, but (CP) does not suffer from this problem. Its step sizes are select as to satisfy a simple condition that involves the norm of the operator L. The algorithm consists of the following iterations

$$x^{k+1} = \mathbf{prox}_{\alpha_1 f}\left(x^k - \alpha_1 L^* z^k\right) \tag{5.53a}$$

$$z^{k+1} = \mathbf{prox}_{\alpha_2 g^*}\left(z^k + \alpha_2 L(2x^{k+1} - x^k)\right) \tag{5.53b}$$

The parameters, $\alpha_1, \alpha_2 > 0$, are primal and dual step sizes. Iterations defined by (5.53a) converge to a solution, assuming there is one, if

$$\alpha_1 \alpha_2 \|L\|^2 < 1 \tag{5.54}$$

is satisfied (CP11, Theorem 1).

Optimality conditions for this method can be stated as

$$0 \in \begin{bmatrix} \partial f & \\ & \partial g^* \end{bmatrix} + \begin{bmatrix} & L^* \\ -L & \end{bmatrix} = F(x^*, u^*) \tag{5.55}$$

Let us denote with $Z = X \times Y$ the primal-dual space and define operator $P : Z \to Z$ as

$$P = \begin{bmatrix} \frac{1}{\alpha_1} I & -L^* \\ -L & \frac{1}{\alpha_2} I \end{bmatrix}, \tag{5.56}$$

which is indeed positive definite as long as (5.54) is satisfied. This operator induces a new inner product on the space, which is given by $\langle z, z' \rangle_P = \langle z, Pz' \rangle$ and the corresponding induced norm $\|z\| = \langle z, z' \rangle_P$. We can now define a new operator T as

$$T = (P + F)^{-1} P. \tag{5.57}$$

Chamboll-Pock can be seen as a fixed point iteration of this operator T. Moreover, the *residual* operator is defined as

$$R = I - T$$

and it is *firmly nonexpansive* (see eq.(2.18)). Finding a fixed point of T is equivalent to finding a zero of R. We rely on this property in the next subsection where we describe how to use SuperMann scheme to find a zero of the above residual operator R following (STSP17a).

Since we already know how to project onto g efficiently, we make use of the extended Moreau identity to project onto g^*. The identity reads

$$\mathbf{prox}_{\lambda f}(x) = x - \lambda\, \mathbf{prox}_{\frac{1}{\lambda} f^*}(\frac{x}{\lambda}). \tag{5.58}$$

for $\lambda > 0$. In case of the second order cone this boils down to

$$\mathbf{prox}_{\sigma \delta^*_{\mathcal{K}_s}}(v) = v - \sigma \left(\mathbf{\Pi}_s(\frac{v+b}{\sigma}) - b \right). \tag{5.59}$$

Projection onto a nonnegative orthant changes to a projection onto a non-positive one

$$\mathbf{prox}_{\sigma \delta^*_{\mathcal{K}_+}}(v) = v - \sigma \Pi_+(\frac{v}{\sigma}) = [v]_- = \Pi_-(v). \tag{5.60}$$

5.6.2 SuperMann

Recently , authors in (TP19) have proposed a SuperMann algorihtmic scheme for finding fixed points of nonexpansive operators. The benefit of this scheme is that it guarantees, under certain assumptions, super-linear convergence and can be used as a wrapper around other operator splitting schemes. The downside is additional overhead in each iteration and a bit more complex algorithm , but usually, the speed-ups are well worth it, see (SMP19).

Let T be an α averaged update with $\alpha \in (0, 1)$. The classical Krasnoselskii-Mann (KM) scheme for finding a fixed point of T is given by

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k T x_k \tag{5.61}$$

which converges weakly to a fixed point of T as long

$$\lambda_k > 0, \quad \sum_k \lambda_k(1-\lambda_k) = \infty.$$

Sequence generated by the above algorithm is *Fejer* monotone, meaning

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 - \lambda_k(1/\alpha - \lambda_k)\|Rx_k\|^2 \quad \forall z \in \mathbf{fix}\, T. \tag{5.62}$$

The KM scheme is very general and it encompasses all of the operator based methods. Unfortunately, KM method suffers from well documented drawbacks like sensitivity to ill-conditioning and its Q-linear convergence. However, SuperMann scheme remedies those shortcoming by taking the KM step along a direction d_k. Authors in (TP19) introduce a new interpretation of KM iteration as a projection

$$x_{k+1} = (1 - 2\alpha\lambda)x_k + 2\alpha\lambda\, \mathbf{\Pi}_{C_{x_k}}\, x_k \tag{5.63}$$

with C_{x_k} being the half-space

$$C_{x_k} = \{y \in Z \mid \|Rx\|^2 - 2\alpha\langle Rx, x - y\rangle\}.$$

Building on the previous insight they propose a Generalized KM projection taken along a direction d_k given with

$$x_{k+1} = x_k - \lambda\frac{\rho_k}{\|Rw_k\|^2}Rw_k \tag{5.64}$$

where $w_k = x_k + \tau_k d_k$ for $\tau_k > 0$ and a positive $\rho_k = \|Rw_k\|^2 - 2\alpha\langle Rw, w - x\rangle$. In order for ρ_k be positive $\rho_k \geq \sigma\|Rw_k\|\|Rx_k\|$ for $\sigma \in (0,1)$ has to be satisfied. Generalized KM updates are taken to ensure the convergence of the general scheme and they are referred to as the *safeguard updates*. However, for the speed we want to accept the $x_{k+1} = w_k = z_k + \tau_k d_k$ updates, where d_k is a "good" direction. These updates are called *educated* updates and are accepted if $\|Rw_k\| \leq c\|Rx_k\|$ where $c \in (0,1)$, i.e. if the norm of the residual of the candidate solution is sufficiently better than the current point.

5.6.3 SuperMann on Chambolle-Pock (SPOCK)

In what follows, we shall operate on space equipped with the inner product $\langle \cdot, \cdot \rangle_P$, where P is given by (5.56). Chamboll-Pock operator T introduced previously is non-expansive with respect to inner product induced by P. SuperMann acceleration of Chambolle-Pock has been successfully applied to image processing (STSP17a) and we follow procedure outlined in that paper.

Steps of the method are outlined below in Algorithm (4). In lines 3-5 we compute the iterate of the Chambolle-Pock operator T and calculate the residual at a current point. We shall address the if condition a bit later. Next, we choose a direction d_k update our candidate point in line 7. We then calculate the residual of that point and check if conditions for an *educated* update are met. If so, we accept the candidate point and exit the loop. If not, we check the condition for *safeguard update*, namely if $\rho_k > 0$ so we can take the GKM update and exit the loop. Otherwise, we do one iteration of line-search and check the conditions again. This line search is guaranteed to finish and eventually we shall take a GKM update for any direction d_k (provided that it is not a good direction for educated updates).

We also e need to discuss how to choose a good direction d_k. The convergence of the scheme hinges on the method of direction d_k. In the original paper, properties of the SuperMann algorithmic scheme have been proven using Brodyen methods to select direction, but this approach is too memory intensive for any practical purpose. Authors in (STSP17a) suggest using restarted Broyden method. Here, however, we have found that using Anderson's acceleration (WN11) gives very good results with a low memory footprint. Anderson's acceleration method adapted to our problem is sketched below in Algorithm (5).

Next, we consider how to precondition this method to make it more robust to ill-conditioning.

Algorithm 4 SPOCK with Anderson acceleration

Input: $\alpha_1, \alpha_2 > 0, \alpha_1\alpha_2\|L\|^2 < 1, \sigma, c, q, \beta \in (0,1), \lambda \in (0,2), \epsilon > 0, m \in \mathbb{N}_{[1,..m^{\max}]}$

for $k = 0, 1, \cdots,$ **do**
  if last update was not educated **then**
    $\bar{x}_k = \mathbf{prox}_{\alpha_1 f}(x_k - \alpha_1 L^* u_k)$
    $\bar{u}_k = \mathbf{prox}_{\alpha_2 g^*}(u_k + \alpha_2 L(2\bar{x}_k - x_k))$
    $r_k = (x_k - \bar{x}_k, u_k - \bar{u}_k)$
  end if
  if $\|r_k\|_\infty < \epsilon$ **then**
    return $(\bar{x}_k, \bar{u}_k)$
  end if
  Calculate direction d_k using Anderson acceleration method
  loop $= 1, \tau_k = 1$
  while loop **do**
    $(w_k, v_k) = (x_k, u_k) + \tau_k d_k$
    $\bar{w}_k = \mathbf{prox}_{\alpha_1 f}(w_k - \alpha_1 L^* v_k)$
    $\bar{v}_k = \mathbf{prox}_{\alpha_2 g^*}(v_k + \alpha_2 L(2\bar{w}_k - w_k))$
    $\tilde{r}_k = (w_k - \bar{w}_k, v_k - \bar{v}_k)$
    if $\|\tilde{r}_k\|_P \leq r_{\text{safe}}$ and $\|\tilde{r}_k\|_P \leq c_0\|r_k\|_P$ **then**
      loop $= 0$
      $r_{\text{safe}} = \|\tilde{r}_k\|_P + q^k$
      $(x_{k+1}, u_{k+1}) = (w_k, v_k)$
      $(\bar{x}_{k+1}, \bar{u}_{k+1}) = (\bar{w}_k, \bar{v}_k)$
      $r_{k+1} = \tilde{r}_k$
    else
      $\rho_k = \|\tilde{r}_k\|_P^2 - \tau_k d_k^\top P \tilde{r}_k$
      if $\rho_k \geq \sigma\|\tilde{r}_k\|_P\|r_k\|_P$ **then**
        loop $= 0$
        $\eta_k = \lambda\rho_k / \|\tilde{r}_k\|_P^2$
        $(x_{k+1}, u_{k+1}) = (x_k, u_k) - \eta_k \tilde{r}_k$
      else
        $\tau_k = \beta\tau_k$
      end if
    end if
  end while
end for

Algorithm 5 Anderson's method for calculating d_k

Input: $x_k, u_k, x_{k-1}, u_{k-1}, r_k, r_{k-1}, Z_{k-1}, \Xi_{k-1}, m \in \mathbb{N}_+$

Update matrices of past measurement

$z_k = (x_k, u_k) - (x_{k-1}, u_{k-1})$

$\xi_k = (r_k - r_{k-1})$

$Z_k = [z_k \; z_{k-1} \ldots z_{k-m+1}]$

$\Xi_k = [\xi_k \; \xi_{k-1} \ldots \xi_{k-m+1}]$

$t_k = \mathbf{argmin}_{t_k} \|\Xi^k t_k - r_k\|^2$

$d_k = -r_k - (Z_k - \Xi_k) t_k$

return d_k

5.7 Preconditioning

It is a well known issue with first order methods that they tend to perform poorly on ill-conditioned problems. To accelerate the convergence of the first order algorithms we usually resort to various preconditioning schemes (Ben02; GB17). For the sake of computational simplicity we shall restrict our our attention to a class of diagonal preconditioners. Using diagonal preconditioners usually preserves simplicity of closed form solutions of proximal operator just as in non scaled case. Finding an optimal diagonal preconditioner can be cast as and SDP problem (BEFB94, Chapter 3), but here we focus on easy-to-compute heuristics which still perform well in practice. Note however, that (LXY18) recently proposed an interesting method (which shows promising results) using non-diagonal preconditioners and inexact calculations of proximal operators (RC18) as non-diagonal preconditioner ruins the "prox friendliness" of iterates.

5.7.1 Simple preconditioning method for Chambolle-Pock

For Chambolle-Pock authors describe the following simple approach (PC11) which also guarantees convergence of the algorithm. Given a positive definite matrix $X \in \mathbf{S}_{++}$ we can define inner product as $\langle x, y \rangle_X = \langle Xx, y \rangle$ and norm as $\|x\|_X = \langle x, x \rangle_X^{\frac{1}{2}}$. We can now write the *extended* proximal operator, in the notation of (LXY18), as

$$\mathbf{prox}_f^X(v) = \underset{x \in \mathbb{R}^n}{\mathbf{argmin}} \{ f(x) + \tfrac{1}{2} \|x - v\|_X^2 \}. \tag{5.65}$$

Extended proximal operator 5.65 is tied to the standard proximal operator by choosing $X = \gamma^{-1} I$ with $\gamma > 0$.

We also have a generalization of Moreau's identity (CR11) which is given by

$$v = \mathbf{prox}_f^X(v) + X^{-1}\, \mathbf{prox}_{f^*}^X(Xv). \tag{5.66}$$

Let us define two diagonal and positive definite matrices M, D. The authors in (PC11, Lemma 2) suggest a following simple heuristics to calculate elements of the scaling matrices

$$D_i = \frac{1}{\sum_{j=1}^{n} |A(i,j)|^{2-\alpha}}, \quad \forall i \in \mathbb{N}_{[1,n]} \tag{5.67}$$

$$M_i = \frac{1}{\sum_{j=1}^{m} |A(j,i)|^{\alpha}}, \quad \forall i \in \mathbb{N}_{[1,n]} \tag{5.68}$$

for any $\alpha \in [0,2]$, where M_i and D_i stand for i-th diagonal element. In general, the algorithm will converge to a solution if matrices M and D are chosen such that condition

$$\|M^{\frac{1}{2}} A D^{\frac{1}{2}}\|^2 < 1 \tag{5.69}$$

holds (PC11, Lemma 1).

General updates at each step now adhere to the following scheme

$$x^{k+1} = \mathbf{prox}_f^{D^{-1}} \left(x^k - DA^* z^k\right) \tag{5.70a}$$

$$z^{k+1} = \mathbf{prox}_f^{L^{-1}} \left(z^k + MA(2x^{k+1} - x^k)\right) \tag{5.70b}$$

This simple but effective heuristic allows us to compute scaling matrices without writing down the full operator L. Because of the relatively simple structure of L we can calculate scaling matrices directly using the above method. Furthermore, on a large scale problem, computing the norm of the operator L could be challenging which is altogether avoided here. Note as well, that operator P given in the previous chapter now takes the form of

$$P = \begin{bmatrix} D^{-1} & -L^* \\ L & M^{-1} \end{bmatrix}, \tag{5.71}$$

which is positive definite if (5.69) holds. The rest of the algorithm remains the same, except for the CP oracle of course which is now calculated as given in (5.70).

We can use matrix D directly with Chambolle-Pock scheme, but scaling with general M will lead to more difficult projections onto second order cone as we are scaling elements of a single SOC constraint with different values. In effect, we will ruin the closed form solution for such projections. Other constraints, such as equality constraints or inequalities are easier to handle, but SOC projections requires finding a root of a potentially non-convex function for every projection, a step which we will try to avoid here. Later, we briefly discuss how to do this if case the our problem is very ill-conditioned and this step is required. However, in general, we shall strive to avoid this approach.

A simple heuristic we propose is to select diagonal elements of matrix L that belong to the same SOC constraint, and set them all equal to the minimum element of that set. In this way the norm of scaled operator L will be no larger than one. Another very popular method which is successfully used in quadratic programs , see (SBG$^+$17) and (HTP19) for example, is described in (KRU14). Note that we have no guarantees on the magnitude of the norm of operator L, an important ingredient in Chambolle-Pock algorithm.

5.7.2 Scaling second-order cone constraints

Here, we will briefly discuss how to deal when we actually scale the elements of a SOC with different values. We follow the reasoning in (Bau96, Remark 3.3.22) to arrive at a nonlinear equation which gives the projection.

Let us denote with $D = \mathrm{diag}(d_i)$ where $d_i > 0$ is the i-th scaling constant predetermined beforehand by some method and let $d > 0$ be the scaling constant of the last element of the vector we wish to project onto a second order cone. Denote with $\bar{D} = \mathrm{diag}(D, d)$.

The extended proximal operator $\mathbf{prox}^{\bar{D}}_{I_{SOC}}$ given by (5.65) is equivalent to the optimization problem

$$\underset{x,t}{\mathbf{minimize}} \quad \frac{1}{2}\|x-y\|_D + \frac{1}{2}\|t-s\|_d \tag{5.72a}$$

$$\mathbf{subject\ to} \quad \|x\|_2 \leq t \tag{5.72b}$$

Optimality conditions for this problem are

$$(0,0) \in (D(x-y), d(t-s)) + \mu\,(\partial f(x), -1) \tag{5.73}$$

with $\mu \geq 0$. If we assume that $(x, t) \notin \mathbf{epi}(f(x))$ we can conclude that $\mu > 0$. From the optimality conditions we can see that $\mu = d(t - s)$. From complementary slackness we have that $f(x) = t$ at the optimum, hence $\mu = d(f(x) - s)$. Derivative of $f(x)$ with respect to x is $(x_i/\|x\|)$ for all i. Now it needs to hold

$$D(x - y) + d(f(x) - s)\left(\frac{1}{f(x)}\right), \tag{5.74}$$

hence for every element of vector d we have

$$d_i x_i + (d - sd/f(x))x_i = d_i y_i \tag{5.75}$$

or

$$x_i = \frac{d_i}{(d_i + d - s/f(x))} y_i. \tag{5.76}$$

Plugging it back into $f(x)$ we have

$$f(x) = \sqrt{\sum_{i=1}^{n} \frac{d_i^2 y_i^2}{(d_i + d - sd/f(x))^2}}. \tag{5.77}$$

If we further introduce the shortcut $k = f(x)$ we obtain

$$k = \sqrt{\sum_{i=1}^{n} \frac{d_i^2 y_i^2}{(d_i + d - ds/k)^2}}. \tag{5.78}$$

We can look for the solution of this equation numerically with Newton's method for example because we need a very accurate solution to make the overall algorithm viable. It can be noted that for $s > 0$ the function above has vertical asymptotes at points

$$k = \frac{ds}{d_i + d}, \quad \forall i \in \mathbb{N}_{[1,\ldots,n]} \tag{5.79}$$

where n is the length of vector x. The solution is to the right of the rightmost asymptote. This is important for newton type algorithms that might get stuck between the two asymptotes if the initial point is chosen badly. In case $s < 0$ the function has no asymptotes for k in $\mathbb{R}_+$.

Once we compute optimal value k^*, the projection onto second order cone in the new distance denoted with diagonal positive-definite matrix D and positive scalar d is

$$x_i = \frac{d_i y_i}{d_i + d - sd/k^*} \tag{5.80a}$$

$$t = k^*. \tag{5.80b}$$

5.8 Simulation results

In this section we will briefly present some preliminary results regarding convergence of SuperMann on Chambolle-Pock method that performs the best in practice.

For our test system we have chosen a two state Markovian system where state matrices are chosen randomly and we solve an optimal control problem using $\mathrm{AV@R}_\alpha$ nested risk measure with quadratic cost. Parameter α was set to 0.8.

In Fig. (5.8) we can see the effect of the prescaling method explained previously. We will present two criteria in figure; i) FPR which stands for fixed point residual and ii) an inifnity norm distance between MOSEK solution and our solution. We present the distance to MOSEK solution to verify that our algorithmic scheme indeed does converge to a true solution. Here, by solution, we mean the pair $(\mathbf{x}, \mathbf{u})$ as defined in (5.25a) and (5.25b), i.e., collection of all the state and control actions of the risk-averse problem. The steps of the algorithm were iterated until the fixed point residual dropped below 10^{-8}. Residaul of an operator T is given as

$$R = I - T.$$

To drive the point even harder, we will make the problem even more badly scaled with by selecting $P = 200$ and setting all the other matrices to be unitary.

5.9 Conclusions

This chapter offers a splitting tailor made for risk-averse problems to exploit the rich structure these problems possess. We have also presented some preliminary results which show that the method is sound in terms of accuracy with respect to state of the art solvers such as MOSEK (MOS16). In our future work we shall implement the proposed method on a GPU to make it competitive in terms of speed of execution and hopefully surpass generals purpose solvers.

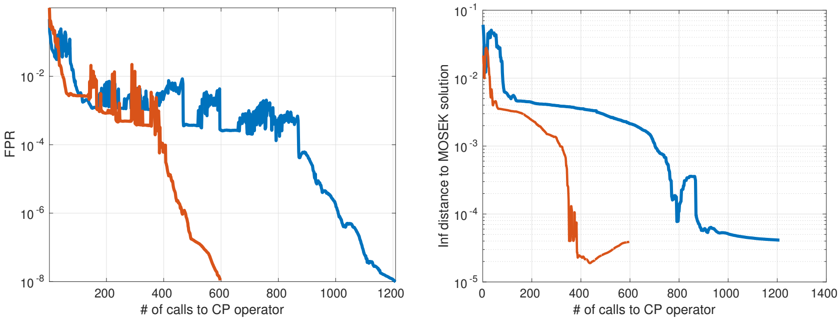

Figure 12: Convergence results for a two state Markovian system with $N = 5$. Results are shown for SuperMann method applied on Chambolle-Pock algorithm. We can clearly see the effect that scaling the problem has on convergence of the algorithm. Matrices were set to $Q_1 = Q_2 = 2I, R_1 = R_2 = 2I$ and $P = 20I$

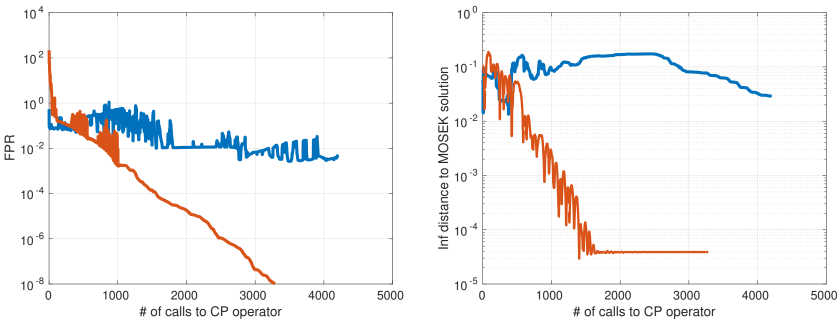

Figure 13: Convergence results for a two state Markovian system with $N = 5$. Results are shown for SuperMann method applied on Chambolle-Pock algorithm for a badly scaled problem. Matrices were set to $Q_1 = Q_2 = I, R_1 = R_2 = I$ and $P = 200I$. We can see that nonscaled version struggles to make any progress.

Chapter 6

Conclusions

Control of nonlinear and uncertain system is of paramount importance in control theory because of its vast implication for real-world systems. Precisely due to the complexity of dealing with such problems, we have striven to offer new theoretical framework for some open corner problems. Mainly, we show how to make stochastic approaches more robust the underlying assumptions on uncertainty, offer new numerical schemes for solving such problems and offer a theoretical framework for economic model predictive control in presence of uncertainty. Below we provide a summary of results obtained in this thesis.

6.1 Main contributions of the thesis

This thesis addresses several open problems in the are of control of nonlinear and uncertain systems. In particular we have obtained:

- **Novel theoretical framework for risk-averse MPC**
  Uncertain system are inherent in real-world operation of many system, particularly emerging technologies as robotics and large scale networked control. What is usually tacitly assumed is the uncertainty is modeled perfectly (which gives rise to stochastic approach) or it is not modeled at all (robust approach). Here, we have offered a sort of interpolation approach between the two extremes

using the theory of risk measure. What risk measures allow is to formulate a stabilizing MPC control law which accounts for the uncertainty in the underlying uncertainty allowing the system designer to balance between safety and performance concerns. We also show how to design a controller for the nonlinear system using linearization.

- **Computational methods for general risk-averse problems** Even thought risk-averse problems can be tackled using standard convex optimization software, we demonstrate that these problems posses a rich structure that we can exploit to devise very efficient and massively parallelisable methods to solve them. We show how to reformulate risk-averse problems in such way that they are suitable for parallel implementation. In doing so, we have leveraged recent advances in proximal algorithms to arrive at a robust and fast numerical.

- **Novel theoretical results regarding stochastic economic MPC** This chapter offers a theoretical framework for the control of Markovian switching systems using EMPC. We first introduce the notion of stochasticity into economic MPC framework and derive results akin to its deterministic counterpart. W also provided design guidelines based on the system linearization procedure.

Appendix A

Appendix

A.1 Scaled system dynamics projection

Here we give scaled version the algorithm used to projecto onto system dynamics.

Algorithm 6 Scaled projection algorithm precompute (off-line)

Input: $A_j, B_j, D^x_j, D^u_i \quad \forall i \in \mathbf{nodes}(1, N-1), \forall j \in \mathbf{nodes}(2, N)$
  $H_i \leftarrow D^x_i, \quad \forall i \in \mathbf{nodes}(N)$
  for $k = N-1 : -1 : 1$ **do**
    for $i \in \mathbf{nodes}(k)$ **do**
      $\hat{B}^{-1}_i \leftarrow (D^u_i + \sum_{j \in \mathbf{ch}(i)} B'_j H_j B_j)^{-1}$
      $K_i \leftarrow -\hat{B}^{-1}_i (\sum_{j \in \mathbf{ch}(i)} B'_j H_j A_j)$
      $A_{ij} \leftarrow A_j + B_j K_i, \quad \forall j \in \mathbf{ch}(i)$
      $H_i \leftarrow D^x_i + K'_i D^u_i K_i + \sum_{j \in \mathbf{ch}(i)} (A'_{ij} H_j A_{ij})$
    end for
  end for

Algorithm 7 Scaled Projection algorithm (on-line)

Input: $A_{ij}, B_j, K_i, \hat{B}_i^{-1} \quad \forall i \in \mathbf{nodes}(1, N-1), \forall j \in \mathbf{ch}(i)$

{Backward iteration}
for $k = N-1 : -1 : 1$ **do**
  for $i \in \mathbf{nodes}(k)$ **do**
    $f_i \leftarrow \hat{B}_i^{-1}(D_i^u v_i - \sum_{j \in \mathbf{ch}(i)} B_j' h_j)$
    $h_i \leftarrow -D_i^x z_i - K_i' D_i^u v_i + \sum_{j \in \mathbf{ch}(i)} A_{ij}' h_j$
  end for
end for
{Forward iteration}
for $k = 1 : N-1$ **do**
  for all $i \in \mathbf{nodes}(k)$ **do**
    $u^i = K_i x^i + f_i$
    for all $j \in \mathbf{ch}(i)$ **do**
      $x^j = A_j x^i + B_j u^i = A_{ij} x^i + B_j f_i$
      $x^j = A_{ij} x^i + B_j f_i$
    end for
  end for
end for

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