Journal of Philosophical Logic
https://doi.org/10.1007/s10992-024-09774-3

Patches, Patchworks, and Epsilon Terms: A Neo-Carnapian Account of Theoretical Terms in Science

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Received: 11 January 2024 / Accepted: 18 August 2024

Abstract

In the last decades, scientific laws and concepts have been increasingly conceptualized as a patchwork of contextual and indeterminate entities. These patchwork constructions are sometimes claimed to be incompatible with traditional views of scientific theories and concepts, but it is difficult to assess such claims due to the informal character of these approaches. In this paper, we will show that patchwork approaches pose a new problem of theoretical terms. Specifically, we will demonstrate how a toy example of a patchwork structure might trivialize Carnap's semantics for theoretical terms based upon epsilon calculus. However, as we will see, this new problem of theoretical terms can be given a neo-Carnapian solution, by generalizing Carnap's account of theoretical terms in such a way that it applies also to patchwork constructions. Our neo-Carnapian approach to theoretical terms will also demonstrate that the analytic/synthetic distinction is meaningful even for patchwork structures.

Keywords Theoretical terms · Patchwork concepts · Scientific patchworks · Epsilon calculus · Carnap

1 Introduction

Theoretical terms have posed a considerable challenge to empiricist views of scientific inquiry since the beginning of the twentieth century. Specifically, finding an adequate semantics for these terms has proven to be particularly difficult. A well-known formal solution to this problem was developed by Carnap, who defined theoretical terms by means of Hilbert's epsilon operator. Carnap saw that the indeterminate way of

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Published online: 14 September 2024

reference of the epsilon operator could be used to account for the open-endedness of theoretical terms in science. Thanks to the expressive power of epsilon terms, Carnap managed to give an adequate, on empiricist grounds, characterization of theoretical terms. He, in fact, managed to separate, in a theory-dependent way, the empirical part of a theory from its non-empirical one, making the former synthetic and the latter analytic.

Yet, in the last decades, the received view of scientific theories and laws in philosophy of science has changed substantially. The so-called practice turn in philosophy of science pushed philosophers to reconsider central assumptions of the semantic-centered orthodox view of scientific theories. As a consequence of this turn, pragmatist approaches to scientific theories have increasingly conceptualized scientific laws and concepts as fundamentally contextual and indeterminate entities. A paradigmatic example of this tendency are the so-called patchwork approaches to scientific concepts, i.e., approaches that model scientific concepts as complex patchworks of partly-connected, locally valid patches of usage. Although patchwork approaches rarely explicitly engage with the traditional literature on theoretical terms, nor they spell out in detail the semantic implications of their approach, some supporters of patchwork approaches claimed that the mere existence of patchwork constructions in science is enough to falsify traditional views of scientific theories and concepts. In this paper, we will formally reconstruct the semantic behaviors described by patchwork approaches and we will show that the existence of patchwork constructions has important consequences for the semantics of theoretical terms.

Specifically, we will argue that patchwork approaches to scientific concepts pose a new problem of theoretical terms, i.e., the issue of finding a semantics for theoretical terms consistent with the contextual behaviors described by patchwork approaches. We will do that by demonstrating that the kind of context-dependency and polysemy that, according to patchwork approaches, scientific concepts exhibit might trivialize Carnap's semantics for theoretical terms. However, we will show that this new problem of theoretical terms can be given a neo-Carnapian solution by modifying Carnap's epsilon-based approach. More specifically, our neo-Carnapian account of theoretical terms modifies Carnap's in two central ways: it redefines the Ramsey sentence of a theory as the conjunction of the Ramsey sentences of the single patches (crucially constrained by the satisfaction of the inter-patches relations) and, then, it redefines theoretical terms over the disjunction of single-patches-related definitions. We will show how our neo-Carnapian account is able to model the semantic behavior of theoretical terms for a wide class of patchwork structures. Our neo-Carnapian account of theoretical terms will also demonstrate that, in contrast to what some patchwork theorists claimed, the analytic/synthetic distinction is meaningful also for patchwork constructions.

The aim of this paper is two-fold. First, we seek to generalize Carnap's epsilon-based account of theoretical terms in science to a wider range of semantic behaviors, in order to defend the viability of its philosophical assumptions, also in relation to contemporary pragmatic accounts of scientific theories and concepts. Our second aim is to offer a neo-Carnapian view of patchworks, providing a formal reconstruction of the semantic implications that the existence of patchwork constructions has for our understanding of theoretical terms in science.

In Section 2, we will present Carnap's mature epsilon-based account of theoretical terms in science. In Section 3, we will turn to pragmatist understandings of scientific theories and concepts, focusing specifically on so-called patchwork approaches. In Section 4, we will show how a straightforward formalization of the semantic behaviors described by patchwork approaches might trivialize Carnap's account of theoretical terms, posing a general challenge to traditional accounts of theoretical terms. In Section 5, we will present our neo-Carnapian solution of this new problem of theoretical terms. In Section 6, we generalize our neo-Carnapian account of patchworks, proving some of its properties and comparing it to existing alternatives. Section 7 concludes.

2 Carnap's Account of Theoretical Terms in Science

In this section, we will present Carnap's [10–12, 35] mature account of theoretical terms in science. More specifically, we will focus on Carnap's [11, 35] epsilon-based formulation of his mature account, a formulation that uses the expressive power of epsilon calculus to explicitly define theoretical terms.

Theoretical terms, as understood by Carnap, are non-empirical and non-logical terms of a scientific theory, such as "force", "gene", "electro-magnetic field", and the like. These terms usually figure in the most general and fundamental laws of a given scientific theory and they usually refer to non-observable entities. The meaning of theoretical terms is not acquired in isolation, but it is instead holistically determined via the laws of the scientific theory in which they figure, so that, say, the meaning of "force" in classical mechanics cannot be specified without reference to Newton's laws.[1] Moreover, the meaning of a theoretical term always appears under-specified by the laws of a theory, making the term fundamentally open-ended. Scientists can, in fact, always add new laws to a given theory, thereby further specifying (and thus changing) the meaning of the theoretical terms that figure in these laws. This is, for instance, what happened in the eighteenth and nineteenth century for the theoretical term 'force' in classical mechanics, the meaning of which was repeatedly specified by domain-specific laws that hold for certain kinds of force (e.g., elastic, frictional, viscous forces, etc.). These two semantic properties of theoretical terms, i.e., the holism and the open-endedness of their meaning, are, according to Carnap [10], the reason why theoretical terms cannot be explicitly defined via nor reduced to empirical or observational terms.

Theoretical terms pose a challenge to any empiricist philosophy of science such as Carnap's, since, despite not being definable nor reducible to empirical terms, they figure in the most important laws of our best scientific theories. As such, theoretical terms seem to break Carnap's empiricist ideal that the factual content of a scientific

[1] This characterization of theoretical terms based on the holism of their meaning is not the only one possible. For instance, theoretical terms could also be characterized in terms of their specific measurements procedures, as it was done by model-theoretic structuralists (cf. [6]). For a general discussion of the two approaches and their relations, see [3].

theory is somehow identifiable with its empirical content. The mature Carnap [10, 12] still held that the empirical content of a scientific theory can be divided from its theoretical content, in such a way that the latter is analytic, i.e., true in virtue of form or meaning alone, and the latter is synthetic, i.e., not analytic. In other words, Carnap considered the empirical content of a scientific theory to be its factual, world-determined part, and the theoretical content to be instead its conventional, theory-determined part. Theoretical terms seem to break this neat division, as they figure in the main laws of a given theory (and thus arguably pertain to its factual content), but cannot be reduced to the empirical content of the theory (due to their holistic and open-ended meaning).

Carnap's specific understanding of the problem of theoretical terms aims at giving an adequate semantics of these terms in a way that restores this conceptual division between these three important epistemological and semantic distinctions about the content of a scientific theory: factual/conventional, empirical/theoretical, and synthetic/analytic. More formally, Carnap reconstructs scientific theories in formal languages (cf. [26]), where he divides the non-logical language of a scientific theory in an observational and a theoretical part. The observational language $\mathcal{L}_{\mathrm{O}}$ contains only observational sentences, i.e., descriptive sentences in which only observational terms occur, while the theoretical language $\mathcal{L}_{\mathrm{T}}$ contains only theoretical sentences, i.e., descriptive sentences in which also theoretical terms occur. The theoretical sentences that make up the theoretical language of a scientific theory are of two kinds: theoretical postulates and correspondence rules. Theoretical postulates are theoretical sentences that contain only theoretical terms (i.e., no observational term occurs), while correspondence rules (also called operational rules or coordinative definitions) are theoretical sentences that contain both theoretical terms and observational terms. As such, Carnap semantically represents a scientific theory as a combination of two axiomatic components, i.e., the conjunction of its observational postulates, denoted by A_{O}, and the conjunction of all the theoretical postulates and correspondence rules $\varphi(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are all the theoretical terms occurring in it, denoted by TC:

$$A_{\mathrm{O}},\ TC \qquad \text{(Form I)}$$

Given these definitions, we can reframe Carnap's [11, pp. 159-160] problem of theoretical terms as the task of isolating the factual/empirical content of TC from its conventional/theoretical content, making the former synthetic and the latter analytical. Carnap's solution starts from a technical intuition of Ramsey [36], i.e., the so-called ramsification of a scientific theory. Ramsey noticed that one could substitute, *salva veritate*, each predicate of a given theory (e.g. $\mathrm{P}_1, \mathrm{P}_2, \ldots$) with a new variable of the same type (e.g. $X_1, X_2, \ldots$), bound by an existential quantifier. So that, instead of a given property, one could have a quantified variable (interpreted as "there is something that [description of the property]"). As already noticed by Hempel [22], Carnap [11, 12, 35] highlighted that ramsification can play an important role for empiricist philosophy of science, because, if one applies this technique to the theoretical terms of a given scientific theory, one obtains an observationally equivalent version of the theory without any theoretical term. Specifically, Carnap [11, p. 159] noticed that if we apply the ramsification technique to TC, we obtain what is usually called the Ramsey Sentence of a theory:[2]

$$R := \exists x_1 \cdots x_n \, \varphi(x_1, \ldots, x_n)$$

Carnap considers the Ramsey Sentence as expressing the empirical content of (the theoretical language of) a scientific theory. This is because the Ramsey Sentence of the theory has exactly the same observational consequences of TC. If the empirical content of TC is represented by the Ramsey Sentence, the theoretical content is represented by what is now called the Carnap sentence (or the theoretical postulate) of the theory, i.e., the following conditional:

$$A_{\mathrm{T}} := R \rightarrow TC$$

The Carnap Sentence of a theory is then the conditional that has the Ramsey sentence of the theory as its antecedent and TC as its consequent. It can be interpreted as stating that "if the Ramsey sentence is true, we must understand the theoretical terms in such a way that the entire theory is true" [12, p. 270]. Note that the Carnap Sentence does not state an empirical claim, but only a semantic one, since it only states that if the Ramsey sentence is true, then its witnesses are the theoretical terms of TC. Moreover, the Carnap sentence does not entail any observational consequence (except the logically true ones) and it can be thus considered fully analytical. This is why Carnap [11, p. 160] refers to the Carnap Sentence also as the analytic postulate A_{T} of a given scientific theory. Given these definitions, a scientific theory can thus be represented, according to Carnap, in the following way:

$$A_{\mathrm{O}},\ R,\ A_{\mathrm{T}} \qquad \text{(Form II)}$$

The combination of the Carnap sentence and the Ramsey Sentence can be considered a first formulation of Carnap's mature solution to the problem of theoretical terms. In fact, through these two sentences Carnap achieves the long wanted separation of the factual content of a scientific theory from its conventional content. The two sentences are an equivalent reformulation of the theory, since the Carnap sentence, just like the Ramsey Sentence, is a logical consequence of the theory TC and the two sentences together entail the theory TC by *modus ponens*. In this reformulation, the factual/empirical content of the theory, encoded in the Ramsey sentence, is sharply separated from the conventional/theoretical content, encoded in the Carnap sentence. Moreover, the Ramsey sentence is evidently synthetic in character, expressing all the observational consequences of the theory, while the Carnap sentence is clearly analytical, expressing a purely semantic, meaning-defining claim. In this way, Carnap restores the neat division between the factual/conventional, the empirical/theoretical, and the synthetic/analytic content of a scientific theory.

[2] Here, for simplicity, we apply ramsification to first-order terms (and not on predicates), in order to work at the first-order level.

This formulation of Carnap's mature account of theoretical terms, despite its many positive sides, is still not a fully ideal solution to the problem of theoretical terms. The combination of Ramsey Sentence and Carnap sentence does not, in fact, provide an explicit definition of theoretical terms.[3] Such a long-sought definition of theoretical terms was achieved by Carnap in his last formulation of his mature account of theoretical terms. This other formulation starts from the previous one, but it crucially relies on Hilbert's epsilon operator.

The epsilon operator ε, first introduced by Hilbert [5, 23], is a variable-binding device that expresses indeterminate denotation of witnesses for a formula, that is, it allows to pick an object of the domain, if any exists, that satisfies a certain formula. When this operator is applied to a given formula $\varphi(x)$, the resulting term $\varepsilon x\, \varphi(x)$ is understood as "an object x, if any, satisfying $\varphi(x)$" and x refers to an arbitrary witness in the domain for the property denoted by $\varphi(x)$, if such an object exists. If, instead, no object in the domain satisfies $\varphi(x)$, an arbitrary object in the domain is chosen. Carnap's [11, p. 156] characterization of the epsilon operator ε relies on the following axiomatization:[4]

- $\exists x\, \varphi(x) \rightarrow \varphi(\varepsilon x\, \varphi(x))$ (Critical axiom)
- $\forall x\, (\varphi \leftrightarrow \psi) \rightarrow \varepsilon x\, \varphi = \varepsilon x\, \psi$ (Extensionality axiom)

For convenience, we follow Carnap [11, p. 160] and denote by $\mathbf{t}$ the tuple of all the theoretical terms occurring in TC, i.e. $\langle t_1, \ldots t_n \rangle$, and with a slight abuse of notation abbreviate $\varphi(t_1, \ldots t_n)$ by $\varphi(\mathbf{t})$. Given the previous axiomatization of the epsilon operator, Carnap [11, p. 161] explicitly defines $\mathbf{t}$ as follows, for $\mathbf{x}$ denoting a variable for tuples:

$$A(\mathbf{t}) := \mathbf{t} = \varepsilon \mathbf{x}\, \varphi(\mathbf{x})$$

From $A(\mathbf{t})$ we can thus obtain the explicit definition of each theoretical term $t_i \in \mathbf{t}$:

$$A(t_i) := t_i = \varepsilon y\, (\exists x_1 \ldots x_n\, \mathbf{t} = \langle x_1, \cdots, x_n \rangle \wedge y = x_i)$$

In this way, the theoretical postulate A_{T} can be rephrased as the conjunction of $A(\mathbf{t})$ and that of the explicit definitions of the single theoretical terms:

$$A_{\varepsilon} := A(\mathbf{t}) \wedge \bigwedge_{i=1}^{n} A(t_i)$$

This epsilon-based formulation of Carnap's theoretical postulate has all the advantages of the Carnap sentence, but it also gives an explicit definition of theoretical

[3] There are other philosophically problematic aspects of the non-epsilon-based formulation of theoretical terms, such as its lack of compositionality, and some technical ones, such as the need of an additional axiom. Since we will focus in this paper on the epsilon-based formulation, we will not discuss these issues here. For details see [16–18, 24, 25, 28, 37, 38, 43].

[4] The Critical axiom ensures that epsilon terms can be introduced as witnesses of existentially quantified statements, while the Extensionality axiom ensures the same choices of objects are performed for witnesses of equivalent formulas (i.e., denoting the same property).

terms. With this new epsilon-based theoretical postulate, Carnap's mature solution to the problem of theoretical terms can be presented as:

$$A_{\mathrm{O}},\ R,\ A_{\varepsilon} \qquad \text{(Form III)}$$

Carnap was so happy and so surprised about this result that, as the transcript of the Santa Barbara lecture shows (cf. [35, p. 168]), he could for a time not believe that the epsilon operator would allow an explicit definition of theoretical terms. Yet, the epsilon operator, thanks to its reference-determining mechanism, allows indeed the explicit definition of under-determined and open-ended entities such as scientific theoretical terms.

With this last formulation, Carnap completes his solution to the problem of theoretical terms, separating the empirical and the theoretical content of the theory, consistently with the synthetic/analytic and the factual/conventional divisions, and explicitly defining theoretical terms. The empirical content of a scientific theory can be reconstructed as the Ramsey Sentence of that theory. This empirical content is factual, synthetic, it does not involve any reference to theoretical entities, entailing all and only the observational consequences of the original theory. The theoretical content of a scientific theory can be instead reconstructed via the conjunction of epsilon-based definitions of the theoretical terms of the theory. These epsilon-based definitions of theoretical terms do not entail any observational consequence (aside from the logically true ones), and they imply only conventional truths about the theoretical terms, respecting the under-determination, the holism, and the open-endedness of the meaning of such terms.

3 The Pragmatic Turn: from Theories to Patchworks

We saw how Carnap achieved, at last, a fully satisfactory account of theoretical terms, thanks to the expressive power of the epsilon operator. In this section, we will shift our focus from the problem of theoretical terms to recent pragmatic accounts of scientific laws and concepts.

Philosophy of science has changed a lot in the last sixty years. One of the most significant conceptual shifts from the kind of philosophy of science practiced in Carnap's time is the so-called practice or pragmatic turn. With this umbrella term, philosophers usually refer to the increasing attention dedicated by the philosophical community since the 1960s to the practical and pragmatic aspects of scientific inquiry. This increasing attention dedicated by philosophers to the practical and pragmatic aspects of science made them reconsider some central assumptions of the received semantic-centered philosophical image of science. In this section, we will focus on certain important modifications that the pragmatic turn determined for the received view of scientific theories and concepts.

One of the most important changes caused by the pragmatic turn for our understanding of science involves the alleged universality and generality of scientific laws. For a large part of the twentieth century, the received view of the internal structure of a scientific theory wanted it to be centered around a group of general and universal laws that, via suitable restrictions or additional parameters, could be easily applied to any phenomenon belonging to the domain of the theory.[5] From the 1980s onward, several pragmatist approaches in philosophy of science criticized this alleged universality and generality of scientific laws (e.g., [7, 13, 14, 19, 29, 30, 41, 42]).[6] A paradigmatic example of these pragmatist critiques to the universality and generality of scientific laws is Cartwright's [13, 14] contextualist picture of how scientists apply a general scientific law to a specific problem. In applying a general scientific law to a given specific domain, Cartwright (Cartwright 1983, pp. 21-73, Cartwright 1999, pp. 23-74, 179-233) argued through a series of detailed examples from physics and economics, scientists often drastically modify this general law, up to the point that different applications of the same law can differ so much to describe the world in incompatible ways. As such, the received view in philosophy of science of scientific theories, which wanted them to be structured around a group of general and universal laws applicable to any phenomenon in the domain of the theory via suitable restrictions or additional parameters, is untenable. According to pragmatist approaches to scientific theories, then, scientific practice involves complex contextual adjustments and modifications of scientific theories to the specific problem and domain under focus. Our best scientific theories are not structured as a neat hierarchy of laws of growing generality, but as a patchwork of laws, locally-valid in a specific domain of application and related to each other in a complex way (cf., [7, 14, 41, 42]).

If scientific laws have to be often contextually adjusted in scientific practice, the same holds, according to pragmatist views of science, for scientific concepts. The inferences afforded by a given scientific concept often vary from application to application or from scale to scale, as well as the related measurement techniques (cf. [7, 8, 41, 42]). Semantically, the same scientific term can have different referents, different meanings, and can figure in different inferences in different parts of the same scientific theory. This is the main idea behind so-called patchwork approaches to scientific concepts (e.g., [15, 21, 34, 41]), i.e., a group of pragmatic theories which argues that scientific concepts are structured as a complex cluster of partially-connected local domains of usages. These different usages of a term are loosely connected with each other via multiple kinds of relationships, local and general ones. This patchwork structure is what is behind the polysemic behavior that many scientific concepts often exhibit (cf. [34, 40, 41]). In recent years, many important scientific concepts have been shown to exhibit a patchwork structure. The list of recognized patchworks includes concepts from very different sciences such as force [41, 42], hardness [41], species [32], gold [9], neural column [20], homology [31], and attention [39]. In all these examples of

[5] Perhaps the clearest example of this received view can be seen at work in model-theoretic structuralism, where the relationship between the models of the most general laws of a theory and their more applied counterparts is a subset relation (i.e., the specialization relation, in structuralist lingo), (see [6, pp. 168-176]). The example of structuralism is particularly relevant for us, because, as stressed by several authors, Carnap's mature philosophy of science can be seen as closely related to structuralist reconstructions of scientific theories (cf. [1, 27, 38]).

[6] Here we are lumping together rather different authors who, in different times, different contexts, and for different reasons, contested the alleged universality and generality of scientific laws. Despite their differences, these authors all share a common critical aim and, as such, we pragmatically group them together here.

patchwork structures, the semantic content of scientific concepts changes consistently with the demands of particular applications and domains. Conceptual change, usually assumed by the received view of scientific theory to be exclusively a diachronic phenomenon caused by scientific revolutions, is, according to patchwork approaches, a common phenomenon that happens also synchronically within the same theory.[7]

Scientific concepts and laws have been conceptualized by pragmatic philosophy of science as extremely contextual and structurally-complex entities, the semantic plasticity of which allows scientists to tinker their theories to the practical demands of science and its applications. How does such a picture of scientific laws and concepts affect our received view of theoretical terms in science? Although pragmatist approaches rarely connect their proposal directly to the problem of theoretical terms, their picture of scientific concepts as contextual entities seems to be extremely relevant for the problem of theoretical terms. If, in fact, scientific laws and concepts are contextual and polysemic entities, as pragmatic approaches argue, then any adequate account of theoretical terms in science must take this contextuality and polysemy into account. To see this connection, recall that, as we mentioned in the last section, the meaning of theoretical terms is assumed to be holistically (semi-)determined by the general laws of the theory in which they figure. If the laws and the concepts of a scientific theory are complex patches of localized usages, then the meaning of theoretical terms is also determined by an analogous patchwork.

More precisely, from a traditional Carnapian perspective, we can see patchwork approaches as characterizing theoretical terms not only as theory-dependent terms, but actually, in certain cases, as patches-dependent terms. That is, patchwork approaches can be interpreted as implying that the meaning of theoretical terms can be not only dependent on the theory to which they belong, but also on a proper part of it. This is because the postulates determining their meaning (both the theoretical postulates and the correspondence rules) can be just locally valid, i.e., valid only within a certain sub-domain of the theory, namely, a patch. This local validity of meaning postulates for theoretical terms is the semantic representation of the contextuality of scientific laws and concepts stressed by the aforementioned pragmatic approaches in philosophy of science. Moreover, these theoretical postulates, valid in different patches, can be globally incompatible with each other. From this perspective, then, consistency is a requirement only for the single patches, which can therefore be treated as mini-theories. As such, the relationships between the patches are often not straightforward, in that the interpretation of theoretical terms can drastically differ from patch to patch.

What happens to a standard view of theoretical terms, such as Carnap's, when we allow the meaning postulates for theoretical terms to be only locally valid? Although, as we already stressed, pragmatist approaches rarely explicitly engage with the problem of theoretical terms, nor they spell out the exact semantic implications of their proposal, some proponents of these approaches seem to believe that the existence of scientific patchworks is a fatal blow to traditional views of scientific theories and concepts. For example, Wilson [41] claims that the mere existence of patchwork structures in

[7] Carnap explicitly stresses this diachronic trans-theoretic character of conceptual change at the end of his 1956 discussion of theoretical terms, where he states that changing a theoretical postulate would amount to nothing less than a scientific revolution (cf. [10, p. 51]).

science is enough to reject the validity of the analytic/synthetic distinction in science and, with that, Carnap's whole account of scientific theories and theoretical terms:

> "Logical positivists such as Rudolf Carnap (4, iv) were quite taken with Poincare's conventionalism and believed that major hunks of their implicitly defining axiomatic schemes enjoy that status. Because Quine essentially smudges the positivist's tidy 'theories' into messier 'webs of belief' (5, v), he famously remarks that our opinions represent a 'gray lore, black with fact and white with convention'. He believes that the two shades can't be sorted out due to the rough hewn and holistic manner in which the web enlarges (his famous critique of the analytic/synthetic distinction rests upon this basis). However, we now see that what goes wrong with conventionalism is often more localized than this: as a single statement, '$F = ma$' can enter into sundry computational recipes in different ways and this fact alone induces it to span patches where its former property correlations have become grayed through attribute dragging. In my opinion, the basic validity of Quine's rejection of analytic/synthetic clarities can be sustained without appealing to holism of any kind, but simply in terms of the non-classical looseness of predicate/world ties that tolerates diverse forms of patch-to-patch prolongation" [41, p. 372]

Carnap's empiricist account of theoretical terms as definable via the postulates of a theory is flawed, according to Wilson, simply because scientific laws and concepts are often not stable across the different parts of a theory. That is, the different patches of a single scientific patchwork are often connected among each other and to the world in different ways. The "looseness" of these connections invalidates Carnap's empiricist division of a theory into a synthetic, empirical part and an analytic, theoretical part. Or so Wilson claims. In fact, as we mentioned before, due to the lack of formality of most patchwork accounts, the implications of these approaches for a formal semantics of theoretical terms like Carnap's are not so clear. In the next section, we will clarify the implications of patchwork approaches for formal accounts of theoretical terms, by formalizing a toy example of a patchwork structure and by showing how such a toy example is enough to pose a new problem of theoretical terms.

4 The New Problem of Theoretical Terms

We saw how patchwork approaches to scientific laws and concepts sketch a semantic structure of scientific theories far more complex than what traditional semantics for theoretical terms in science, such as Carnap's, assumed. The meaning of many important theoretical terms in science seems to be partly dependent on the specific part of the theory under focus, i.e., the patch. This patch-dependency has been claimed to be enough to refute Carnap's account of theoretical terms and its division of a scientific theory into an analytic part and a synthetic part.

In this section, we will show that a patch-dependent picture of theoretical terms poses indeed a new problem of theoretical terms. We will do that by virtue of a formal toy example of a patchwork construction inspired by Wilson's [41] main example of a scientific patchwork, namely, the semantic of 'force' in classical mechanics. We will show that such a simple formal structure can trivialize Carnap's mature account of theoretical terms.

As our formal toy example of a patchwork, we will take the patchwork $\mathcal{P}$, containing the three patches P_1, P_2 and P_3. We assume a simple base theoretical language $\mathcal{L}_T$ for all patches, containing only two theoretical terms t and t'. Each patch P_i validates some theoretical sentence $\varphi_i(t, t')$. For simplicity, we assume that the observational language is the same in all the patches. For what concerns the two theoretical terms of the language, we assume that P_1 and P_2 share the same interpretation of t, but have a different one for t'. We do not make any assumption on P_3. Schematically, we can represent the patchwork as a couple $\langle \mathcal{S}, \sim \rangle$, where $\mathcal{S}$ is the set of patches $\{P_1, P_2, P_3\}$ and $\sim$ is a primitive ternary relation on patches and theoretical terms $P_h \overset{t_j}{\sim} P_k$, which holds if and only if two patches $P_i, P_k \in \mathcal{S}$ have the same interpretation of a given theoretical term t_j in all intended models.[8] Given our assumptions on the interpretation of theoretical terms in different patches, our patchwork $\mathcal{P}$ can be represented as follows:

$$\mathcal{P} := \langle \mathcal{S}, \sim \rangle, \quad \text{for } \mathcal{S} = \{P_1, P_2, P_3\},\ P_1 \overset{t}{\sim} P_2 \text{ and } P_1 \overset{t'}{\nsim} P_2$$

This simple formal patchwork, in which two patches (i.e., P_1 and P_2) have a different interpretation of a given theoretical term (i.e., t'), while having the same one for another term (i.e., t) is a simplified formalization of the semantic behavior that, according to patchwork approaches, many scientific theories exhibit. This can be seen, for instance, by looking at Wilson's [41, pp. 157-165, 175-182] reconstruction of the 'force' patchwork in classical mechanics. Wilson [41, pp. 175-176], in fact, argues that the meaning of the term 'force', as commonly interpreted in Newton laws and in the related Newtonian patch of classical mechanics, is radically different from the interpretation of 'force' that one can see at work in more recent patches of classical mechanics, such as the viscous fluids one (cf. [41, pp. 158-159]). Our toy example above can be seen as a minimal, heavily simplified, formalization of a proper part of the classical mechanics patchwork, where P_1 represents the Newtonian patch, P_2 the viscous fluids patch, t represent the theoretical term 'weight', and t' represents the theoretical term 'force'.

Two things concerning our example should be noted. First, our third patch P_3 serves only generalization purposes and, as such, can be seen as representing another arbitrary patch of the classical mechanics patchwork. Secondly, our formal toy example of a patchwork reconstructs, within a broadly Carnapian view of scientific theories, a specific semantic behavior that scientific concepts, according to patchwork theories, might exhibit, namely, the case of a given theoretical term that is interpreted in different ways by different patches of the same patchwork. This case is just one of the several semantic phenomena that, according to patchwork approaches, scientific

[8] Note that such a definition of the relation $\sim$ allows for the case in which two patches neither share nor differ, in all intended models, in the interpretation of a theoretical term.

theories might exhibit (cf. [15, 21, 33, 41]).[9] We focus here on this rather minimal example because, as we shall see in what follows, this will be enough to trivialize Carnap's account and, thus, to substantiate Wilson's claim that the existence of scientific patchworks is incompatible with Carnap's account of scientific theories and theoretical terms.

Equipped with our formal toy example of a patchwork, let us see whether Carnap's account of theoretical terms can be applied to it. Recall that Carnap's account starts by considering TC, i.e., the conjunction of all the theoretical sentences (postulates and correspondence rules) of a theory. In our patchwork $\mathcal{P}$, this corresponds to the conjunction of all the theoretical postulates of the three patches, i.e., $\varphi_1(t, t') \wedge \varphi_2(t, t') \wedge \varphi_3(t, t')$. Carnap's account would then ramsify this conjunction, in order to isolate its empirical content. Yet, such a move is already problematic for our simple patchwork. Since, in fact, no interpretation of t' is shared by all patches, the conjunction of their theoretical postulates might be false, which would have the effect of making also its ramsification false. Things get worse if one looks at what would the Carnap sentence for such patchwork look like. Since the ramsification would be false, the conditional that has it as an antecedent, i.e., the Carnap sentence, would be trivially true. Even worse would be the fate of Carnap's epsilon definitions of theoretical terms. In fact, recall that epsilon terms, when applied to a false formula, refer to an arbitrary element of the domain. As such, Carnap's epsilon-based definitions of theoretical terms for our patchwork would just pick arbitrary elements in the patchwork domain as their references.

While none of the above issues necessarily produces an inconsistency, Carnap's account of theoretical terms is completely trivialized by our simple toy example of patchwork $\mathcal{P}$. The Ramsey sentence of the patchwork, i.e., its empirical content in Carnap's account, turns out to be false. The Carnap sentence, i.e., its conventional content in Carnap's account, is trivially true. The theoretical terms defined by the epsilon terms pick up arbitrary elements in the domain. As such, a simple difference in the interpretation of one theoretical term between two patches of a patchwork, a phenomenon that according to patchwork approaches is quite common in scientific inquiry, is enough to make Carnap's solution to the problem of theoretical terms crumble.

More exactly, we can now describe more precisely which problems scientific patchworks pose to traditional accounts of theoretical terms like Carnap's.[10] The existence of semantic structures like patchworks, in which theoretical terms might be interpreted differently among patches, poses both philosophical and technical problems to Carnap's account. On the philosophical side, the patchwork example above showed us that, since the theoretical postulates and the correspondence rules associated with a theoretical term might change substantially from one patch to the other, it is unclear

[9] Note that Novick [33] conceptualizes the conceptual complexity of a patchwork in a somewhat different light than the other patchwork theorists, supporting a neutral theory of conceptual complexity. Nevertheless, for the purposes of our paper, we group Novick's theory with other patchwork approaches.

[10] Note that, although in this paper we focus on Carnap's account of theoretical terms, the contextuality of patchworks arguably create problems for all traditional accounts of theories and theoretical terms. For instance, the aforementioned model-theoretic structuralist account of scientific theories [6] has been found to be, at least in its orthodox form, inconsistent with patchwork constructions (cf. [15]).

whether the factual/synthetic/empirical content of a patchwork can be divided from its conventional/analytic/theoretical content. This seems to substantiate Wilson's claim, quoted in the last section, that the mere existence of scientific patchworks is enough to reject any analytic/synthetic distinction in a scientific theory. On the technical side, instead, the main issue that patchwork constructions pose to Carnap's account is the problem of finding a way to contextualize its definitions of the Ramsey sentence and the Carnap sentence of a theory, as well as its epsilon-based definitions of theoretical terms. In fact, as the toy example of a patchwork has shown, Carnap's standard definitions do not seem to sit well with the kind of contextuality involved in patchwork constructions. Fixing these philosophical and technical problems will be the focus of the next section.

5 A Neo-Carnapian Solution to the New Problem of Theoretical Terms

We saw how even very simple patchwork constructions, like the one represented by our toy formal example, are able to trivialize Carnap's account of theoretical terms. However, in this section, we will show that Carnap's approach can be generalized in such a way that makes it able to account for the quirky semantic behaviors of theoretical terms in patchwork constructions. This generalization of Carnap's approach will also demonstrate that the analytic/synthetic distinction is meaningful even for patchwork constructions.

The first step of our generalization of Carnap's approach to patchwork constructions starts by noticing that Carnap's approach works perfectly at the level of the single patch. That is, if we treat a single patch as a mini-theory, isolating it from the rest of the patchwork, we can obtain perfectly valid Ramsey sentences, Carnap sentences, and even epsilon definitions of theoretical terms. In order to see this, we can look again at our toy example of a three-patches patchwork that we presented in the last section. Let A_{O} denote the conjunction of the observational postulates shared by the three patches, whereas $TC^{P_1}, TC^{P_2}, TC^{P_3}$ denote the conjunction of theoretical and correspondence postulates $\varphi_1(t, t')$, $\varphi_2(t, t')$ and $\varphi_3(t, t')$ for P_1, P_2 and P_3 respectively. Similar to what was done before, we define the couple $\mathbf{t} := \langle t, t' \rangle$. We can schematically provide, indexed over a given patch $i \in \{1, 2, 3\}$, the Ramsey sentence, the Carnap sentence, and the epsilon-based definitions of theoretical terms:

- $TC^i := \varphi_i(t, t')$
- $R^i := \exists x x' \, \varphi_i(x, x')$
- $A^i_{\mathrm{T}} := R^i \rightarrow TC^i$
- $A(\mathbf{t})^i := \mathbf{t} = \varepsilon \mathbf{x} \, \varphi_i(\mathbf{x})$
- $A(t) := t = \varepsilon y \, (\exists x x' \, \mathbf{t} = \langle x, x' \rangle \wedge y = x)$
- $A(t') := t' = \varepsilon y \, (\exists x x' \, \mathbf{t} = \langle x, x' \rangle \wedge y = x')$
- $A^i_{\varepsilon} := A(\mathbf{t})^i \wedge A(t) \wedge A(t')$

Since, as we stressed in Section 3, the single patches are assumed to be consistent, the above definitions are perfectly fine and do not risk to be trivialized by the semantic behavior we saw in Section 4. As such, we can write every patch in Carnap's Form

III, achieving therefore the separation of analytic and synthetic content of the patch and explicitly defining its theoretical terms:

$$A_{\mathrm{O}}, \ R^i, \ A^i_{\varepsilon} \qquad \text{(Form III of } P_i\text{)}$$

In this way, the different patches P_1, P_2 and P_3 of our patchwork can be represented as three independent Carnapian theories. Yet, this is not an adequate representation of the patchwork $\mathcal{P}$. The three Carnapian theories related to the three patches are, in fact, completely independent from one another, up to the point that their implications, their intended models, and their theoretical terms have no interrelationships whatsoever. Recall, instead, that in our patchwork $\mathcal{P}$, the theoretical terms occurring in the patches P_1 and P_2 are importantly related in their interpretation, as our relation $\sim$ specifies. Thus, in order to adequately represent the patchwork $\mathcal{P}$, and the theoretical terms that figure in it, we need a way of making these three Carnapian theories relate to each other in a meaningful way. More generally, representing the interrelationships between patches of a given patchwork is a crucial point for adequately representing patchwork constructions. The complex interrelationships between the patches are, in fact, a constitutive trait of patchwork constructions and of the related polysemy phenomena, as supporters of patchwork approaches have repeatedly stressed (cf. [14, 21, 34, 41, 42]). Although the patches of a patchwork are not so strictly connected to each other as orthodox accounts of theories would demand, they are still interdependent components of one, single theory.

We have then to find a way to somehow integrate the Carnapian reconstruction of the single patches into a more coherent whole. A natural way of integrating independent logical claims is to conjoin them. Can we just conjoin the empirical and theoretical contents of the single patches? Unfortunately, this straightforward tentative solution faces familiar problems. In fact, consider the following form using conjunctions:

- $TC^{\cap} := \bigwedge_{i=1}^{3} \varphi_i(t, t')$
- $R^{\cap} := \exists x x' \bigwedge_{i=1}^{3} \varphi_i(x, x')$
- $A^{\cap}_{\mathrm{T}} := R^{\cap} \rightarrow TC^{\cap}$
- $A(\mathbf{t})^{\cap} := \mathbf{t} = \varepsilon \mathbf{x} \bigwedge_{i=1}^{3} \varphi_i(\mathbf{x})$
- $A^{\cap}_{\varepsilon} := A(\mathbf{t})^{\cap} \wedge A(t) \wedge A(t')$

This form, as we already discussed informally in the last section, fails to adequately represent our patchwork $\mathcal{P}$, forcing P_1 and P_2 to share the interpretation of t', and, as such, risking being trivialized. Yet, the idea of conjoining the single contents of the patches is not completely unfruitful. In fact, by conjoining the specific Ramsey sentence of the single patches, i.e., $\bigwedge_{i=1}^{3} R^{P_i}$, one obtains a consistent claim, which holds in all models of the patchwork, since patches are locally consistent. Yet, this claim is too weak to represent an adequate Ramsey sentence for the patchwork $\mathcal{P}$, because it does not satisfy the fact that P_1 and P_2 share a part of their empirical claim, namely, the witness of the existential variable corresponding to t. That is, our tentative conjunction $\bigwedge_{i=1}^{3} R^{P_i}$ does not represent the empirical part of the relationship between the interpretation of the theoretical terms in the two patches. As such, despite this limited success on conjoining the specific Ramsey sentences of the single patches, we need to find another way to adequately integrate the patches.

In order to enforce the global consistency of the theoretical content of the patchwork, one might be tempted to disjoin the theoretical contents of the single patches. Philosophically, such a disjunctive option could be justified by noticing that, in a certain sense, different patches can be seen as constituting different alternative interpretations of the theoretical terms of a patchwork. Yet, also this second tentative solution does not work. In order to see this, consider the following form using disjunctions:

- $TC^{\cup} := \bigvee_{i=1}^{3} \varphi_i(t, t')$
- $R^{\cup} := \exists x x' \bigvee_{i=1}^{3} \varphi_i(x, x')$
- $A_{\mathrm{T}}^{\cup} := R^{\cup} \to TC^{\cup}$
- $A(\mathbf{t})^{\cup} := \mathbf{t} = \varepsilon\mathbf{x} \bigvee_{i=1}^{3} \varphi_i(\mathbf{x})$
- $A_{\varepsilon}^{\cup} := A(\mathbf{t})^{\cup} \wedge A(t) \wedge A(t')$

Disjoining the theoretical contents of the patches assures indeed consistency, since all disjuncts are consistent. Yet, this solution is too weak as a representation of the theoretical content of the patchwork. The above disjunctive forms, in fact, imply only statements already entailed by each disjunct, that is, each single-patch theoretical content. As such, this disjunctive integration preserves only the empirical and theoretical content common to all patches, if such a content exists. Since P_1 and P_2 disagree over t', the full empirical and theoretical content of $\mathcal{P}$ is not captured by our disjunctive integration. A more fine-grained integration of the different patches of the patchwork is needed.

Although these two tentative ways of integrating the empirical and theoretical contents of the single patches of our patchworks into a more coherent whole failed, we achieved some limited successful integrations. We saw in fact how the conjunction of the single-patch Ramsey sentences and the disjunction of the theoretical contents of the single patches are both consistent. These two limited successes of our tentative integrations will be the ingredients of our neo-Carnapian solution to the new problem of theoretical terms.

Our solution starts by noticing that both our conjunction of single-patch Ramsey sentences and our disjunction of the theoretical contents of the patches fail to adequately capture (respectively) the empirical and the theoretical content of our patchwork $\mathcal{P}$ for the same reason; that is, because they do not represent the relationships between the different patches of the patchwork. We need a way of integrating the relationships between the patches in these partial solutions. These relationships are, as we have seen in Section 3, a crucial part of the patchwork depiction of the semantics of a scientific theory. Let us start with integrating the inter-patches relationships in the Ramsey sentence. This can be done in two steps. First, for convenience, we rewrite our tentative conjunction $\bigwedge_{i=1}^{3} R^{P_i}$ of Ramsey sentences in an equivalent prenex form, so that each variable occurring in each Ramsey sentence is still quantified separately. Then, we explicitly state the inter-patches relationships between the existentially bound variables by means of identity:

$$R^* := \exists x_1 x_1' x_2 x_2' x_3 x_3' \, (\bigwedge_{i=1}^{3} \varphi_i(x_i, x_i') \wedge x_1 = x_2 \wedge x_1' \neq x_2')$$

This tinkered version of our conjunction of single-patches Ramsey sentence is not only consistent (as its non-tinkered original version), but also includes the empirical content of the inter-patches relationships of the patchwork $\mathcal{P}$, i.e., the fact that P_1 and P_2 share the witness of the existential variable corresponding to t and the fact that they do not share the witness of the variable corresponding to t'. As such, this tinkered Ramsey sentence, just like Carnap prescribed, represents the empirical content of the whole patchwork, as it corresponds to the structured union of the empirical contents of the single patches P_1, P_2 and P_3.

The second step of our neo-Carnapian solution constructs a suitable Carnap sentence for the whole patchworks by taking, as its antecedent, the above tinkered Ramsey sentence, and as its consequent, the disjunction of theoretical contents of the single patches:

$$A_{\mathrm{T}}^* := R^* \rightarrow TC^{\cup}$$

This Carnap sentence for the whole patchwork, just like Carnap prescribed, does not entail any observational content, thus being fully analytical. Moreover, our Carnap sentence, together with our Ramsey sentence, makes us obtain back the disjunctive integration of the theoretical content of the patchwork. Given that the observational content of all patches is now preserved for $\mathcal{P}$ anyway, the disjunction can play the role of just providing the set of consequences (if any) of $\mathcal{P}$ under global consistency requirements. With our Ramsey sentence and our Carnap sentence, we can finally divide the empirical content of the whole patchwork from its theoretical content, obtaining what Carnap calls a Form II of the theory for the patchwork:

$$A_{\mathrm{O}}, \; R^*, \; A_{\mathrm{T}}^* \qquad \text{(Form II of } \mathcal{P}\text{)}$$

Yet, this form does not express the whole theoretical content of the patchworks, because the Carnap sentence does not express the inter-patches semantic relationships between theoretical terms. Differently from the case of the Ramsey sentence, these semantic interrelationships cannot be simply added to the Carnap sentence via stating the related identities, because there is no way of contextualizing the occurrences of theoretical terms in TC without trivializing the essence of the patchwork construction. Indexing or other ways of explicitly marking the different patches-occurrences of theoretical terms would, in fact, result in an inadequate reconstruction of the patchwork construction, as they would inevitably eliminate the polysemy and indeterminacy that patchwork concepts exhibit.

The solution to our issue resorts to epsilon terms. Specifically, we need to integrate our disjunctive theoretical content with an epsilon-based definition of the inter-patches semantic relationships between theoretical terms. In order to do so, we will need some additional notation. Let us abbreviate the formula R^* by $\exists \mathbf{x}\, \varphi^*(\mathbf{x})$, for $\mathbf{x}$ being a variable for tuples. The variable $\mathbf{x}$ hence indicates a tuple representing the bound variables $x_1, x_1', x_2, x_2', x_3, x_3'$ as they occur and are related by the identity relation in R^*. Thanks to this new notation, the following definition of the tuple $\mathbf{t}$ can be thus obtained:

$$A(\mathbf{t})^* := \mathbf{t} = \varepsilon \mathbf{x}\, \varphi^*(\mathbf{x})$$

Crucially, the explicit definition of the single theoretical terms t and t' will be different from the previous characterization. The new formulation enables the choice of a witness among those satisfying the single patches account while respecting the relation $\sim$ among patches and theoretical terms as they were encoded in $\exists \mathbf{x}\, \varphi^*(\mathbf{x})$:

- $A(t)^* := \; t = \varepsilon y\, (\exists x_1 x_1' x_2 x_2' x_3 x_3'\, \mathbf{t} = \langle x_1, x_1', x_2, x_2', x_3, x_3' \rangle \wedge \bigvee_{i=1}^{3} y = x_i)$
- $A(t')^* := \; t' = \varepsilon y\, (\exists x_1 x_1' x_2 x_2' x_3 x_3'\, \mathbf{t} = \langle x_1, x_1', x_2, x_2', x_3, x_3' \rangle \wedge \bigvee_{i=1}^{3} y = x_i')$

The theoretical postulate A_{T} can thus be rephrased as the conjunction of $A(\mathbf{t})$ and that of the explicit definitions of the single theoretical terms:

$$A_\varepsilon^* := \; A(\mathbf{t})^* \wedge A(t)^* \wedge A(t')^*$$

Informally, the epsilon term defining t allows us to choose among different interpretations based on the definitions of t for each patch P_i, respecting inter-patches relationship as well. We can thus still provide an explicit definition of all theoretical terms occurring in $\mathcal{P}$ by making them more open-ended, but ensuring that each of their choice satisfies at least a context of interpretation among those of P_1, P_2 and P_3. Altogether, the final Carnapian form of our patchwork, i.e., Form III of $\mathcal{P}$, can be represented as

$$A_{\mathrm{O}}^*, \;\; R^*, \;\; A_\varepsilon^* \qquad \text{(Form III of } \mathcal{P}\text{)}$$

This form represents our neo-Carnapian solution to the new problem of theoretical terms for the patchwork $\mathcal{P}$. This is because such a formulation correctly divides the empirical content of the patchwork $\mathcal{P}$ from its theoretical content, respecting all Carnap's desiderata for such division. In particular, in this form, the empirical content of the patchwork comes out as purely synthetic and factual in character, being observationally equivalent to the union of the empirical contents of the single patches (together with the empirical consequences of their interrelationships). The theoretical content of the patchwork $\mathcal{P}$ comes out, instead, as purely analytical and conventional in character, since it does not entail any other consequence not implied by the patchwork itself. Theoretical terms, moreover, are explicitly defined by the epsilon terms, embedding all the semantic interrelationship between the different patches as formal constraints that the referents of the theoretical terms of the patchworks must satisfy. All the desiderata that Carnap imposed on his account of theoretical terms are thus satisfied by our neo-Carnapian solution.

This neo-Carnapian account of theoretical terms demonstrates that the analytic/synthetic distinction, together with the related separation of an empirical and factual part of a scientific theory from its theoretical and conventional part, can be maintained also for scientific patchworks. Thus, in contrast to what some patchwork theorists claim (see Section 3), the existence of patchwork constructions in science is not a sufficient reason to reject the validity of the analytic/synthetic distinction. Nevertheless, patchwork constructions, due to the context-dependency and polysemy that they allow, force standard accounts of theoretical terms to generalize their formal reconstructions in order to take into account this additional layer of indeterminacy. This is exactly what our neo-Carnapian account of theoretical terms does: it generalizes Carnap's definitions to account for the contextuality and polysemy that theoretical terms might enjoy in patchworks.

6 Generalization of Our Proposal and Comparison with Other Approaches

In this section, we will generalize our neo-Carnapian account of a patchwork to a wide class of patchwork structures. We will also prove that our account is, in a certain respect, conservative over Carnap's original account. Finally, we will briefly compare our account with another recent formal semantics for scientific patchworks.

Our neo-Carnapian account of a patchwork that we presented in the last section can be straightforwardly generalized to any formal patchwork construction $\mathcal{P}$ represented by the couple $\langle \mathcal{S}, \sim \rangle$. In particular, consider a finite set $\mathcal{S}$ of patches $P_1, \ldots, P_m$. For each patch P_i, we assume that the conjunction of theoretical postulates and correspondence rules $TC^i := \varphi_i(t_1, \ldots, t_n)$ contains n theoretical terms $t_1, \ldots, t_n$.[11] In order to see how our solution can be generalized to such a patchwork schema, we need a schematic way of representing the relation $\sim$ in the Ramsey sentence R^*. Our solution starts by noticing that exactly n times m bound variables are required to separately quantify the n theoretical terms as occurring over the m patches in $\mathcal{S}$, for any patch P_i. Therefore, variables $x_{1_i}, \ldots x_{n_i}$ will occur bound in the quantification of each TC^i. This means that, given any distinct patches P_h and P_k in $\mathcal{S}$ and a theoretical term t_j, the relations $P_h \overset{t_j}{\sim} P_k$ and $P_h \overset{t_j}{\nsim} P_k$ will be represented in our generalized Ramsey sentence as $x_{j_h} = x_{j_k}$ and $x_{j_h} \neq x_{j_k}$ respectively. We denote by ρ the conjunction of these identity and negated identity formulas, thus obtaining the following general form:

$$R^* := \exists x_{1_1} \cdots x_{n_m} \left(\bigwedge_{i=1}^{m} \varphi_i(x_{1_i}, \ldots, x_{n_i}) \wedge \rho \right)$$

Similarly, a general form for the explicit epsilon definition of theoretical terms can be given by (again) abbreviating R^* as $\exists \mathbf{x}\, \varphi^*(\mathbf{x})$, so that the variable over tuple $\mathbf{x}$ represents the bound variables $x_1, \ldots, x_{n_m}$. We can thus define the tuple $\mathbf{t}$ as before, and then proceed by defining any theoretical term t_j by selecting a witnesses from all single patches account P_i standing in the relation $\sim$:

[11] For simplicity, we consider the case in which all theoretical terms occur in the conjunction of theoretical postulates and correspondence rules of all patches. The further generalization to a case in which not all theoretical terms occur in the postulates of each patch is straightforward as well.

- $A(\mathbf{t})^* := \mathbf{t} = \varepsilon \mathbf{x}\, \varphi^*(\mathbf{x})$
- $A(t_j)^* := t_j = \varepsilon y\, (\exists x_{1_1} \cdots x_{n_m}\, \mathbf{t} = \langle x_1, \ldots x_{n_m}\rangle \wedge \bigvee_{i=1}^{m} y = x_{j_i})$
- $A^*_\varepsilon := A(\mathbf{t})^* \wedge \bigwedge_{j=1}^{n} A(t_j)^*$

Our generalized neo-Carnapian approach to patchwork captures the semantic behavior of any patchwork construction that can be represented through the couple $\langle \mathcal{S}, \sim\rangle$. This arguably includes several important examples of patchwork constructions, such as the aforementioned case of the force patchwork in classical mechanics (cf. [15, 41, 42]). Further semantic behaviors that patchwork constructions might exhibit, such as instances where theoretical terms are connected by relations weaker than identity, can be also represented in our account by including other relations than $\sim$ in the definition of a patchwork.[12]

It is important to stress that the modifications that our neo-Carnapian account of patchworks makes to Carnap's original account of theoretical terms are quite conservative in character. This can be seen by noticing that we used the same technical tools that Carnap used, without changing logic nor augmenting the base theory needed for our reconstruction. Furthermore, we can prove that our solution is able to recover Carnap's original formulation as a specific case. If, in fact, a patchwork is a traditional theory (i.e., what Wilson calls a flat structure facade, cf. [41, p. 379]), then the reconstruction of this patchwork given by our account is equivalent to the one of Carnap's original account. More precisely, traditional theories can be taken as patchworks $\mathcal{C} = \langle \mathcal{S}, \sim\rangle$ where $\sim$ is such that the interpretation of any theoretical term coincides across all patches in $\mathcal{S}$. We can take, then, Carnap's original account to coincide with our early attempt of providing a representation of a patchwork by simply conjoining the theoretical postulates of all its patches, i.e., A_{O}, $R^\cap$, $A^\cap_\varepsilon$. Let us prove this equivalence:

Theorem *Let $\mathcal{C} = \langle \mathcal{S}, \sim\rangle$ be a traditional theory, i.e., a patchwork such that, for all distinct patches P_h and P_k in $\mathcal{S}$, and for all theoretical terms t_j, $P_h \overset{t_j}{\sim} P_k$. Then:*

$$A_{\mathrm{O}}, R^\cap, A^\cap_\varepsilon \quad \textit{iff} \quad A_{\mathrm{O}}, R^*, A^*_\varepsilon.$$

Proof Let $\mathcal{S}$ in the definition of the patchwork $\mathcal{C}$ be the set of patches $P_1, \ldots, P_m$, and $t_1, \ldots, t_n$ the set of theoretical terms occurring in their postulates. We hence define $A^\cap_{\mathrm{O}}$, $R^\cap$ and $A^\cap_\varepsilon$ as follows:

- $R^\cap := \exists x_1 \cdots x_n \bigwedge_{i=1}^{m} \varphi_i(x_1, \ldots, x_n)$
- $A(\mathbf{t})^\cap := \mathbf{t} = \varepsilon \mathbf{x} \bigwedge_{i=1}^{m} \varphi_i(\mathbf{x})$
- $A(t_j) := t_j = \varepsilon y\, (\exists x_1 \cdots x_n\, \mathbf{t} = \langle x, x'\rangle \wedge y = x_j)$

[12] In particular, any relation holding among two patches P_h and P_k and a theoretical term t_j can be represented as a binary relation R in the Ramsey sentence R^*, i.e., as the occurrence of $x_{j_h} \mathsf{R} x_{j_k}$ in ρ.

- $A_\varepsilon^\cap := A(\mathbf{t})^\cap \bigwedge_{j=1}^{n} A(t'_j)$

By definition of $\sim$ in $\mathcal{C}$, the conjunction of all the identity statements of the kind $x_{j_h} = x_{j_k}$ occurs in its Ramsey sentence R^*. Hence, By classical logic alone, R^* can be equivalently simplified into $R^\cap$. For the same reason, any choice of disjunct $y = x_{j_i}$ in the definition of $\mathbf{t_j}$ in $A(\mathbf{t_j})^*$ coincides. Since by the Extensionality axiom of epsilon calculus the same witness are chosen over formulas expressing the same properties, A_ε^* and $A_\varepsilon^\cap$ provide equivalent definitions for the theoretical terms occurring in $\mathcal{C}$. □

This theorem shows that our neo-Carnapian account of theoretical terms reduces to Carnap's original one in the special case when a patchwork is nothing but a traditional theory. In this way, our solution can be seen as generalizing Carnap's original solution to a wider class of scientific theories, i.e., a class that includes both traditional theories and patchwork constructions. From a philosophical point of view, this result can be interpreted as showing that traditional accounts of theoretical terms are valid only within a narrow class of scientific theories, namely, the "flat" ones. From this perspective, the existence of scientific patchworks shows us that scientific theories form a wider and more diverse class of semantic structures that what traditional accounts of scientific theories and concepts previously assumed.

Finally, let us briefly compare our proposal with another recent attempt at building a semantics for scientific patchworks. Andreas [2] proposes a paraconsistent semantics that accounts for global inconsistencies that scientific patchworks might exhibit. In his semantics, the theoretical terms of a possibly inconsistent scientific theory are characterized by combinations of maximally consistent sets of applications of laws in which they figure. That is, the meaning of theoretical terms in Andreas' semantics is not anymore given via universal laws, but it is instead defined through consistent sets of instances of these laws. This results into a paraconsistent account of scientific laws that manages to recover much of their inferential applications within a theory. With respect to Andreas' approach, our proposal accounts for the global inconsistencies that patchworks constructions might exhibit without resorting to a change of logic and without departing so much from Carnap's original definitions. That is, our neo-Carnapian account of theoretical terms manages to give an adequate semantics for patchworks by encoding in the Ramsey sentence and in the epsilon-based definitions of theoretical terms the contextual character that they might exhibit in scientific patchworks. In this way, our account shows that we do not need to go paraconsistent for accounting for the behavior of theoretical terms in scientific patchworks.

7 Conclusion

Let us recap the main steps of the present work. We started by presenting Carnap's solution to the problem of defining theoretical terms based on the epsilon calculus. We then recalled how contemporary patchwork approaches to scientific laws and concepts showed that many important scientific theories are internally organized as a complex patchwork of semi-connected patches of local usages. We also recalled how some patchwork theorists claimed that the mere existence of patchwork constructions in science falsifies traditional accounts of scientific concepts and theories. By virtue of a toy formal example of a patchwork, we showed that patchwork constructions are indeed able to trivialize traditional accounts of theoretical terms such as Carnap's. We then modified Carnap's account to adequately account also for scientific patchworks. Our neo-Carnapian account of theoretical terms modifies Carnap's original account in two fundamental ways: we take as the empirical content of a theory the conjunction of the Ramsey Sentences of the single patches of a theory (crucially constrained by the satisfaction of the inter-patches relations) and we define the theoretical terms of the theory, and thus its theoretical content, over the disjunction of single-patches-related definitions. Thanks to our neo-Carnapian account of theoretical terms, we demonstrated how the division between the analytic/theoretic/conventional part of a scientific theory and its synthetic/empirical/factual part is meaningful even for scientific patchworks. We then generalized our neo-Carnapian account of patchwork to a wide class of patchworks and we showed how it can recover Carnap's original account as a specific case.

In this way, we vindicated Carnap's ideal that any scientific theory can be adequately divided, contextually, in a empirical/synthetic/factual part and a theoretical/analytical/conventional one, even if it consists of complex polysemic semantic structures like patchworks.

Our approach lends itself to be further generalized and extended in various way. A natural possibility would be to extend our neo-Carnapian approach with the help of modal tools, expanding thus the technical possibilities of the epsilon calculus further. This could be done, for example, by building on the modal semantics for theoretical terms presented by Andreas and Schiemer in [4]. Such a modal semantics generalizes Carnap's epsilon approach to theoretical terms with modal tools. Since our approach uses the same formal tools as Carnap's, our approach can be also represented in Andreas' and Schiemer's modal semantics. Such a modal extension of our proposal could perhaps be able to account for complex, hyperintensional features of scientific reasoning that are informally argued for in contemporary pragmatist philosophy of science.

Acknowledgements We would like to thank Hannes Leitgeb and two anonymous reviewers for helpful comments on earlier drafts of this paper. We are also indebted to audiences in Milan, Vienna, Lisbon, and Berlin for constructive feedback.

Author Contributions the authors declare that they contributed equally to this manuscript.

Funding Open access funding provided by Scuola IMT Alti Studi Lucca within the CRUI-CARE Agreement. The work of MDB on this paper was funded by the Emmy-Noether project "From Perception to Belief and Back Again", Deutsche Forschungsgemeinschaft (BR 5210/1-1).

Availability of Data and Material Not applicable.

Code Availability Not applicable.

Declarations

Ethical Approval Not applicable.

Informed Consent Not applicable.

Competing Interests The authors declare that they have no competing interests.

References

1. Andreas, H. (2014). Carnapian Structuralism. *Erkenntnis, 79*, 1373–1391.
2. Andreas, H. (2018). Modular Semantics for Theories: An Approach to Paraconsistent Reasoning. *Journal of Philosophical Logic, 47*(5), 877–912.
3. Andreas, H. (2021). Theoretical Terms in Science. *The Stanford Encyclopedia of Philosophy*, (Fall 2021 Edition), Edward N. Zalta (Ed.), https://plato.stanford.edu/archives/fall2021/entries/theoretical-terms-science/.
4. Andreas, H., & Schiemer, G. (2016). A Choice-Semantical Approach to Theoretical Truth. *Studies in History and Philosophy of Science Part A, 58*, 1–8.
5. Avigad, J., & Zach, R. (2020). The Epsilon Calculus. In E. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy*, Fall 2020.
6. Balzer, W., Moulines, C. U., & Sneed, J. (1987). *An Architectonic for Science*. The structuralist Program. D: Reidel Publishing Company, Dordrecht.
7. Batterman, R. (2001). *The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence*. Oxford: Oxford University Press.
8. Batterman, R. (2013). The Tyranny of Scales. *The Oxford Handbook of Philosophy of Physics* (pp. 255–286). Oxford: Oxford University Press.
9. Bursten, J. R. (2018). Smaller than a Breadbox: Scale and Natural Kinds. *British Journal for Philosophy of Science, 69*(1), 1–23.
10. Carnap, R. (1956). The Methodological Character of Theoretical Concepts. In H. Feigl & M. Scriven (Eds.), *The Foundations of Science and the Concepts of Psychology and Psychoanalysis* (pp. 38–76). Minneapolis: University of Minnesota Press.
11. Carnap, R. (1961). On the use of Hilbert's ε-Operator in Scientific Theories. In Y. Bar-Hillel (Ed.), *Essays in the Foundation of Mathematics* (pp. 156–164). Jerusalem: Magnes Press.
12. Carnap, R. (1966). *Philosophical foundations of physics: An introduction to the philosophy of science*. New York: Basic Books.
13. Cartwright, N. (1983). *How the Laws of Physics Lie*. Oxford: Oxford University Press.
14. Cartwright, N. (1999). *The Dappled World: A Study of the Boundaries of Science*. Cambridge: Cambridge University Press.
15. De Benedetto, M. (2021). Taming conceptual wanderings: Wilson-Structuralism. *Synthese, 199*, 13225–13246.
16. Demopoulos, W. (2007). Carnap on the Rational Reconstruction of Scientific Theories. In M. Friedman & R. Creath (Eds.), *The Cambridge Companion to Carnap* (pp. 248–272). Cambridge: Cambridge University Press.
17. Demopoulos, W., & Friedman, M. (1985). Bertrand Russel's The Analysis of Matter: Its Historical Context and Contemporary Interest. *Philosophy of Science, 52*(4), 621–639.
18. Glymour, C. N. (1980). *Theory and Evidence*. Princeton (NJ): Princeton University Press.
19. Hacking, I. (1983). *Representing and Intervening: Introductory Topics in the Philosophy of Natural Science*. Cambridge: Cambridge University Press.
20. Haueis, P. (2021). The Death of the Cortical Column? Patchwork Structure and Conceptual Retirement in Neuroscientific Practice. *Studies in History and Philosophy of Science Part A, 85*, 101–113.

21. Haueis, P. (2021b). A generalized patchwork approach to scientific concepts. *The British Journal for the Philosophy of Science*. https://www.journals.uchicago.edu/doi/10.1086/716179.
22. Hempel, C. G. (1958). The Theoretician's Dilemma: A Study in the Logic of Theory Construction. In M. Scriven, G. Maxwell, & H. Feigl (Eds.), *Concepts, Theories, and the Mind-Body Problem, Minnesota Studies in the Philosophy of Science 2* (pp. 37–98). Minneapolis: University of Minnesota Press.
23. Hilbert, D., & Bernays, P. (1939). *Grundlagen der Mathematik, 2*. Berlin: Springer.
24. Ketland, J. (2004). Empirical Adequacy and Ramsification. *The British Journal for the Philosophy of Science, 55*(2), 287–300.
25. Ketland, J. (2009). Empirical Adequacy and Ramsification, II. In A. Hieke, H. Leitgeb (Eds.), *Reduction, Abstraction, Analysis: Proceedings of the 31st International Ludwig Wittgenstein Symposium in Kirchberg 2008* (pp. 29–45). Ontos Verlag, Frankfurt.
26. Lutz, S. (2012). Artificial Language Philosophy of Science. *European Journal for Philosophy of Science, 2*(2), 181–203.
27. Lutz, S. (2012b). On a Straw Man in the Philosophy of Science: A Defense of the Received View. *HOPOS: The Journal of the International Society for the History of Philosophy of Science, 2*(2), 77–120.
28. Lutz, S. (2017). Carnap on Empirical Significance. *Synthese, 194*(1), 217–252.
29. Mitchell, S. (1997). Pragmatic Laws. *Philosophy of Science, 64*(4), 468–479.
30. Mitchell, S. (2009). *Unsimple Truths: Science, Complexity, and Policy*. Chicago (IL): University of Chicago Press.
31. Novick, A. (2018). The Fine Structure of 'Homology'. *Biology and Philosophy, 33*(1.2). https://doi.org/10.1007/s10539-018-9617-3
32. Novick, A., & Doolittle, W. F. (2021). 'Species' Without Species. *Studies in History and Philosophy of Science Part A, 87*, 72–80.
33. Novick, R. (2023). The Neutral Theory of Conceptual Complexity. *Philosophy of Science, 1–17*. https://doi.org/10.1017/psa.2023.25
34. Novick, R., & Haueis, P. (2023). Patchworks and operations. *European Journal for Philosophy of Science, 13*(15). https://doi.org/10.1007/s13194-023-00515-y
35. Psillos, S. (Ed.) (2000). Rudolf Carnap's 'Theoretical concepts in science'. *Studies in History and Philosophy of Science, 31*, 151–172.
36. Ramsey, F. P. (1929). Theories. In Braithwaite (Ed.), *Foundations of Mathematics and Other Logical Essays* (pp. 212–236). Kegan Paul, London, 1931.
37. Schurz, G. (2005). Semantic Holism and (Non-)Compositionality in Scientific Theories. In M. Werning (Ed.), *The Compositionality of Meaning and Content* (vol I pp. 271–284). Frankfurt, Ontos Verlag.
38. Schurz, G. (2014). Criteria of theoreticity: Bridging statement and non-statement view. *Erkenntnis, 79*, 1521–1545.
39. Taylor, H. (2023). Attention as a patchwork concept. *European Journal for Philosophy of Science, 13*(3), 1–25.
40. Taylor, H., & Vickers, P. (2015). Conceptual fragmentation and the rise of eliminativism. *European Journal for Philosophy of Science, 7*(1), 17–40.
41. Wilson, M. (2006). *Wandering Significance: An Essay on Conceptual Behavior*. Oxford: Clarendon Press.
42. Wilson, M. (2017). *Physics Avoidance: And Other Essays in Conceptual Strategy*. Oxford: Oxford University Press.
43. Winnie, J. A. (1970). Theoretical Analyticity. In R. C. Buck, & R. S. Cohen (Eds.), *In Memory of Rudolf Carnap: Proceedings of the Biennial Meeting of the Philosophy of Science Association 1970* (pp. 289–305). Springer Netherlands, Dordrecht.