SUPPLEMENTARY MATERIAL
Assessing frustration in real-world signed networks: a statistical theory of balance

APPENDIX A.
REPRESENTING BINARY, UNDIRECTED, SIGNED NETWORKS

The three functions $a_{ij}^{-}=[a_{ij}=-1]$, $a_{ij}^{0}=[a_{ij}=0]$ and $a_{ij}^{+}=[a_{ij}=+1]$ have been defined via the Iverson's brackets notation. Iverson's brackets work in a way that is reminiscent of the Heaviside step function, i.e. $\Theta[x]=[x>0]$; in fact,

$$a_{ij}^{-}=[a_{ij}=-1]=\begin{cases}1, & \text{if}\quad a_{ij}=-1\\ 0, & \text{if}\quad a_{ij}=0,+1\end{cases} \tag{1}$$

(i.e. $a_{ij}^{-}=1$ if $a_{ij}<0$ and zero otherwise),

$$a_{ij}^{0}=[a_{ij}=0]=\begin{cases}1, & \text{if}\quad a_{ij}=0\\ 0, & \text{if}\quad a_{ij}=-1,+1\end{cases} \tag{2}$$

(i.e. $a_{ij}^{0}=1$ if $a_{ij}=0$ and zero otherwise),

$$a_{ij}^{+}=[a_{ij}=+1]=\begin{cases}1, & \text{if}\quad a_{ij}=+1\\ 0, & \text{if}\quad a_{ij}=-1,0\end{cases} \tag{3}$$

(i.e. $a_{ij}^{+}=1$ if $a_{ij}>0$ and zero otherwise). These new variables are mutually exclusive, i.e. $\{a_{ij}^{-},a_{ij}^{0},a_{ij}^{+}\}=\{(1,0,0),(0,1,0),(0,0,1)\}$ and sum to 1, i.e. $a_{ij}^{-}+a_{ij}^{0}+a_{ij}^{+}=1$. The matrices $\mathbf{A}^{+}\equiv\{a_{ij}^{+}\}_{i,j=1}^{N}$ and $\mathbf{A}^{-}\equiv\{a_{ij}^{-}\}_{i,j=1}^{N}$ remain naturally defined, inducing the relationships $\mathbf{A}=\mathbf{A}^{+}-\mathbf{A}^{-}$, i.e. $a_{ij}=a_{ij}^{+}-a_{ij}^{-}$, $\forall\, i\neq j$ and $|\mathbf{A}|=\mathbf{A}^{+}+\mathbf{A}^{-}$, i.e. $|a_{ij}|=a_{ij}^{+}+a_{ij}^{-}$, $\forall\, i\neq j$.

APPENDIX B.
MINIMISATION OF THE BAYESIAN INFORMATION CRITERION

Signed Stochastic Block Model (SSBM) and Bayesian Information Criterion (BIC)

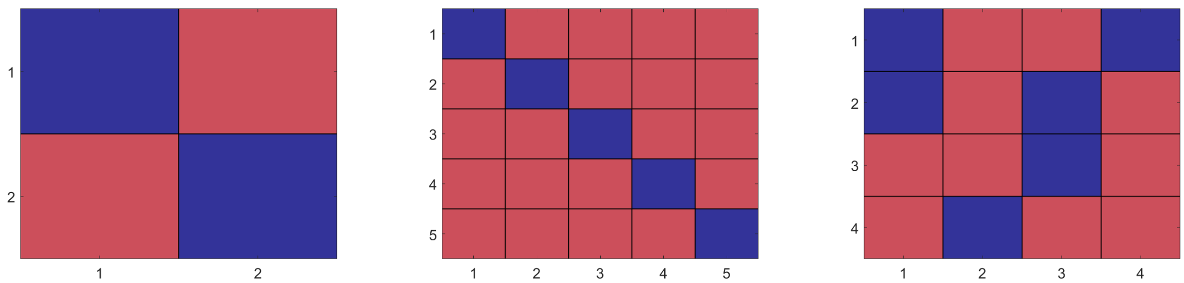


FIG. 1: Three, ideal partitions recoverable upon minimising BIC: one compatible with the strong balance theory ($k = 2$ with $p^+_{11} > p^-_{11}$, $p^+_{22} > p^-_{22}$ and $p^+_{12} < p^-_{12}$ - left panel), one compatible with the weak balance theory ($k > 2$ with $p^+_{rr} > p^-_{rr}$, $r = 1 \dots 5$, $p^+_{rs} < p^-_{rs}$, $r, s = 1 \dots 5$, $\forall\, r < s$ - middle panel), one compatible with the relaxed balance theory ($p^+_{rr} \leq p^-_{rr}$ for some, diagonal blocks and $p^+_{rs} \geq p^-_{rs}$ for some, off-diagonal blocks - right panel).

Let us, first, recall the derivation of the SSBM. It is defined by the Hamiltonian

$$\begin{aligned} H(\mathbf{A}) &= \sum_{r\leq s}[\alpha_{rs}L^+_{rs}(\mathbf{A}) + \beta_{rs}L^-_{rs}(\mathbf{A})] \\ &= \sum_{r\leq s}\left\{\alpha_{rs}\left[\sum_{i=1}^N\sum_{j(>i)}\delta_{g_i r}\delta_{g_j s}a^+_{ij}\right] + \beta_{rs}\left[\sum_{i=1}^N\sum_{j(>i)}\delta_{g_i r}\delta_{g_j s}a^-_{ij}\right]\right\} \\ &= \sum_{i=1}^N\sum_{j(>i)}[\alpha_{g_ig_j}a^+_{ij} + \beta_{g_ig_j}a^-_{ij}] \end{aligned} \tag{4}$$

leading to

$$\begin{aligned} Z = \sum_{\mathbf{A}\in\mathbb{A}} e^{-H(\mathbf{A})} = \sum_{\mathbf{A}\in\mathbb{A}}\prod_{i=1}^N\prod_{j(>i)} e^{-[\alpha_{g_ig_j}a^+_{ij} + \beta_{g_ig_j}a^-_{ij}]} &= \prod_{i=1}^N\prod_{j(>i)}\sum_{a_{ij}=-1,0,+1} e^{-[\alpha_{g_ig_j}a^+_{ij} + \beta_{g_ig_j}a^-_{ij}]} \\ &= \prod_{i=1}^N\prod_{j(>i)}[1 + e^{-\alpha_{g_ig_j}} + e^{-\beta_{g_ig_j}}]. \end{aligned} \tag{5}$$

As a consequence,

$$\begin{aligned} P_{\text{SSBM}}(\mathbf{A}) = \frac{e^{-H(\mathbf{A})}}{Z} = \frac{\prod_{i=1}^N\prod_{j(>i)} e^{-[\alpha_{g_ig_j}a^+_{ij} + \beta_{g_ig_j}]}}{\prod_{i=1}^N\prod_{j(>i)}[1 + e^{-\alpha_{g_ig_j}} + e^{-\beta_{g_ig_j}}]} &\equiv \prod_{i=1}^N\prod_{j(>i)}\frac{x_{g_ig_j}^{a^+_{ij}}y_{g_ig_j}^{a^-_{ij}}}{1 + x_{g_ig_j} + y_{g_ig_j}} \\ &\equiv \prod_{i=1}^N\prod_{j(>i)}(p^+_{g_ig_j})^{a^+_{ij}}(p^0_{g_ig_j})^{a^0_{ij}}(p^-_{g_ig_j})^{a^-_{ij}} \end{aligned} \tag{6}$$

having posed $e^{-\alpha_{g_ig_j}} \equiv x_{g_ig_j}$, $e^{-\beta_{g_ig_j}} \equiv y_{g_ig_j}$, $p^+_{ij} \equiv x_{g_ig_j}/(1 + x_{g_ig_j} + y_{g_ig_j})$, $p^-_{ij} \equiv y_{g_ig_j}/(1 + x_{g_ig_j} + y_{g_ig_j})$, $p^0_{ij} \equiv 1/(1 + x_{g_ig_j} + y_{g_ig_j})$; let us notice that

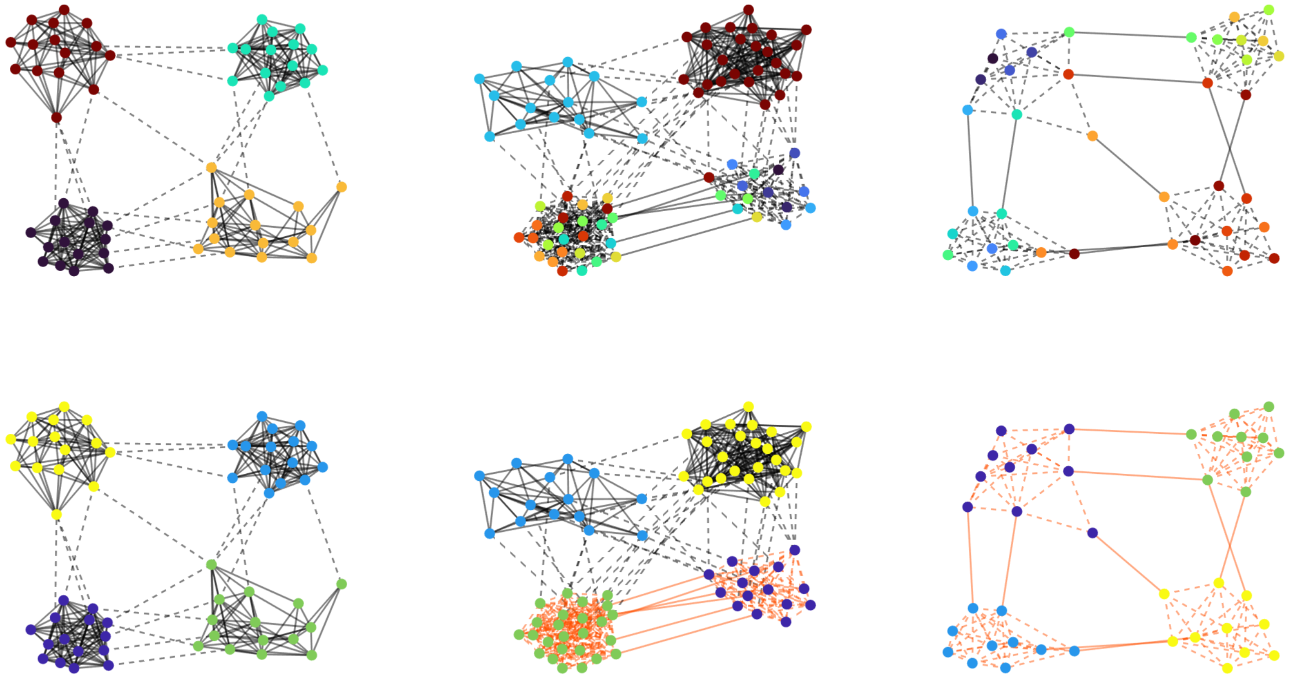


FIG. 2: Top panels: partitions recovered upon minimising $F(\boldsymbol{\sigma})$. Bottom panels: partitions recovered upon minimising BIC (orange links would be classified as misplaced according to the F-test). Minimising BIC leads us to find a community structure whose definition depends on the 'signed density' of connections: such a structure coincides with the one recovered upon minimising $F(\boldsymbol{\sigma})$ only if the former is k-balanced, i.e. satisfies the relationships $p_{rr}^{-}=0$, $r=1\dots k$ and $p_{rs}^{+}=0$, $\forall\, r<s$.

$$\begin{aligned}
P_{\text{SSBM}}(\mathbf{A}) &= \prod_{i=1}^{N}\prod_{j(>i)}\prod_{r=1}^{k}\prod_{s(\geq r)}[(p_{rs}^{+})^{a_{ij}^{+}}(p_{rs}^{0})^{a_{ij}^{0}}(p_{rs}^{-})^{a_{ij}^{-}}]^{\delta_{g_i r}\delta_{g_j s}}\\
&= \prod_{r=1}^{k}\prod_{s(\geq r)}\prod_{i=1}^{N}\prod_{j(>i)}[(p_{rs}^{+})^{\delta_{g_i r}\delta_{g_j s}a_{ij}^{+}}(p_{rs}^{0})^{\delta_{g_i r}\delta_{g_j s}a_{ij}^{0}}(p_{rs}^{-})^{\delta_{g_i r}\delta_{g_j s}a_{ij}^{-}}]\\
&= \prod_{r=1}^{k}\prod_{s(\geq r)}[(p_{rs}^{+})^{\sum_{i=1}^{N}\sum_{j(>i)}\delta_{g_i r}\delta_{g_j s}a_{ij}^{+}}(p_{rs}^{0})^{\sum_{i=1}^{N}\sum_{j(>i)}\delta_{g_i r}\delta_{g_j s}a_{ij}^{0}}(p_{rs}^{-})^{\sum_{i=1}^{N}\sum_{j(>i)}\delta_{g_i r}\delta_{g_j s}a_{ij}^{-}}]\\
&= \prod_{r=1}^{k}(p_{rr}^{+})^{L_{rr}^{+}}(p_{rr}^{0})^{L_{rr}^{0}}(p_{rr}^{-})^{L_{rr}^{-}}\cdot\prod_{r=1}^{k}\prod_{s(>r)}(p_{rs}^{+})^{L_{rs}^{+}}(p_{rs}^{0})^{L_{rs}^{0}}(p_{rs}^{-})^{L_{rs}^{-}}\\
&= \prod_{r=1}^{k}(p_{rr}^{+})^{L_{rr}^{+}}(p_{rr}^{-})^{L_{rr}^{-}}(1-p_{rr}^{+}-p_{rr}^{-})^{\binom{N_r}{2}-L_{rr}}\cdot\prod_{r=1}^{k}\prod_{s(>r)}(p_{rs}^{+})^{L_{rs}^{+}}(p_{rs}^{-})^{L_{rs}^{-}}(1-p_{rs}^{+}-p_{rs}^{-})^{N_r N_s-L_{rs}}\\
&= \prod_{r=1}^{k}\frac{x_{rr}^{L_{rr}^{+}}y_{rr}^{L_{rr}^{-}}}{[1+x_{rr}+y_{rr}]^{\binom{N_r}{2}}}\cdot\prod_{r=1}^{k}\prod_{s(>r)}\frac{x_{rs}^{L_{rs}^{+}}y_{rs}^{L_{rs}^{-}}}{[1+x_{rs}+y_{rs}]^{N_r N_s}}
\end{aligned}\tag{7}$$

where $L_{rr}=L_{rr}^{+}+L_{rr}^{-}$, $L_{rs}=L_{rs}^{+}+L_{rs}^{-}$, $p_{rr}^{+}=x_{rr}/(1+x_{rr}+y_{rr})$, $p_{rr}^{-}=y_{rr}/(1+x_{rr}+y_{rr})$, $r=1\dots k$ and $p_{rs}^{+}=x_{rs}/(1+x_{rs}+y_{rs})$, $p_{rs}^{-}=y_{rs}/(1+x_{rs}+y_{rs})$, $\forall\, r<s$. Its log-likelihood reads

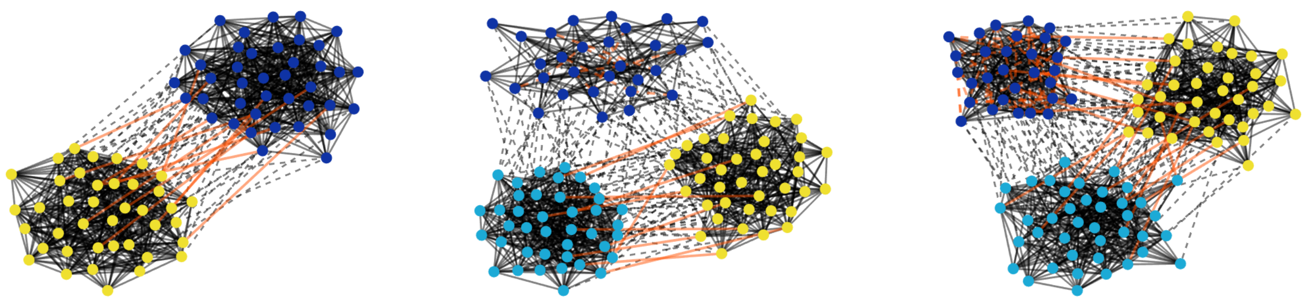


FIG. 3: Minimising $F(\boldsymbol{\sigma})$ can lead to a number of ambiguous situations, such as returning configurations that are neither traditionally nor relaxedly balanced (orange links are classified as misplaced according to the F-test; however, they can neither be arranged into homogeneous blocks). Within our, novel, statistical framework these configurations are unambiguously classified as balanced according to the statistical variant of the traditional balance theory.

$$\ln \mathcal{L}_{\text{SSBM}} = \sum_{r=1}^{k} \left[L_{rr}^{+}(\mathbf{A}) \ln x_{rr} + L_{rr}^{-}(\mathbf{A}) \ln y_{rr} - \binom{N_r}{2} \ln[1 + x_{rr} + y_{rr}] \right] + \sum_{r=1}^{k} \sum_{s(>r)} [L_{rs}^{+}(\mathbf{A}) \ln x_{rs} + L_{rs}^{-}(\mathbf{A}) \ln y_{rs} - N_r N_r \ln[1 + x_{rs} + y_{rs}]] \tag{8}$$

and its maximisation leads to recover the conditions $p_{rr}^{+} = 2L_{rs}^{+}(\mathbf{A})/N_r(N_r - 1)$, $p_{rr}^{-} = 2L_{rs}^{-}(\mathbf{A})/N_r(N_r - 1)$, $r = 1 \dots k$ and $p_{rs}^{+} = L_{rs}^{+}(\mathbf{A})/N_r N_s$, $p_{rs}^{-} = L_{rs}^{-}(\mathbf{A})/N_r N_s$, $\forall\, r < s$.

Let us, now, recall that BIC is defined as

$$\text{BIC} = \kappa \ln n - 2 \ln \mathcal{L} \tag{9}$$

where $\mathcal{L}$ is a model likelihood and κ indicates the number of parameters entering into its definition. Here, we have posed $\mathcal{L} \equiv \mathcal{L}_{\text{SSBM}}$, $\kappa \equiv \kappa_{\text{SSBM}} = k(k+1)$ (i.e. $k + k(k-1)/2$ parameters to be tuned on the set of values L_{rs}^{+}, $\forall\, r \leq s$ and $k + k(k-1)/2$ parameters to be tuned on the set of values L_{rs}^{-}, $\forall\, r \leq s$) and $n = N(N-1)/2$.

Figure 1 shows three, ideal partitions: one compatible with the strong balance theory, one compatible with the weak balance theory and one compatible with the relaxed balanced theory.

Comparing BIC minimisation with F minimisation

Let us, now, carry out another comparison between the recipe prescribing to minimise $F(\boldsymbol{\sigma})$ and the one prescribing to minimise BIC. As fig. 2 shows. the partitions that are recovered upon minimising BIC match the planted ones, a result confirming that BIC is sensitive to the 'signed density' of the modules. As a consequence, the partitions recovered upon minimising it coincide with the ones recovered upon minimising $F(\boldsymbol{\sigma})$ only if the former ones are k-balanced, i.e. satisfy the relationships $p_{rr}^{-} = 0$, $r = 1 \dots k$ and $p_{rs}^{+} = 0$, $\forall\, r < s$.

Minimising $F(\boldsymbol{\sigma})$ can lead to a number of ambiguous situations, such as *i)* returning configurations that are neither traditionally nor relaxedly balanced; *ii)* returning more than one frustrated configuration.

Figure 3 depicts the first situation. Nodes of the same colour are those put together as a consequence of minimising $F(\boldsymbol{\sigma})$: although the presence of negative links between and within modules makes the recovered partitions 'traditionally' frustrated, the original formulation of the relaxed balance theory would lead us to conclude that they are 'relaxedly' frustrated as well; only within our, novel, statistical framework these configurations can be unambiguously classified as balanced according to the statistical variant of the traditional balance theory.

Figure 4 depicts the second situation: the minimisation of $F(\boldsymbol{\sigma})$ can return more than one frustrated configuration; our BIC-based test, however, 'prefers' the one on the left, classifying it as balanced according to the statistical variant of the traditional balance theory.

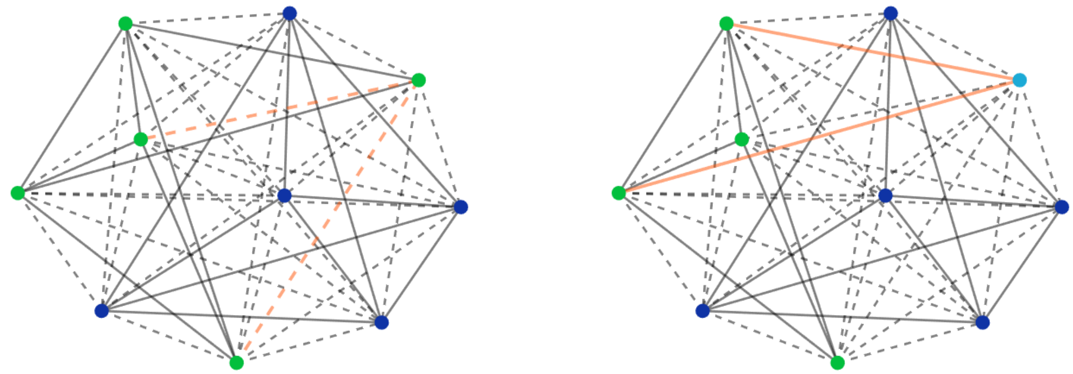


FIG. 4: Minimising $F(\boldsymbol{\sigma})$ can lead to a number of ambiguous situations, such as returning more than one configuration in correspondence of which $F(\boldsymbol{\sigma})$ attains its minimum. In this, particular case, the Slovenian Parliament admits two, different arrangements of nodes characterised by $F(\boldsymbol{\sigma}) = 2$ (orange links are classified as misplaced according to the F-test). Our BIC-based test, however, 'prefers' the one on the left, classifying it as balanced according to the statistical variant of the traditional balance theory.

Minimising BIC on configurations generated by the Signed Random Graph Model (SRGM)

Let us, first, recall the derivation of the SRGM. It is defined by the Hamiltonian

$$H(\mathbf{A}) = \alpha L^+(\mathbf{A}) + \beta L^-(\mathbf{A}) = \sum_{i=1}^{N} \sum_{j(>i)} [\alpha a_{ij}^+ + \beta a_{ij}^-] \tag{10}$$

leading to

$$\begin{aligned} Z = \sum_{\mathbf{A}\in\mathbb{A}} e^{-H(\mathbf{A})} = \sum_{\mathbf{A}\in\mathbb{A}} \prod_{i=1}^{N} \prod_{j(>i)} e^{-[\alpha a_{ij}^+ + \beta a_{ij}^-]} &= \prod_{i=1}^{N} \prod_{j(>i)} \sum_{a_{ij}=-1,0,+1} e^{-[\alpha a_{ij}^+ + \beta a_{ij}^-]} \\ &= \prod_{i=1}^{N} \prod_{j(>i)} [1 + e^{-\alpha} + e^{-\beta}] = [1 + e^{-\alpha} + e^{-\beta}]^{\binom{N}{2}}. \end{aligned} \tag{11}$$

As a consequence,

$$P_{\text{SRGM}}(\mathbf{A}) = \frac{e^{-H(\mathbf{A})}}{Z} = \frac{e^{-[\alpha L^+(\mathbf{A}) + \beta L^-(\mathbf{A})]}}{[1 + e^{-\alpha} + e^{-\beta}]^{\binom{N}{2}}} \equiv \frac{x^{L^+(\mathbf{A})} y^{L^-(\mathbf{A})}}{[1 + x + y]^{\binom{N}{2}}} \equiv (p^-)^{L^-(\mathbf{A})} (p^0)^{L^0(\mathbf{A})} (p^+)^{L^+(\mathbf{A})} \tag{12}$$

where $p^+ = x/(1 + x + y)$, $p^- = y/(1 + x + y)$ and $p^0 = 1/(1 + x + y)$. Its log-likelihood reads

$$\ln \mathcal{L}_{\text{SRGM}} = L^+(\mathbf{A}) \ln x + L^-(\mathbf{A}) \ln y - \binom{N}{2} \ln[1 + x + y] \tag{13}$$

and its maximisation leads to recover the conditions $p^+ = 2L^+(\mathbf{A})/N(N-1)$, $p^- = 2L^-(\mathbf{A})/N(N-1)$.

Figure 5 shows three configurations generated by the SRGM. As such a model does not carry any information about a network modular structure, no groups of nodes should be recognised. This is precisely the output of our BIC-based recipe, returning a single community gathering all nodes together (i.e. $k = 1$), irrespectively from the choice of the parameters (i.e. be $p^+ < p^-$, $p^+ \simeq p^-$ or $p^+ > p^-$). Minimising $F(\boldsymbol{\sigma})$ (or maximising $Q(\boldsymbol{\sigma})$ - see Appendix C), instead, leads to recover a number of modules $k \geq 1$ (in the case $L^+ < L^-$, $k_{\text{F}} = 25$ and $k_{\text{Q}} = 13$; in the case $L^+ \simeq L^-$, $k_{\text{F}} = 6$ and $k_{\text{Q}} = 5$; in the case $L^+ > L^-$, $k_{\text{F}} = k_{\text{Q}} = 1$).

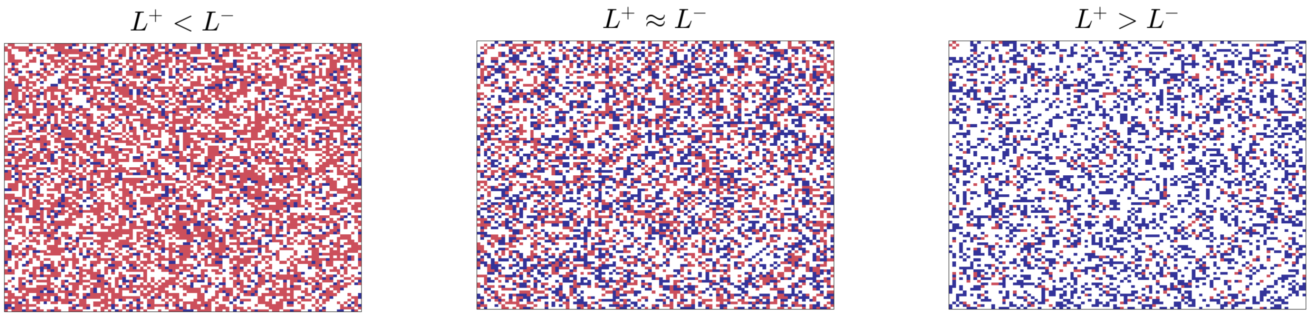


FIG. 5: Three configurations generated by the SRGM (positive links are coloured in blue; negative links are coloured in red). Since this model does not carry any information about a network modular structure, no groups of nodes should be detected: minimising BIC, in fact, always returns a single community gathering all nodes together, irrespectively from the choice of the parameters.

Minimising BIC on negative and positive cliques

Let us, now, consider a negative clique composed by N nodes; evaluating BIC on the partition defined by k modules returns the value $k(k+1)\ln[N(N-1)/2]$; since $N \geq k \geq 1$, keeping all nodes together is the most convenient choice. The same result holds true if we consider a positive clique composed by N nodes, the reason lying in the completely symmetric role played by negative and positive links, both contributing to the density of the (potential) network modules.

Minimising BIC on complete, signed graphs

When dealing with complete graphs, the signs come into play in a quite peculiar fashion. Let us, in fact, consider a complete graph of size $N = N_1 + N_2$, constituted by two, negative cliques having, respectively, N_1 and N_2 nodes and such that each node of a clique is connected to each node of the other via a positive link. Let us, now, pose $k > 2$ and consider the following inequality

$$k(k+1)\ln\binom{N}{2} - 2\ln\mathcal{L}_{\text{SSBM}} > 6\ln\binom{N}{2} \tag{14}$$

stating that evaluating BIC on a generic partition defined by $k > 2$ modules returns a value that is strictly larger than the value of BIC calculated on the bi-partition whose modules coincide with the cliques themselves (in fact, $N \geq k > 2$ and $\ln\mathcal{L}_{\text{SSBM}} > 0$).

Let us, now, compare the bi-partition induced by the negative cliques with the partition induced by imposing the presence of just one module: in this, last case, evaluating BIC returns the value

$$\begin{aligned}
\text{BIC}(k=1) &= 2\ln\binom{N}{2} - 2\ln\left[(p^+)^{L^+}(p^-)^{L^-}\right] \\
&= 2\ln\binom{N}{2} - 2\ln\left[\left(\frac{2N_1N_2}{N(N-1)}\right)^{N_1N_2}\left(\frac{N_1(N_1-1)+N_2(N_2-1)}{N(N-1)}\right)^{\binom{N_1}{2}+\binom{N_2}{2}}\right] \\
&= 2\ln\binom{N}{2} - 2\ln\left[\left(\frac{2N_1(N-N_1)}{N(N-1)}\right)^{N_1(N-N_1)}\left(\frac{N_1(N_1-1)+(N-N_1)(N-N_1-1)}{N(N-1)}\right)^{\binom{N_1}{2}+\binom{N-N_1}{2}}\right]
\end{aligned} \tag{15}$$

where we have used the relationship $N_2 = N - N_1$. As depicted in fig. 6, splitting nodes according to the partition induced by the signs is always 'more convenient' than partitioning them in a different way, a result suggesting that signs keep playing a role as long as the information embodied by the network density becomes irrelevant.

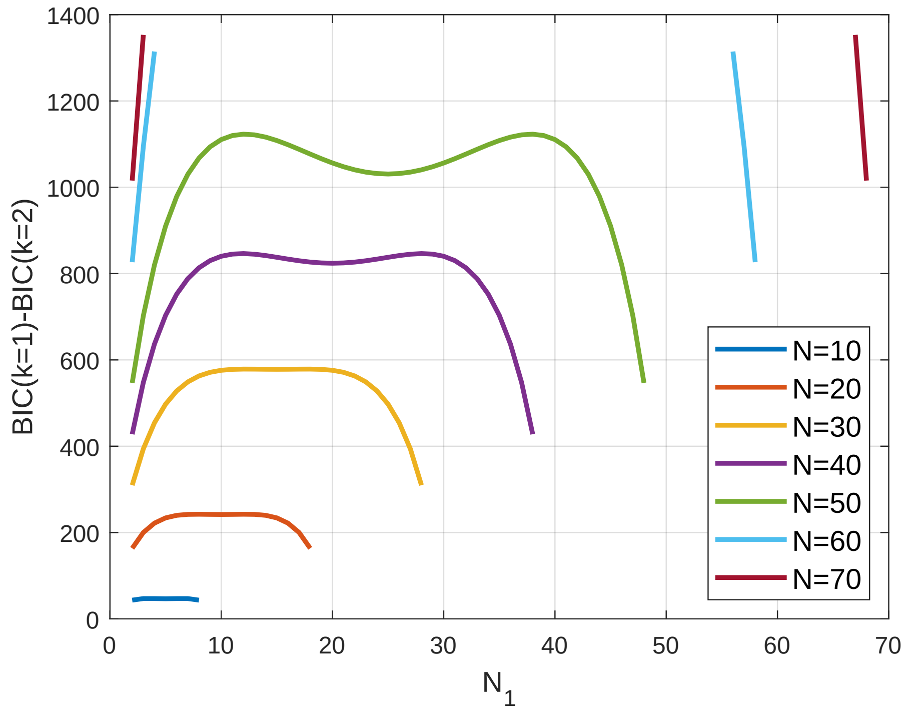


FIG. 6: Plotting the difference between BIC($k = 1$) and the value of BIC characterised by the bi-partition whose modules coincide with the cliques themselves, as a function of $2 < N_1 < N - 2$, reveals it to be always positive. This result confirms that such a bi-partition is the one in correspondence of which BIC attains its minimum value.

Purely numerical experiments seem to suggest that an analogous result holds true for any number of cliques, i.e. the partition defined by (the modules that coincide with) the cliques themselves is the one attaining the minimum value of BIC (see also fig. 7).

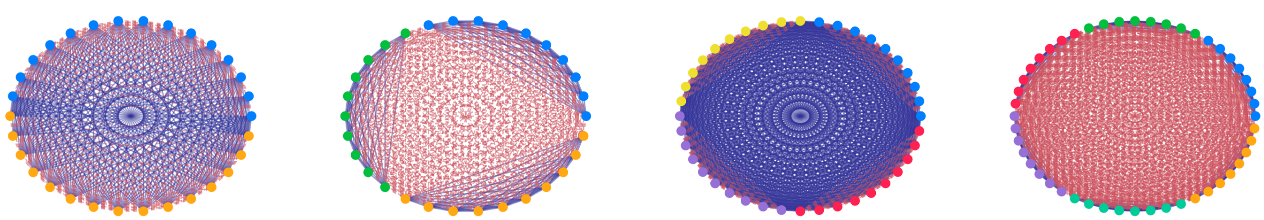


FIG. 7: Results of testing BIC minimisation on four, complete, signed graphs balanced according to the traditional balance theory (second panel: a 3-balanced configuration; fourth panel: a 6-balanced configuration) and maximally frustrated according to the traditional balance theory but perfectly balanced according to the relaxed balance theory (first panel: $k = 2$; third panel: $k = 4$). Minimising BIC leads to the partition induced by signs, since always 'more convenient' than any, other partition. Positive links are coloured in blue; negative links are coloured in red.

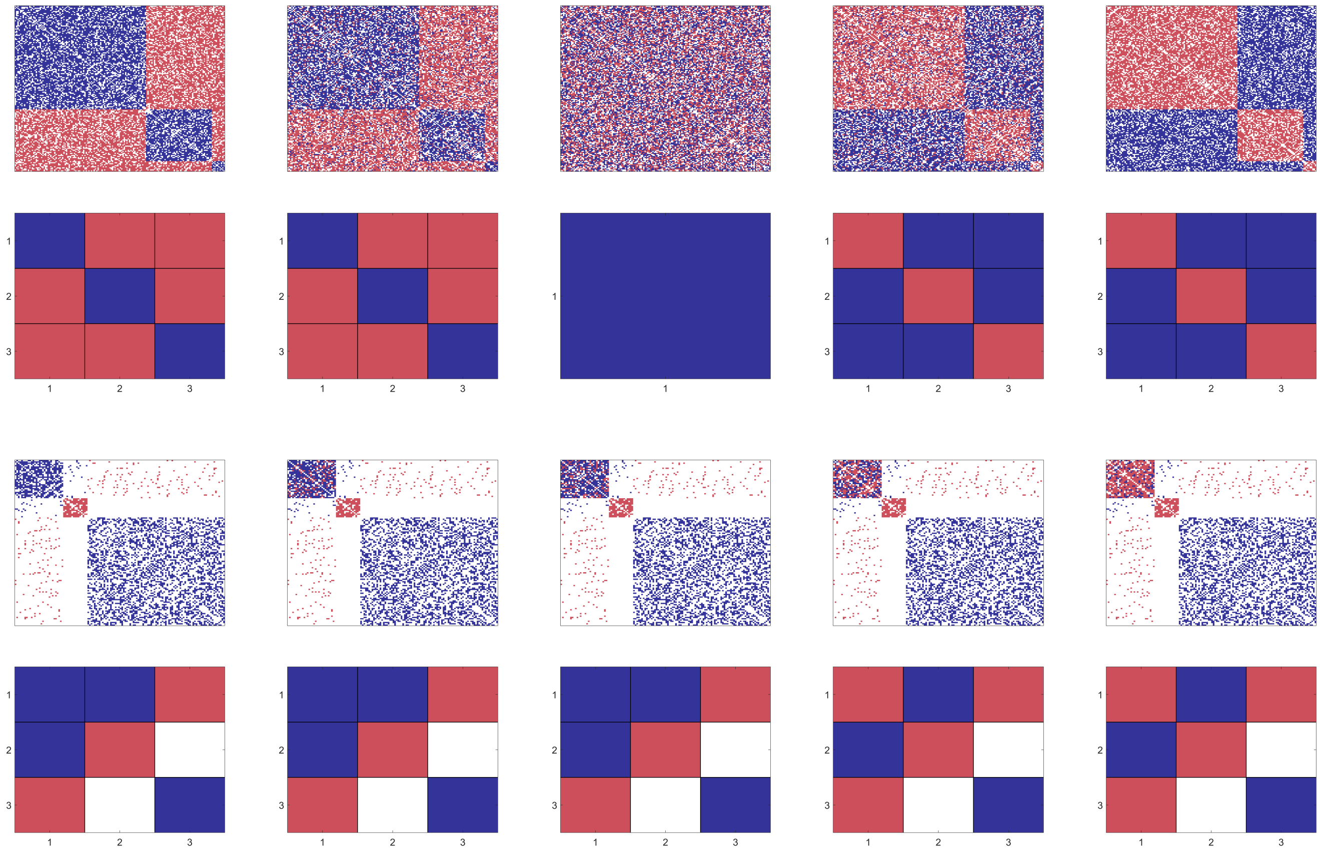


FIG. 8: Partitions recovered upon minimising BIC on, five, synthetic (dense: upper panels; sparse: lower panels) configurations generated via complementary probability matrices. By varying a single parameter, the $C = 3$ blocks can be designed to obey either the traditional or the relaxed balance theory. Upper panels: the sizes of the blocks are 100, 50 and 10 while the parameters have been set to the values $c = 0.7$ and $\alpha = 0.7, 0.6, 0.35, 0.1, 0$. Lower panels: the sizes of the blocks are 30, 15 and 85 while the parameters have been set to the values $c = 0.8$ and $\alpha = 0.8, 0.6, 0.4, 0.2, 0$.

Minimising BIC on more, synthetic configurations

More realistic, planted partitions can be obtained as follows. Let us consider a matrix with C communities, inducing a $C \times C$ block matrices. Let us, now, pose $p_\bullet^+ = \alpha$, $p_\bullet^- = c - \alpha$, $p_\circ^+ = c - \alpha$, $p_\circ^- = \alpha$ with α playing the role of internal density of positive links and c playing the role of link density: more formally,

$$\mathbf{P}^+ = \begin{bmatrix} \alpha & c-\alpha & c-\alpha \\ c-\alpha & \alpha & c-\alpha \\ c-\alpha & c-\alpha & \alpha \end{bmatrix}, \quad \mathbf{P}^- = \begin{bmatrix} c-\alpha & \alpha & \alpha \\ \alpha & c-\alpha & \alpha \\ \alpha & \alpha & c-\alpha \end{bmatrix}. \tag{16}$$

Considering probability matrices that are complementary allows us to span a wide spectrum of different configurations, by perturbing the entire network structure at the same time (from this perspective, the considered benchmark is similar-in-spirit to the so-called Aldecoa's 'relaxed caveman' benchmark): by fixing c and letting α range from 0 to c, in fact, one can move from traditionally balanced networks to relaxedly balanced networks, through configurations that are compatible with the SRGM. The upper panels of fig. 8 have been produced by considering $C = 3$ blocks of size 100, 50, 10 and setting $c = 0.7$, $\alpha = 0.7, 0.6, 0.35, 0.1, 0$.

The lower panels of fig. 8 have been produced by considering the setting

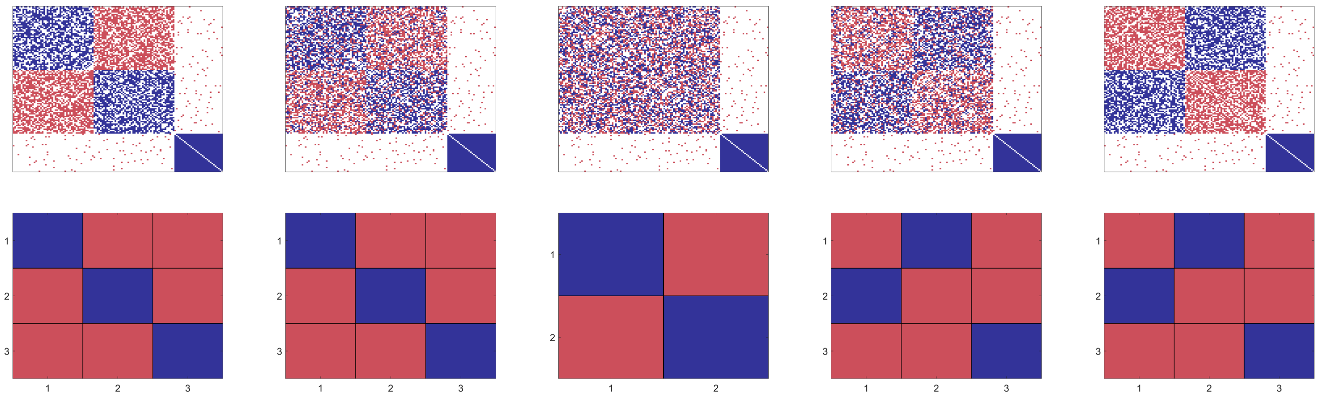


FIG. 9: Partitions recovered upon minimising BIC on, five, synthetic configurations generated via complementary probability matrices. By varying a single parameter, the $C = 3$ blocks (whose sizes are 50, 50 and 30) can be designed to obey either the traditional or the relaxed balance theory. The parameters have been set to the values $c = 0.6$ and $\alpha = 0.6, 0.45, 0.3, 0.15, 0$.

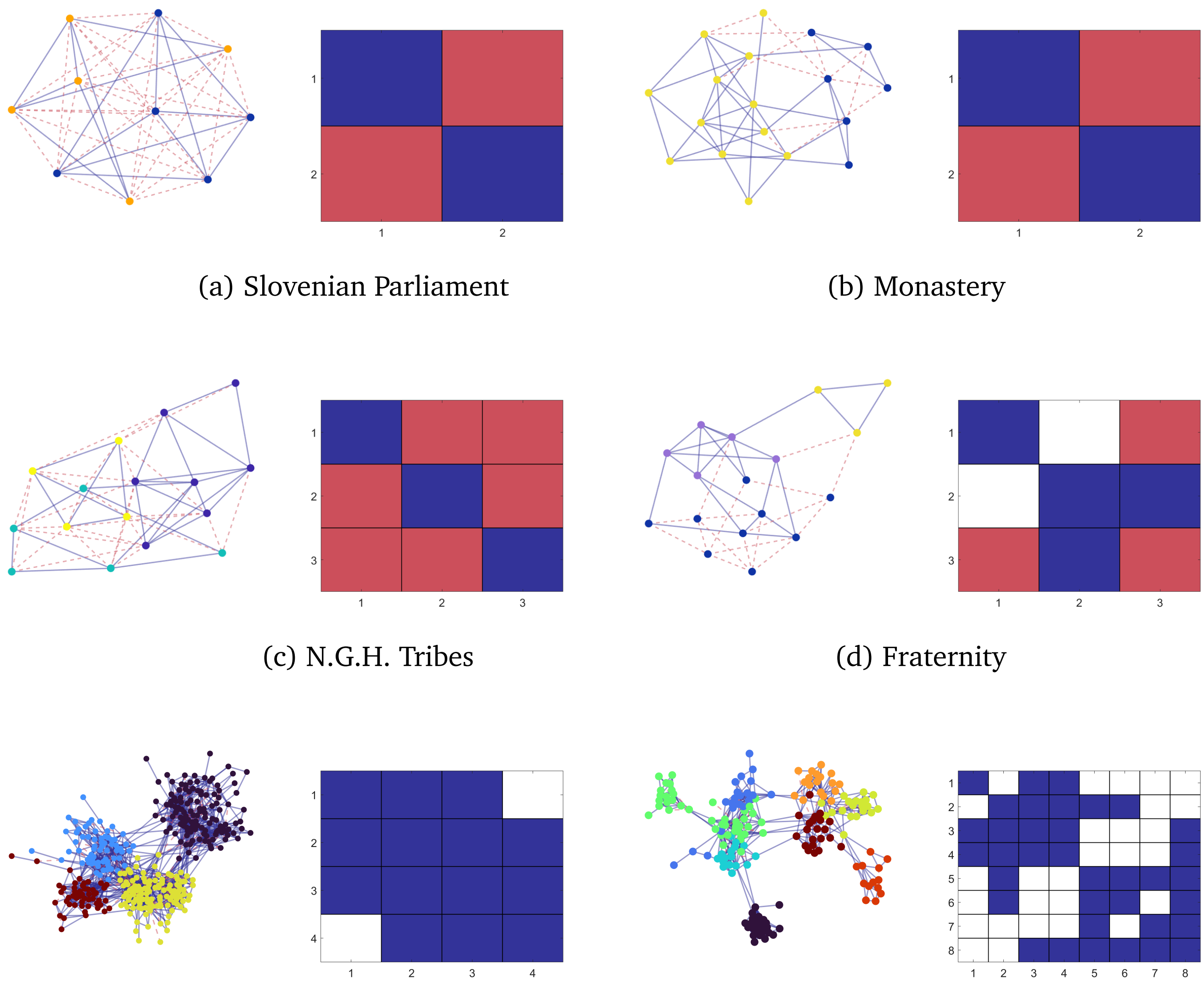
$$\mathbf{P}^{+} = \begin{bmatrix} \alpha & 0.05 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \quad \mathbf{P}^{-} = \begin{bmatrix} c-\alpha & 0 & 0.05 \\ 0 & 0.8 & 0 \\ 0.05 & 0 & 0 \end{bmatrix}, \tag{17}$$

identifying $C = 3$ blocks whose sizes are 30, 15 and 85, with $c = 0.8$ and where the only varying parameter is $\alpha \in \{0.8, 0.6, 0.4, 0.2, 0\}$: the partitions recovered by minimising BIC are always consistent with the planted ones, irrespectively from the density of (the blocks constituting) the considered configuration.

Finally, the panels of fig. 9 have been produced by considering the setting

$$\mathbf{P}^{+} = \begin{bmatrix} \alpha & c-\alpha & 0 \\ c-\alpha & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}^{-} = \begin{bmatrix} c-\alpha & \alpha & 0.03 \\ \alpha & c-\alpha & 0.03 \\ 0.03 & 0.03 & 1 \end{bmatrix}, \tag{18}$$

identifying $C = 3$ blocks whose sizes are 50, 50 and 30, with $c = 0.6$ and where the only varying parameter is $\alpha \in \{0.6, 0.45, 0.3, 0.15, 0\}$: the partitions recovered by minimising BIC are always consistent with the planted ones across a wide range of the parameters values.



(a) Slovenian Parliament (b) Monastery

(c) N.G.H. Tribes (d) Fraternity

(e) Spanish School 1 (f) Spanish School 2

FIG. 10: Partitions recovered upon minimising BIC on the Slovenian Parliament [1], Monastery, N.G.H. Tribes, Fraternity [2] and Spanish Schools [3] datasets. A generic block, indexed as rs, is coloured in blue if $L^+_{rs} > L^-_{rs}$, in red if $L^+_{rs} < L^-_{rs}$ and in white if $L^+_{rs} = L^-_{rs}$. Minimising BIC leads to recover partitions that obey either the statistical variant of the traditional balance theory or the statistical variant of the relaxed balance theory. Positive links are coloured in blue; negative links are coloured in red.

Minimising BIC on more, real-world configurations

Let us, now, apply our recipe to a number of real-world, signed configurations, i.e. Slovenian Parliament [1], Monastery, N.G.H. Tribes, Fraternity [2] and Spanish Schools [3]. Since the last dataset originally displayed directed interactions, we have made it undirected by applying the following set of rules: '+/+' becomes '+' and '−/−' becomes '−' (i.e. if any two agents have the same opinion, the undirected connection preserve the sign); '+/−' becomes '−' (i.e. if any two agents have opposite opinions, the undirected connection has a negative sign); '+/0' and '−/0' become '0' (i.e. any interaction that is not reciprocated disappears in the undirected version of the network).

As fig. 10 shows, minimising BIC leads to recover partitions that obey either the statistical variant of the traditional balance theory (in its strong or weak form) or the statistical variant of the relaxed balance theory (a blue block is characterised by a majority of positive links; a red block is characterised by a majority of negative links). On the

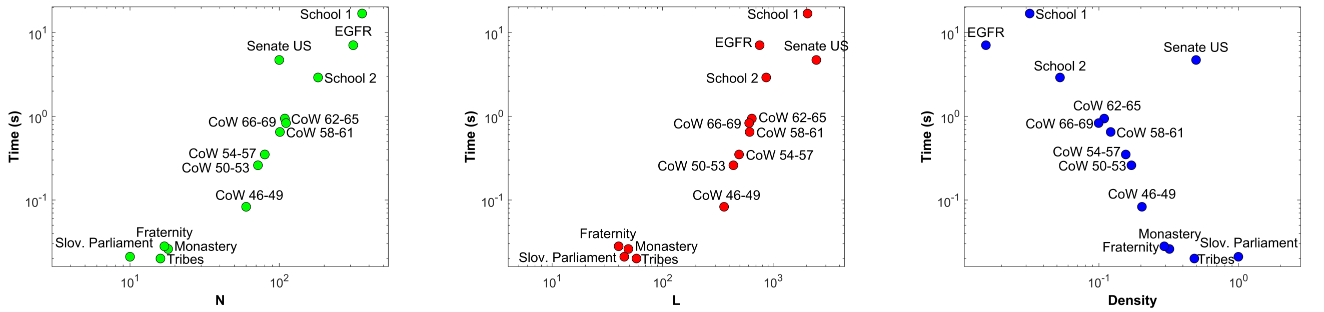


FIG. 11: The computational complexity of the algorithm to minimise BIC rises with N and L but decreases with the link density.

other hand, US Senate, EGFR, Macrophage, Bitcoin Alpha and Bitcoin OTC datasets [4] are characterised by $k = 1$ (either because $p^+ \simeq p^-$ or because $p^+ \gg p^-$).

The community structure characterising synthetic and real-world, signed networks has been also explored in [5–13].

From a purely computational perspective, the complexity of the algorithm to minimise BIC rises with N and L although it decreases with the link density $c(\mathbf{A}) = 2L(\mathbf{A})/N(N-1)$, as fig. 11 shows - in fact, the denser the configuration, the faster the algorithm.

APPENDIX C. MORE ON SIGNED MODULARITY

A deterministic theory of balance can be turned into a statistical theory of balance by answering the following question: is it possible to define a reference level of misplaced links by means of which discerning frustrated graphs from balanced ones? The answer is affirmative and calls for comparing the empirical amount of frustration of a given, signed configuration with the one predicted by a properly-defined reference model: in formulas, one may define a partition as *statistically balanced* if satisfying the relationship $F(\boldsymbol{\sigma}) < \langle F(\boldsymbol{\sigma}) \rangle$.

A quantity embodying such a comparison already exists: it is the *signed modularity*, reading

$$\begin{aligned} Q(\boldsymbol{\sigma}) &= \sum_{i=1}^{N} \sum_{j(>i)} [(a_{ij}^{+} - p_{ij}^{+}) - (a_{ij}^{-} - p_{ij}^{-})] \delta_{\sigma_i, \sigma_j} \\ &= L_{\bullet}^{+} - \langle L_{\bullet}^{+} \rangle - (L_{\bullet}^{-} - \langle L_{\bullet}^{-} \rangle) \\ &= (L^{+} - L_{\circ}^{+}) - \langle L^{+} - L_{\circ}^{+} \rangle - (L_{\bullet}^{-} - \langle L_{\bullet}^{-} \rangle) \\ &= -(L_{\circ}^{+} + L_{\bullet}^{-}) + \langle L_{\circ}^{+} + L_{\bullet}^{-} \rangle + L^{+} - \langle L^{+} \rangle \\ &= -F(\boldsymbol{\sigma}) + \langle F(\boldsymbol{\sigma}) \rangle + L^{+} - \langle L^{+} \rangle \end{aligned} \tag{19}$$

with obvious meaning of the symbols (the addendum $L^{+} - \langle L^{+} \rangle$ is just an offset not depending on the specific partition and amounting at zero for any model reproducing the total number of positive links) [14]. Since the total number of positive links is preserved under any model considered in the present paper, we obtain

$$Q(\boldsymbol{\sigma}) = -F(\boldsymbol{\sigma}) + \langle F(\boldsymbol{\sigma}) \rangle. \tag{20}$$

$Q(\boldsymbol{\sigma})$ has been widely employed to spot communities on signed networks, with the positions $p_{ij}^{+} = k_i^{+} k_j^{+} / 2L^{+}$ and $p_{ij}^{-} = k_i^{-} k_j^{-} / 2L^{-}$, $\forall\, i < j$ [6, 14, 15]. Such a recipe, instantiating the Chung-Lu (CL) model, is applicable only in case $p_{ij}^{+} \leq 1$ and $p_{ij}^{-} \leq 1$, $\forall\, i < j$: these conditions, however, do not hold in several cases of interest, an example of paramount importance being provided by sparse networks whose degree distribution is scale-free [16]. In order to overcome the aforementioned limitation, a different framework is needed.

One may follow the analytical approach introduced in [17] and further developed in [18], aimed at identifying the functional form of the maximum-entropy probability distribution that preserves a desired set of empirical constraints, on average. Specifically, this approach looks for the graph probability $P(\mathbf{A})$ that maximises Shannon entropy $S = -\sum_{\mathbf{A} \in \mathbb{A}} P(\mathbf{A}) \ln P(\mathbf{A})$, under constraints enforcing the expected value of a chosen set of properties. The formal solution to this problem is the exponential probability $P(\mathbf{A}) = e^{-H(\mathbf{A})}/Z$ where the *Hamiltonian* $H(\mathbf{A})$ is a linear combination of the constrained properties and the *partition function* $Z = \sum_{\mathbf{A} \in \mathbb{A}} e^{-H(\mathbf{A})}$ plays the role of normalising constant, the sum running over the set $\mathbb{A}$ of all binary, undirected, signed graphs whose cardinality amounts to $|\mathbb{A}| = 3^{\binom{N}{2}}$. Two examples of models of the kind are the Signed Random Graph Model (SRGM) and the Signed Configuration Model (SCM) [19].

According to the traditional balance theory, several ways exist in which a given configuration can be frustrated. Let us, now, analyse them in detail.

1. *Evaluating frustration due to negative links*

Positive subgraphs connected by negative links. In order to understand how a Q-based test would work, let us consider two subgraphs with, respectively, m and n nodes, positive intra-modular links and negative inter-modular links. Let us denote with $V_{\bullet} = m(m-1)/2 + n(n-1)/2$ the total number of intra-modular pairs of nodes and with Q_0 the value of modularity associated to the partition of the entire graph, except our, two subgraphs; let us also call $L_{\bullet}^{+}$ the total number of positive links within modules and $L_{\circ}^{-}$ the total number of negative links between modules. Then,

$$Q_A = Q_0 + [L_{\bullet}^{+} - p^{+} V_{\bullet}] - [0 - p^{-} V_{\bullet}], \tag{21}$$

$$Q_B = Q_0 + [L_{\bullet}^{+} - p^{+}(V_{\bullet} + mn)] - [L_{\circ}^{-} - p^{-}(V_{\bullet} + mn)] \tag{22}$$

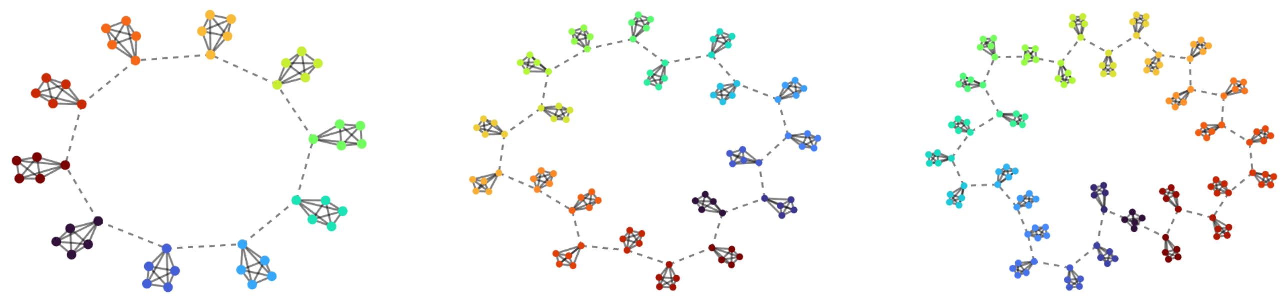


FIG. 12: Partitions recovered upon maximising the signed modularity on three rings of cliques, i.e. a set of 10 (left), 20 (middle), 30 (right) cliques, constituted by 5 nodes each, internally connected by positive links and inter-connected by negative links.

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to be fully consistent with the traditional balance theory, one should require

$$Q_A > Q_B = Q_A - L_\circ^- - (p^+ - p^-)mn, \tag{23}$$

a condition that it is satisfied whenever $L_\circ^- > (p^- - p^+)mn$, i.e. whenever the probability $p_\circ^- \equiv L_\circ^-/mn$ of establishing a negative link within modules is larger than $p^- - p^+ = 2(L^- - L^+)/N(N-1)$. This condition sheds light on the role played by the signed variant of the resolution limit, naturally re-interpretable as a threshold-based criterion for discerning if a given, signed configuration is balanced or not: in words, the 'acceptable' level of frustration, according to which our subgraphs can be safely interpreted as a single community, is represented by $(p^- - p^+)mn$.

2. *Evaluating frustration due to positive links*

Positive subgraphs connected by positive links. Let us, now, consider two subgraphs with, respectively, m and n nodes, positive intra- and inter-modular links; let us also call $L_\bullet^+$ the total number of positive links within modules and $L_\circ^+$ the total number of positive links between modules. Then,

$$Q_A = Q_0 + [L_\bullet^+ - p^+ V_\bullet] - [0 - p^- V_\bullet], \tag{24}$$

$$Q_B = Q_0 + [L_\bullet^+ + L_\circ^+ - p^+(V_\bullet + mn)] - [0 - p^-(V_\bullet + mn)] \tag{25}$$

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to be fully consistent with the traditional balance theory, one should require

$$Q_B = Q_A + L_\circ^+ - (p^+ - p^-)mn > Q_A, \tag{26}$$

a condition that it is satisfied whenever $L_\circ^+ > (p^+ - p^-)mn$, i.e. whenever the probability $p_\circ^+ \equiv L_\circ^+/mn$ of establishing a positive link between modules is larger than $p^+ - p^- = 2(L^+ - L^-)/N(N-1)$. The threshold-based criterion for discerning balance represented by the signed variant of the resolution limit, now, sets the 'acceptable' level of frustration, according to which our subgraphs can be safely interpreted as two, separate communities, at $(p^+ - p^-)mn$.

3. *Evaluating frustration in real-world networks*

Interestingly enough, when studying real-world, signed networks, the relationship $L^+ \gg L^-$ is often (if not always) found to hold true: as a consequence, the condition

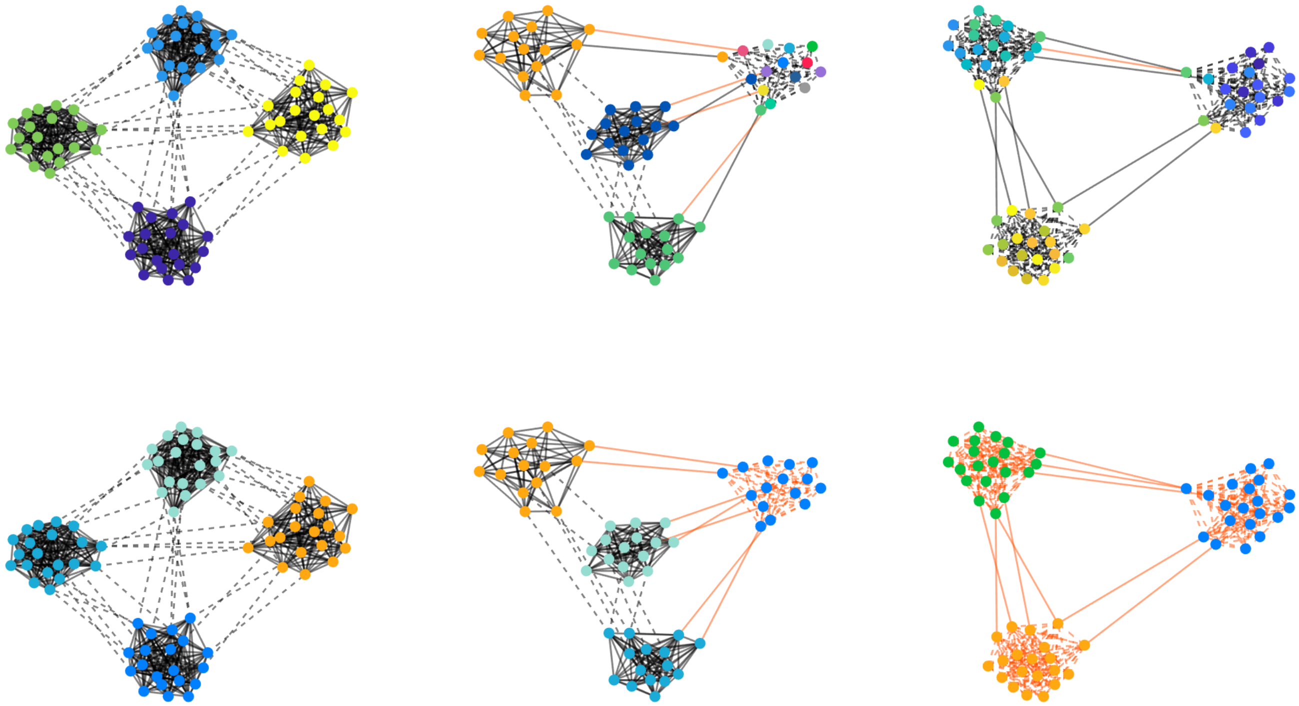


FIG. 13: Top panels: partitions recovered upon maximising $Q(\boldsymbol{\sigma})$. Bottom panels: partitions recovered upon minimising BIC. Left panels: minimising BIC returns partitions that coincide with those returned upon maximising the signed modularity only in case they are k-balanced. Middle and right panels: minimising BIC (bottom panels) leads to recover the planted partitions, balanced according to the relaxed balance theory; maximising $Q(\boldsymbol{\sigma})$ (top panels), instead, leads to the fragmentation of the subgraphs constituted by negative links into singletons. In other words, the Q-based test seeks to recover a configuration obeying the traditional balance theory even when there is none 'by design' (orange links are classified as misplaced according to the F-test).

$$L_\circ^- > (p^- - p^+)mn < 0 \tag{27}$$

is trivially satisfied. Such an evidence has several consequences. In order to discuss them, let us focus on *i)* the case of negative subgraphs connected by negative links; *ii)* the case of negative subgraphs connected by positive links.

Negative subgraphs connected by negative links. Let us consider two subgraphs with, respectively, m and n nodes, negative intra- and inter-modular links; let us also call $L_\bullet^-$ the total number of negative links within modules and $L_\circ^-$ the total number of negative links between modules. Then,

$$Q_A = Q_0 + [0 - p^+ V_\bullet] - [L_\bullet^- - p^- V_\bullet], \tag{28}$$

$$Q_B = Q_0 + [0 - p^+(V_\bullet + mn)] - [L_\bullet^- + L_\circ^- - p^-(V_\bullet + mn)] \tag{29}$$

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to limit the number of links that would be deemed as misplaced according to the traditional balance theory, one should require

$$Q_A > Q_B = Q_A - L_\circ^- - (p^+ - p^-)mn, \tag{30}$$

a condition that it is satisfied whenever $L_\circ^- > (p^- - p^+)mn$, i.e. whenever the probability $p_\circ^- \equiv L_\circ^-/mn$ of establishing a negative link between modules is larger than $p^- - p^+ = 2(L^- - L^+)/N(N-1) < 0$. Hence, it is always convenient to separate negatively connected modules and, if such a line of reasoning is repeated in a hierarchical fashion, it is always convenient to separate negatively connected nodes. Otherwise stated, one should not expect the presence of negative links within blocks since negatively connected modules will always lead to singletons: in this sense, the signed modularity is resolution limit-free.

Negative subgraphs connected by positive links. Let us, now, focus on the case of negative subgraphs connected by positive links and consider two subgraphs with, respectively, m and n nodes, negative intra-modular links and positive inter-modular links; let us also call $L_\bullet^-$ the total number of negative links within modules and $L_\circ^+$ the total number of positive links between modules. Then,

$$Q_A = Q_0 + [0 - p^+ V_\bullet] - [L_\bullet^- - p^- V_\bullet], \tag{31}$$

$$Q_B = Q_0 + [L_\circ^+ - p^+(V_\bullet + mn)] - [L_\bullet^- - p^-(V_\bullet + mn)] \tag{32}$$

with Q_A being the SRGM-induced modularity of the configuration identifying our subgraphs as two, separate communities and Q_B being the SRGM-induced modularity of the configuration identifying our subgraphs as a single community. In order to limit the number of links that would be deemed as misplaced according to the traditional balance theory, one should require

$$Q_B = Q_A + L_\circ^+ - (p^+ - p^-)mn > Q_A, \tag{33}$$

a condition that it is satisfied whenever $L_\circ^+ > (p^+ - p^-)mn$, i.e. whenever the probability $p_\circ^+ \equiv L_\circ^+/mn$ of establishing a positive link between modules is larger than $p^+ - p^- = 2(L^+ - L^-)/N(N-1)$. Now, as a consequence of eq. (27), it is convenient to fragment negatively connected modules into singletons; hence, according to the traditional balance theory, frustration can only occur because of misplaced, positive links appearing between blocks.

Figure 12 depicts the results of the signed modularity maximisation on three rings of cliques: since the relationship $L^+ \gg L^-$ holds true, one should not expect the presence of negative links within blocks (as we said, the signed modularity is resolution limit-free, in this sense). Notice that our exercise is defined in such a way that the numerical value of the generic addendum $(a_{ij}^+ - p_{ij}^+) - (a_{ij}^- - p_{ij}^-)$ is fixed, once and for all, by the choice of the benchmark to be solved: in other words, the definition of modularity does not change with the level of aggregation, being just recomputed (as any other score function) as the partition changes [19].

As fig. 13 shows, minimising BIC returns partitions that coincide with those returned by maximising modularity, or minimising $F(\boldsymbol{\sigma})$, solely in case they are k-balanced (i.e. obey the traditional balance theory). In case subgraphs constituted by negative links are, instead, present, minimising BIC leads to the planted partition induced by gathering such nodes together while maximising $Q(\boldsymbol{\sigma})$ leads to their fragmentation, hence recovering singletons. In other words, the Q-based test (exactly as the F-based test and the G-based test) seeks to recover traditionally balanced configurations even when there is none 'by design'.

From a purely numerical perspective, partitioning nodes by minimising BIC is accomplished as described in Algorithms 1 - 3.

For further details on the signed modularity, see also [14, 15, 20].

Algorithm 1: Pseudocode to partition nodes by minimising BIC - step I

```
function BICBasedCommunityDetectionStepI(N, A)
C ← array of labels of length N, initialised as (1, 2 ... N);
BIC ← UpdateBIC(A, C);
E ← randomly sorted edges;
for (u, v) ∈ E do
    C_0 ← C;
    BIC_0 ← BIC;
    if C(u) ≠ C(v) then
        C_1 ← C;
        for node w ∈ C(u) do
            C_1(w) ← C(v);
        end for
        BIC_1 ← UpdateBIC(A, C_1);
    end if
    if BIC_1 < BIC_0 then
        C ← C_1;
        BIC ← BIC_1
    else
        C ← C_0;
        BIC ← BIC_0
    end if
end for
⇒ repeat the for-loop to improve the chance of finding the best partition
```

Algorithm 2: Pseudocode to partition nodes by minimising BIC - step II

```
function BICBasedCommunityDetectionStepII(N, A)
C ← BICBasedCommunityDetectionStepI(N, A);
BIC ← UpdateBIC(A, C);
E ← randomly sorted edges;
for (u, v) ∈ E do
    C_0 ← C;
    BIC_0 ← BIC;
    if C(u) ≠ C(v) then
        C_1 ← C;
        C_1(u) ← C(v);
        BIC_1 ← UpdateBIC(A, C_1);
        C_2 ← C;
        C_2(v) ← C(u);
        BIC_2 ← UpdateBIC(A, C_2);
    else if C(u) = C(v) then
        C_1 ← C;
        C_1(u) ← randomly sorted community different from C(v);
        BIC_1 ← UpdateBIC(A, C_1);
        C_2 ← C;
        C_2(v) ← randomly sorted community different from C(u);
        BIC_2 ← UpdateBIC(A, C_2);
    end if
    i ← argmin{BIC_0, BIC_1, BIC_2};
    C ← C_i;
    BIC ← BIC_i;
end for
⇒ repeat the for-loop to improve the chance of finding the best partition
```

Algorithm 3: Pseudocode to update BIC

function *UpdateBIC*$(\mathbf{A}, C)$

$k \leftarrow$ number of modules, i.e. number of distinct labels in C;

$\mathbf{P}^{-} \leftarrow k \times k$ matrix whose entry (c_1, c_2) is the probability that a node $u \in C(u) = c_1$ is linked via a -1 to a node $v \in C(v) = c_2$;

$\mathbf{P}^{+} \leftarrow k \times k$ matrix whose entry (c_1, c_2) is the probability that a node $u \in C(u) = c_1$ is linked via a $+1$ to a node $v \in C(v) = c_2$;

$\mathbf{L}^{-} \leftarrow k \times k$ matrix whose entry (c_1, c_2) is *i)* the number of -1s between c_1 and c_2, if $c_1 \neq c_2$; *ii)* the number of -1s within c_1, otherwise;

$\mathbf{L}^{+} \leftarrow k \times k$ matrix whose entry (c_1, c_2) is *i)* the number of $+1$s between c_1 and c_2, if $c_1 \neq c_2$; *ii)* the number of $+1$s within c_1, otherwise;

$\mathbf{n} \leftarrow k \times 1$ array whose c-th entry is the number of nodes belonging to c;

$\mathcal{L} \leftarrow 1$;

for $c = 1 \dots k$ **do**

    $\mathcal{L} = \mathcal{L} \cdot \mathbf{P}^{-}(c,c)^{\mathbf{L}^{-}(c,c)} \mathbf{P}^{+}(c,c)^{\mathbf{L}^{+}(c,c)} (1 - \mathbf{P}^{-}(c,c) - \mathbf{P}^{+}(c,c))^{\binom{\mathbf{n}(c)}{2} - \mathbf{L}^{-}(c,c) - \mathbf{L}^{+}(c,c)}$;

    for $d = c+1 \dots k$ **do**

        $\mathcal{L} = \mathcal{L} \cdot \mathbf{P}^{-}(c,d)^{\mathbf{L}^{-}(c,d)} \mathbf{P}^{+}(c,d)^{\mathbf{L}^{+}(c,d)} (1 - \mathbf{P}^{-}(c,d) - \mathbf{P}^{+}(c,d))^{\mathbf{n}(c)\mathbf{n}(d) - \mathbf{L}^{-}(c,d) - \mathbf{L}^{+}(c,d)}$;

    end for

end for

$\text{BIC} = k(k+1) \ln \binom{N}{2} - 2 \ln \mathcal{L}$

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