



A monolithic DAE formulation for dynamic phase-field fracture: BDF adaptive time integration and 2D/3D benchmarks

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ABSTRACT

We consider dynamic brittle fracture in the framework of phase-field models and revisit its structure from the viewpoint of differential–algebraic equations (DAEs). Building on variational formulations of phase-field fracture, we treat the displacement field and the phase-field variable as primary unknowns of a coupled evolution system. Irreversibility is enforced through a history variable, so that the continuous problem takes the form of an index-1 DAE that preserves the thermodynamic structure of the underlying fracture model.

On this basis, we propose a monolithic time-integration strategy in which the full DAE system is advanced by an implicit, variable-order backward differentiation formula (BDF) scheme. This approach provides adaptive time stepping and automatic control of local truncation errors, while avoiding the additional splitting error and algorithmic dissipation associated with staggered displacement/phase-field updates.

The formulation is implemented in a finite element code that supports both small- and large-displacement kinematics, mixed finite elements, and distributed-memory parallelism, enabling large-scale two- and three-dimensional dynamic fracture simulations. The method is assessed on a set of benchmark problems, including single-edge-notched tension and impact configurations such as the Kalthoff–Winkler test and pre-cracked plates. The numerical results demonstrate that the proposed DAE-based formulation accurately reproduces experimentally observed crack paths and branching patterns, while providing improved robustness and controllable algorithmic dissipation.

1. Introduction

Dynamic failure of brittle solids is governed by highly non-smooth phenomena such as crack initiation, rapid propagation, branching, arrest and interaction with stress waves. Under impact or high-rate loading these processes occur on very short time scales and give rise to complex crack patterns that are challenging to predict quantitatively. Classical experiments on brittle materials have revealed, among others, the transition from straight to branching cracks, micro-branching instabilities, and failure-mode transitions under shear-dominated impact, see e.g. [1–6]. These observations have motivated the developing of continuum and computational models capable of reproducing dynamic crack paths and velocities.

In the framework of classical linear elastic fracture mechanics, crack growth is described in terms of sharp discontinuity surfaces $\Gamma(t)$ embedded in a linear elastic body Ω , and propagation criteria are formulated in terms of the energy release rate and stress intensity factors [7,8]. While this sharp-crack description is powerful at the level of asymptotic fields near a pre-existing crack, its direct use in numerical simulations is challenging when complex topological changes take place. Crack nucleation, merging

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and branching require either *a priori* assumptions on admissible crack paths or sophisticated tracking of discontinuities. A large body of work has been devoted to discrete computational strategies capable of handling dynamic fracture, including interface and cohesive-zone elements [9–13], extended finite element methods (XFEM) [14–16] and elements with embedded discontinuities and branching criteria [17,18]. These formulations typically require substantial algorithmic machinery to handle insertion and branching of discontinuity surfaces, in particular in three dimensions case and under dynamic loading.

A conceptually different route is provided by variational approaches to brittle fracture in the spirit of Francfort and Marigo [19], in which crack evolution is obtained from global energy minimization subject to irreversibility. Within this setting, Ambrosio and Tortorelli introduced an elliptic regularization of the Mumford–Shah functional for free-discontinuity problems [20,21], later rigorously analyzed via Γ -convergence [22–24]. Applied to fracture mechanics, this regularization replaces the sharp crack set Γ by a diffusive damage or phase-field variable that smoothly interpolates between the intact and fully broken states. The resulting approximations have been used in quasi-static brittle fracture [25–27] and can be shown to converge to the sharp Griffith model as the internal length parameter tends to zero.

In parallel, phase-field models for fracture emerged in the physics community as gradient-regularized Ginzburg–Landau systems describing the evolution of an order parameter coupled to elasticity, see e.g. [28–30]. These formulations naturally lend themselves to dynamic fracture because the phase field obeys a time-dependent evolution law, often of Allen–Cahn type, driven by elastic energy and regularized by gradient terms. However, many of these models are fully viscous in nature and not directly connected to the rate-independent Griffith theory in the quasi-static limit.

Bridging the mathematical and physical viewpoints, phase-field and gradient-damage formulations for fracture have been developed within continuum mechanics as thermodynamically consistent models with an internal length scale and clearly defined stored energy and dissipation functionals [31–33]. In the small-strain brittle case, a widely adopted choice is to use a scalar phase-field variable $s \in [0, 1]$ (with $s = 0$ denoting the intact state and $s = 1$ the fully broken state), coupled to the tensile part of the elastic energy through a degradation function $g(s)$. This construction allows to avoid crack growth under compression while preserving the thermodynamic structure of the model.

Dynamic extensions of these quasi-static phase-field models have been proposed along two main lines. A first class of approaches describes fracture via rate-independent energetic evolution, regularized in time by viscous or variational integrators for the elastodynamic field [34–37]. A second class follows more closely the Ginzburg–Landau analogy and prescribes an explicit evolution law for the phase field, possibly with inertia or viscosity, coupled to the elastodynamic equations [28–30]. Within the first class, Miehe and co-workers formulated a thermodynamically consistent, variational phase-field model for brittle fracture, introduced a history variable to enforce irreversibility in a local way, and proposed both monolithic and operator-split finite element implementations for quasi-static and dynamic problems [35,36]. In particular, Hofacker and Miehe [36] showed that all coupled field equations, namely momentum balance, the gradient-type phase-field evolution equation and the driving-force evolution, can be obtained as Euler–Lagrange equations of a mixed rate-type variational principle, and they devised a robust staggered scheme based on a Newmark time integrator for the displacement and a variational update for the phase field.

Subsequent works have extended to dynamical phase-field models in several directions. Borden et al. [38] derived a phase-field approximation to the Lagrangian of dynamic brittle fracture, emphasizing the role of degrading only the tensile part of the elastic energy to avoid spurious cracks under compression. Building on the gradient-damage model of Kuhn and Müller, Schlüter and co-workers proposed and analyzed dynamic phase-field approximations and studied their convergence properties [39]. Ulmer et al. extended the framework to dynamic brittle and ductile fracture with plasticity [40], while Geelen et al. proposed a phase-field formulation for dynamic cohesive fracture [41]. On the algorithmic side, both implicit and explicit time-integration schemes have been investigated. Implicit Newmark or generalized- α schemes coupled with staggered or monolithic solvers have become standard numerical strategies in the mechanics community [35,36], whereas explicit and sub-stepping strategies for the phase field have been explored to improve efficiency in large-scale dynamic simulations [42]. More recently, comprehensive reviews have summarized the state of the art in phase-field fracture, including dynamic applications, and have highlighted open issues related to efficiency, robustness and model calibration [43–45].

From a computational perspective, phase-field fracture models are notoriously expensive. The introduction of an internal length scale and an additional scalar field requires very fine meshes around the crack and small time steps, which quickly leads to systems with tens or hundreds of millions of unknowns in three dimensions. To mitigate these costs, several high-performance computing (HPC) implementations have been proposed in recent years. Massively parallel MPI frameworks have demonstrated quasi-brittle phase-field simulations on structured 3D meshes with billions of degrees of freedom on modern supercomputers, showing near-linear strong scaling on thousands of cores [46]. Parallel domain-decomposition schemes based on FETI-DP have been developed for dynamic phase-field fracture, combining explicit updates for the displacement field with implicit subdomain solves for the phase field in order to reduce the size of the global coupled problem and the communication overhead [47]. Other works have focused on massively parallel implementations with time adaptivity [48], as well as matrix-free multigrid and higher-order finite element solvers specifically tailored to the structure of the phase-field equations [49,50]. Overall, these contributions demonstrate that phase-field models can exploit contemporary HPC architectures.

Within the ecosystem of the open-source finite element library `deal.II`, parallel phase-field fracture has been advanced by Heister and co-workers. Building on earlier developments of primal–dual active-set strategies and parallel adaptive mesh refinement for phase-field fracture [51,52], the `pfm-cracks` framework provides a fully MPI-parallel, adaptively refined implementation in two and three dimensions [53]. More recently, Munshi et al. have presented a detailed guide and open-source implementation of a hybrid phase-field method for three-dimensional brittle fracture in `deal.II`, emphasizing parallel adaptive mesh refinement, mixed formulations and practical aspects of large-scale simulations in three dimensions [54].

Despite these advances, several challenges remain for dynamic phase-field fracture. First, the enforcement of crack irreversibility is typically handled at the time-discrete level by introducing a history variable defined as the maximum tensile energy attained in the past and updating it via a pointwise max operation [35,36,38]. While simple and effective, this construction obscures the underlying continuous-time structure and makes it difficult to analyze the evolution problem in terms of standard dynamical systems. Second, most computational implementations rely on staggered solution schemes that alternately update the displacement and phase-field variables. These schemes are robust but introduce an additional source of algorithmic dissipation and may slow down crack evolution compared to fully coupled monolithic solvers, especially in strongly inertial regimes. Finally, from the numerical analysis point of view, the coupled elastodynamic/phase-field system has the structure of a differential–algebraic equation (DAE), due to the presence of algebraic constraints associated with irreversibility and history variables.

The need for robust computational strategies for dynamic fracture is further motivated by recent developments in advanced manufacturing and multi-material structural design. For instance, additively manufactured bimetallic metamaterials and bi-material lattice architectures have been proposed to combine thermal-protection, load-bearing, and energy-absorption capabilities under demanding thermo–mechanical conditions [55,56]. In such applications, failure may involve the interaction between material heterogeneity, complex geometries, dynamic loading, and coupled dissipative mechanisms. Although the present work focuses on standard benchmark problems for dynamic phase-field fracture, these examples highlight the importance of numerical schemes that are robust, adaptive, and suitable for fully coupled formulations.

The aim of the present work is to revisit dynamic brittle fracture within the phase-field framework by making this DAE structure explicit and by exploiting it both at the modeling and at the algorithmic level. Starting from the mixed variational formulation of [36], we introduce a formulation in which the displacement field, the phase field s , the associated crack-driving force and the energetic history variable are treated as primary unknowns of a coupled evolution system. The irreversibility condition is expressed as an algebraic constraint, so that the continuous problem can be written as a DAE that preserves the underlying thermodynamic structure of the phase-field fracture model.

Based on this formulation, we develop a monolithic time discretization for dynamic phase-field fracture in which the entire DAE system is integrated implicitly using variable-order, A-stable backward differentiation formulas. This approach, which can be implemented with standard stiff DAE solvers, provides adaptive time stepping with automatic control of local truncation errors and avoids the additional splitting error associated with staggered schemes.

In addition to the modeling and time-integration aspects, a further contribution of this work lies in the flexibility and efficiency of the underlying finite element implementation. The proposed DAE-based phase-field formulation is embedded in a numerical code that supports both small- and large-displacement kinematics, admits mixed finite element formulations (which are particularly relevant for nearly incompressible materials and for extending the model to inelastic constitutive laws), is fully parallelized for distributed-memory architectures, enabling large-scale two- and three-dimensional dynamic fracture simulations with efficient use of modern high-performance computing resources.

The proposed DAE formulation is implemented in a finite element framework and assessed on a set of dynamic benchmarks, including a single-edge-notched specimen in tension, quasi-static compression tests, the classical Kalthoff–Winkler impact test, and an impact problem on a pre-cracked plate with a hole. The numerical results indicate that the DAE-based formulation reproduces the experimentally observed crack patterns and branching angles, while providing improved control of algorithmic dissipation and enhanced robustness with respect to time-step selection.

2. Dynamic formulation of the phase-field approach to fracture with spectral decomposition

Let us consider an arbitrary body $\Omega \in \mathbb{R}^{n_{dim}}$ in the Euclidean space of dimension n_{dim} , in which an evolving internal discontinuity Γ_b is postulated. The position of a material point is denoted by the vector $\mathbf{x}$ in the global Cartesian frame within the bulk, whereas $\mathbf{x}_c$ identifies an arbitrary point of Γ_i , whose boundary is denoted by $\partial\Omega \in \mathbb{R}^{n_{dim}-1}$. Body forces are denoted by $\mathbf{f}_b : \Omega \rightarrow \mathbb{R}^{n_{dim}}$. Kinematic and traction boundary conditions are prescribed along the disjoint parts $\partial\Omega_u \subset \partial\Omega$ and $\partial\Omega_t \subset \partial\Omega$, respectively, with $\partial\Omega_t \cup \partial\Omega_u = \partial\Omega$ and $\partial\Omega_t \cap \partial\Omega_u = \emptyset$, yielding:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \partial\Omega_u \quad \text{and} \quad \bar{\mathbf{t}} = \boldsymbol{\sigma} \cdot \mathbf{n} \quad \text{on } \partial\Omega_t, \quad (1)$$

where $\mathbf{n}$ denotes the outward normal unit vector to the body, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor.

In the dynamic framework, the Lagrangian of the system is defined as [57,58]:

$$L(\mathbf{u}, \dot{\mathbf{u}}, s, t) = \frac{1}{2} \int_{\Omega} \rho |\dot{\mathbf{u}}|^2 d\Omega - \left\{ \int_{\Omega} \psi(\boldsymbol{\epsilon}, s) d\Omega + \int_{\Gamma_f} \mathcal{G}_c d\Gamma - \int_{\partial\Omega_t} \bar{\mathbf{t}} \cdot \delta \mathbf{u} d\Gamma \right\}, \quad (2)$$

where ρ is the material density, $\mathcal{G}_c$ is the fracture energy, ψ is the elastic strain energy density, $\delta \mathbf{u}$ is the virtual displacement field, and $\mathbf{u}$, $\dot{\mathbf{u}}$ are, respectively, the real displacement field and its time derivative.

Introducing the phase field regularization through the crack density functional γ defined as [59]:

$$\gamma(s, \nabla_{\mathbf{x}} s) = \frac{1}{2l} s^2 + \frac{l}{2} |\nabla_{\mathbf{x}} s|^2 \quad (3)$$

where l is the internal length scale for the nonlocal diffusive damage, one obtains:

$$L(\mathbf{u}, \dot{\mathbf{u}}, s, t) = \frac{1}{2} \int_{\Omega} \rho |\dot{\mathbf{u}}|^2 d\Omega - \left\{ \int_{\Omega} \psi(\boldsymbol{\epsilon}, s) d\Omega + \int_{\Omega} \mathcal{G}_c \gamma(s, \nabla_{\mathbf{x}} s) d\Omega - \int_{\partial\Omega_t} \bar{\mathbf{t}} \cdot \delta \mathbf{u} d\Gamma \right\}, \quad (4)$$

Regarding the energy density in the bulk $\psi(\epsilon, s)$, the following positive–negative decomposition is introduced [59–61]:

$$\psi(\epsilon, s) = g(s)\psi_+^e(\epsilon) + \psi_-^e(\epsilon), \quad (5a)$$

$$\psi_+^e(\epsilon) = \frac{\lambda}{2} (\langle \text{tr}[\epsilon] \rangle_+)^2 + \mu \text{tr}[\epsilon_+^2], \quad (5b)$$

$$\psi_-^e(\epsilon) = \frac{\lambda}{2} (\langle \text{tr}[\epsilon] \rangle_-)^2 + \mu \text{tr}[\epsilon_-^2], \quad (5c)$$

where λ and μ are the Lamé constants, $\text{tr}[\bullet]$ denotes the trace operator, and $g(s)$ is a degradation function that takes the form [59]

$$g(s) = (1 - s)^2 + k, \quad (6)$$

k being a parameter that defines a residual stiffness to prevent numerical instabilities in the computational implementation, and simultaneously preventing that the resulting system of equations becomes ill-conditioned. In the above equations, the decomposition of the strain tensor into its positive and negative counterparts, $\epsilon = \epsilon_+ + \epsilon_-$, is exploited in order to account for damage under tensile loading only. The spectral decomposition of the positive part of the strain tensor reads $\epsilon_+ = \sum_{i=1}^{n_{dim}} \langle \epsilon^i \rangle_+ \mathbf{n}_\epsilon^i \otimes \mathbf{n}_\epsilon^i$, where ϵ^i and $\mathbf{n}_\epsilon^i$ identify the eigenvalues and the eigenvectors of the strain tensor and $\langle \bullet \rangle_+ = (\bullet + |\bullet|)/2$.

Relying on standard arguments [62], the Cauchy stress tensor is defined as:

$$\sigma := \frac{\partial \hat{\psi}}{\partial \epsilon} = g(s)\sigma_+ + \sigma_-; \quad \text{with } \sigma_\pm = \lambda (\langle \text{tr}[\epsilon] \rangle_\pm) \mathbf{1} + 2\mu \epsilon_\pm, \quad (7)$$

where $\mathbf{1}$ denotes the identity tensor.

Whenever it is needed, we augment the formulation with purely dissipative densities, one acting on the mechanical momentum balance and one on the phase-field evolution:

$$\mathcal{L}_{\text{diss}}^u(\dot{\mathbf{u}}) = \frac{1}{2} \dot{\mathbf{u}} \cdot \mathbb{C}_d \dot{\mathbf{u}}, \quad \mathcal{L}_{\text{diss}}^s(\dot{s}) = \frac{1}{2} \eta_s \dot{s}^2, \quad (8)$$

where $\mathbb{C}_d = \alpha_M \mathbf{M} + \beta_K \mathbf{K}$ is a Rayleigh-type damping operator and $\eta_s > 0$ is a small phase-field viscosity. Their contribution to the balance is through the generalized dissipative forces $\partial_{\dot{\mathbf{u}}} \mathcal{L}_{\text{diss}}^u = \mathbb{C}_d \dot{\mathbf{u}}$ and $\partial_{\dot{s}} \mathcal{L}_{\text{diss}}^s = \eta_s \dot{s}$, yielding the additional terms $\mathbb{C}_d \dot{\mathbf{u}}$ in the momentum equation and $\eta_s \dot{s}$ in the microforce balance. The associated (nonnegative) dissipated power reads

$$\mathcal{D}(t) = \int_{\Omega} (\dot{\mathbf{u}} \cdot \mathbb{C}_d \dot{\mathbf{u}} + \eta_s \dot{s}^2) d\Omega \geq 0. \quad (9)$$

Introducing $\mathcal{L}_{\text{diss}}^u$ and $\mathcal{L}_{\text{diss}}^s$ makes the model more general—able to represent light modal damping in mechanics and viscous regularization in the phase field—and improves the numerical behavior of the monolithic BDF solver used for the time integration. We refer the reader to Section 5 for these details.

This formulation yields the following Euler–Lagrange equation associated with the functional (2), which governs the motion of the body in the dynamic regime:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} + \alpha_M \dot{\mathbf{u}} - \nabla \cdot (\sigma + \beta_K \mathbb{C}_d : \dot{\epsilon}(\mathbf{u})) = \mathbf{0}, \quad \text{on } \Omega \times]0, T[, \quad (10a)$$

$$\eta_s \dot{s} + \left(\frac{2l\psi^+}{G_c} + 1 \right) s - l^2 \nabla^2 s = \frac{2l\psi^+}{G_c}, \quad \text{on } \Omega \times]0, T[\quad (10b)$$

with the initial condition $\mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot)$ on Ω , and boundary conditions according to (1). Introducing $H = 2l\psi^+/G_c$, one has:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} + \alpha_M \dot{\mathbf{u}} - \nabla \cdot (\sigma + \beta_K \mathbb{C}_{\text{damp}} : \dot{\epsilon}(\mathbf{u})) = \mathbf{0}, \quad \text{on } \Omega \times]0, T[, \quad (11a)$$

$$\eta_s \dot{s} + (H + 1) s - l^2 \nabla^2 s = H, \quad \text{on } \Omega \times]0, T[\quad (11b)$$

By introducing the additional vectorial variable $\mathbf{v} = \dot{\mathbf{u}}$, the second-order system (10a) becomes a first order system of PDEs as follows:

$$\begin{cases} \rho \dot{\mathbf{v}} + \alpha_M \mathbf{v} - \nabla \cdot (\sigma + \alpha_K \mathbb{C}_{\text{damp}} : \dot{\epsilon}(\mathbf{v})) &= \mathbf{0} & \text{in } \Omega \times]0, T[, \\ \mathbf{v} &= \dot{\mathbf{u}} & \text{in } \Omega \times]0, T[, \\ \eta_s \dot{s} + (H + 1) s - l^2 \nabla^2 s &= H & \text{in } \Omega \times]0, T[, \end{cases} \quad (12)$$

with the initial condition $\mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot)$ on Ω , $\mathbf{v}(\cdot, 0) = \mathbf{v}_0(\cdot) = \dot{\mathbf{u}}_0(\cdot)$ on Ω and boundary conditions according to (1).

We remark that the introduction of the additional variable $\mathbf{v}$ has the effect of reducing the time derivative order of the system. Thus, the increase in the system unknowns is compensated by a lower order time derivative. As will be shown in the next sections, the main advantage of this approach is that the FEM spatial discretization of the problem will result in a DAE or ODE system written in canonical form. This clearly opens up the opportunity to leverage on several algorithms — implemented in open source software — for the resolution of ODE and DAE system available in the literature, so as to investigate on time advancing schemes which are alternative to the ones commonly used in the phase field community.

3. Weak form and finite element discretization

The following weak form of the system associated to (12) is derived:

$$\begin{cases} \int_{\Omega} \rho \dot{\mathbf{v}} \cdot \boldsymbol{\phi} \, dx + \int_{\Omega} \alpha_M \mathbf{v} \cdot \boldsymbol{\phi} \, dx + \int_{\Omega} \left(\boldsymbol{\sigma} + \alpha_K \mathbb{C}_{\text{damp}} : \boldsymbol{\varepsilon}(\mathbf{v}) \right) : \boldsymbol{\varepsilon}(\boldsymbol{\phi}) \, d\Omega &= \int_{\partial\Omega} \bar{\mathbf{t}} \cdot \boldsymbol{\phi} \, d\Gamma + \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\phi} \, d\Omega \\ \int_{\Omega} \mathbf{v} \cdot \boldsymbol{\phi} \, d\Omega &= \int_{\Omega} \dot{\mathbf{u}} \cdot \boldsymbol{\phi} \, d\Omega \\ \int_{\Omega} \eta_s \dot{s} \cdot \psi \, d\Omega + \int_{\Omega} (H+1) s \psi \, d\Omega + \int_{\Omega} l^2 \nabla s \cdot \nabla \psi \, d\Omega &= \int_{\Omega} H \psi \, d\Omega \end{cases} \quad (13)$$

where a body force term $\mathbf{f}$ has also been introduced. System (13) must hold for all test functions $\boldsymbol{\phi} \in \mathbf{H}_0^1(\Omega, \mathbb{R}^3)$ and $\psi \in \mathbf{H}_0^1(\Omega, \mathbb{R})$.

The finite element discretization method is used to discretize System (13) over space. The time discretization is carried out instead by means of an implicit Backward Difference Formula (BDF) scheme. As such, a monolithic scheme is used to treat the resulting set of nonlinear algebraic equations to be solved at each time step. The numerical scheme described has been implemented making use of the functions of the deal.II open source finite element library [63,64] to achieve a high performance computing framework that uses parallel computing and distributed meshes.

Regarding the spatial discretization, we introduce a space decomposition $\hat{\Omega}$ that is the *reference domain*. Such partition is composed of quadrilateral cells in 2D simulations, or hexahedral cells 3D simulations. We denote with K a generic cell in $\hat{\Omega}$ and $K(t)$ with $t \in \mathbb{R}$ the *deformed cell* at time $t \in \mathbb{R}$ according to the notation in Appendix A.

We denote with $\hat{\Omega}(t)$ the *deformed geometry* at time t . The map that associates each point in $\hat{\Omega}$ to a point in $\hat{\Omega}(t)$ is the map χ defined in (A.1).

On the decomposition $\hat{\Omega}(t)$ at a fixed time $t \in \mathbb{R}$, we look for solutions $(\mathbf{u}_h, \mathbf{v}_h, s_h)$ and $(\dot{\mathbf{u}}_h, \dot{\mathbf{v}}_h, \dot{s}_h)$ in the finite dimensional spaces $U_h, V_h, S_h, \dot{U}_h, \dot{V}_h$ and $\dot{S}_h$ defined as

$$U_h := \left\{ \mathbf{u}_h \in C^1(\hat{\Omega}(t))^3 \mid \mathbf{u}_h|_{K(t)} \in \mathcal{P}_U(K(t))^3, K(t) \in \hat{\Omega}(t) \right\} \equiv \text{span}\{\mathbf{u}_h^i\}_{i=1}^{N_U}, \quad (14a)$$

$$V_h := \left\{ \mathbf{v}_h \in C^1(\hat{\Omega}(t))^3 \mid \mathbf{v}_h|_{K(t)} \in \mathcal{P}_V(K(t))^3, K(t) \in \hat{\Omega}(t) \right\} \equiv \text{span}\{\mathbf{v}_h^i\}_{i=1}^{N_V}, \quad (14b)$$

$$S_h := \left\{ s_h \in C^1(\hat{\Omega}(t)) \mid s_h|_{K(t)} \in \mathcal{P}_S(K(t)), K(t) \in \hat{\Omega}(t) \right\} \equiv \text{span}\{s_h^i\}_{i=1}^{N_S}. \quad (14c)$$

$$\dot{U}_h := \left\{ \dot{\mathbf{u}}_h \in C^1(\hat{\Omega}(t))^3 \mid \dot{\mathbf{u}}_h|_{K(t)} \in \mathcal{P}_{\dot{U}}(K(t))^3, K(t) \in \hat{\Omega}(t) \right\} \equiv \text{span}\{\dot{\mathbf{u}}_h^i\}_{i=1}^{N_{\dot{U}}}, \quad (14d)$$

$$\dot{V}_h := \left\{ \dot{\mathbf{v}}_h \in C^1(\hat{\Omega}(t))^3 \mid \dot{\mathbf{v}}_h|_{K(t)} \in \mathcal{P}_{\dot{V}}(K(t))^3, K(t) \in \hat{\Omega}(t) \right\} \equiv \text{span}\{\dot{\mathbf{v}}_h^i\}_{i=1}^{N_{\dot{V}}}, \quad (14e)$$

$$\dot{S}_h := \left\{ \dot{s}_h \in C^1(\hat{\Omega}(t)) \mid \dot{s}_h|_{K(t)} \in \mathcal{P}_{\dot{S}}(K(t)), K(t) \in \hat{\Omega}(t) \right\} \equiv \text{span}\{\dot{s}_h^i\}_{i=1}^{N_{\dot{S}}}. \quad (14f)$$

Here, $C^1(\hat{\Omega}(t))^d$ is the space of differentiable functions of d components over $\hat{\Omega}(t)$. $\mathcal{P}_U(K)$, $\mathcal{P}_V(K)$, $\mathcal{P}_S(K)$, $\mathcal{P}_{\dot{U}}(K)$, $\mathcal{P}_{\dot{V}}(K)$ and $\mathcal{P}_{\dot{S}}(K)$ indicate the polynomial spaces of degree r_U , r_V , r_S , $r_{\dot{U}}$, $r_{\dot{V}}$ and $r_{\dot{S}}$ respectively, defined on the cells $K(t)$. Finally, N_U , N_V , N_S , $N_{\dot{U}}$, $N_{\dot{V}}$ and $N_{\dot{S}}$ denote the dimensions of each finite dimensional space.

Using the *iso-parametric formulation* representation (see Appendix A for details), in which the same parametrization is used to describe both the surface geometry and the physical variables, we represent the solution via a linear combination in the finite dimensional spaces $U_h, V_h, S_h, \dot{U}_h, \dot{V}_h$ and $\dot{S}_h$.

In this way we can formulate the problem (13) in the following form:

$$\begin{aligned} \int_{\hat{\Omega}(t)} \rho \left(\sum_{i=1}^{N_V} a_i^{(v)} \mathbf{v}_h^i \right) \cdot \boldsymbol{\phi} \, d\Omega + \int_{\hat{\Omega}(t)} \alpha_M \left(\sum_{i=1}^{N_V} a_i^{(v)} \mathbf{v}_h^i \right) \cdot \boldsymbol{\phi} \, d\Omega \\ + \int_{\hat{\Omega}(t)} \boldsymbol{\sigma} \left(\sum_{i=1}^{N_U} a_i^{(u)} \mathbf{u}_h^i \right) : \boldsymbol{\varepsilon}(\boldsymbol{\phi}) \, d\Omega \\ + \int_{\hat{\Omega}(t)} \alpha_K \left(\mathbb{C}_{\text{damp}} : \boldsymbol{\varepsilon} \left(\sum_{i=1}^{N_V} a_i^{(v)} \mathbf{v}_h^i \right) \right) : \boldsymbol{\varepsilon}(\boldsymbol{\phi}) \, d\Omega \\ = \int_{\partial\hat{\Omega}(t)} \bar{\mathbf{t}} \cdot \boldsymbol{\phi} \, d\Gamma + \int_{\hat{\Omega}(t)} \mathbf{f} \cdot \boldsymbol{\phi} \, d\Omega, \end{aligned} \quad (15a)$$

$$\int_{\hat{\Omega}(t)} \left(\sum_{i=1}^{N_V} a_i^{(v)} \mathbf{v}_h^i \right) \cdot \boldsymbol{\phi} \, d\Omega = \int_{\hat{\Omega}(t)} \left(\sum_{i=1}^{N_{\dot{U}}} a_i^{(\dot{u})} \dot{\mathbf{u}}_h^i \right) \cdot \boldsymbol{\phi} \, d\Omega, \quad (15b)$$

$$\begin{aligned} \int_{\hat{\Omega}(t)} \eta_s \left(\sum_{i=1}^{N_S} a_i^{(s)} s_h^i \right) \psi \, d\Omega + \int_{\hat{\Omega}(t)} \ell^2 \left(\sum_{i=1}^{N_S} a_i^{(s)} \nabla s_h^i \right) \cdot \nabla \psi \, d\Omega \\ + \int_{\hat{\Omega}(t)} (H+1) \left(\sum_{i=1}^{N_S} a_i^{(s)} s_h^i \right) \psi \, d\Omega = \int_{\hat{\Omega}(t)} H \psi \, d\Omega. \end{aligned} \quad (15c)$$

for all test functions $\phi \in \mathbf{H}_0^1(\Omega, \mathbb{R}^3)$ and $\psi \in \mathbf{H}_0^1(\Omega, \mathbb{R})$.

We notice that the problem (13) formulated with the discretization (15) is a system of non-linear Differential–Algebraic equations.

All the equations associated with the momentum equation and from the velocity $\mathbf{v}$ result in ODEs. On the other hand, all the equations associated with Dirichlet boundary conditions do not include time derivatives. In addition, whenever the damping terms associated with the phase field variable are not included in the formulation, the discretized phase field equation is not including time derivatives.

Proceeding with the standard finite element procedure, we can select the test functions to be the same shape functions for the solution. In this way, we derive a spatially discretized system to solve in order to get the discrete representation $(\mathbf{u}_h, \mathbf{v}_h, s_h)$ and $(\dot{\mathbf{u}}_h, \dot{\mathbf{v}}_h, \dot{s}_h)$.

In this way, we obtain the following non linear system of equation

$$\begin{cases} \rho M \dot{\mathbf{v}}_h + \alpha_m \rho M \mathbf{v}_h + K(\mathbf{u}_h, s_h) \mathbf{u}_h + \alpha_K K_{\text{damp}} \mathbf{v}_h &= \mathbf{0} \\ M \mathbf{v}_h &= M \dot{\mathbf{u}}_h \\ \eta M \dot{s}_h + M(H) s_h + l^2 K_s s_h &= f_H \end{cases}$$

where M is the mass matrix, $K(\mathbf{u}_h, s_h)$ is the stiffness matrix coming from the (7), K_{damp} is the stiffness matrix of the elastic damping term, $M(H)$ is the mass term with also the contribution of the coefficient $(H+1)$, K_s is the stiffness matrix associated to the s variable and f_H is the term on the right-hand side of the phase field variable equation.

3.1. BDF time discretization scheme

System of Eqs. (15) can be recast in the following form

$$F(t, \mathbf{y}, \dot{\mathbf{y}}) = 0,$$

in which

$$\mathbf{y} = \begin{bmatrix} \mathbf{u}_h \\ \mathbf{v}_h \\ s_h \end{bmatrix} \quad \text{and} \quad \dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{u}}_h \\ \dot{\mathbf{v}}_h \\ \dot{s}_h \end{bmatrix}.$$

This equation is a system of non-linear Differential–Algebraic equations (DAE), which we solve using the IDA package of the SUNDIALS OpenSource library [65]. This package exploits a Backward Difference Formula (BDF) for the time stepping scheme. Specifically, the integration method in IDA is of variable-order, variable-coefficient BDF, in fixed-leading-coefficient form. The method order ranges from 1 to 5, with the BDF of order q given by the multistep formula:

$$\sum_{i=0}^q \alpha_{n,i} \mathbf{y}_{n-i} = h_n \dot{\mathbf{y}}_n \quad (16)$$

where:

- $h_n = t_n - t_{n-1}$ is the time step.
- $\alpha_{n,i}$ are coefficients determined by the order q and step size history.
- $\mathbf{y}_n$ and $\dot{\mathbf{y}}_n$ are the solution and its time derivative at t_n .

Applying the BDF formula to the DAE, this results in a nonlinear system to solve at each time step:

$$\mathbf{R}(\mathbf{y}_n) \equiv \mathbf{F} \left(t_n, \mathbf{y}_n, h_n^{-1} \sum_{i=0}^q \alpha_{n,i} \mathbf{y}_{n-i} \right) = \mathbf{0} \quad (17)$$

which is done via the Newton–Raphson iterative-incremental algorithm. For a frozen time, at each iteration step one has:

$$\mathbf{J}(\mathbf{y}_{n,m+1} - \mathbf{y}_{n,m}) = -\mathbf{R}(\mathbf{y}_{n,m}) \quad (18)$$

where:

- $\mathbf{y}_{n,m}$ is the m th Newton iterate at step n .
- $\mathbf{J}$ is the Jacobian approximation of the residual function.

The Jacobian $\mathbf{J}$ used in Newton iteration is given by:

$$\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{y}} + \alpha \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{y}}}, \quad \text{with} \quad \alpha = \frac{\alpha_{n,0}}{h_n}. \quad (19)$$

This accounts for the DAE's dependence on both $\mathbf{y}$ and $\dot{\mathbf{y}}$, as reflected in the residual vectorial function $\mathbf{R}$.

In our implementation, we used automatic differentiation available in the Sacado package in TRILINOS library [66] to compute the Jacobian $\mathbf{J}$ associated to our formulation.

4. Numerical results

To assess the range of validity of our formulation and its time-integration strategy, we deliberately selected a mix of *quasi-static* and *dynamic* fracture benchmarks. The quasi-static cases (e.g., single-edge-notched tension, perforated plate in compression) probe the model's ability to reproduce load–displacement responses, crack paths, and irreversibility under slow driving. The dynamic cases (e.g., crack propagation from a hole/pre-notch, impact-driven fracture) challenge the method in the presence of high elastic waves and fast transients. Across all tests we keep material and phase-field parameters consistent and use light mechanical damping and a small phase-field viscosity when necessary.

This design allows us to examine the *correlation* between model predictions and reference data over a broad spectrum of operating conditions: from stiffness-dominated, slowly evolving cracks to rapid sub-Rayleigh fronts. Consistent agreement across these qualitatively different regimes indicates that the proposed model/numerical pipeline captures fracture phenomena with a monolithic setup.

4.1. Single edge notched tension

As a first benchmark, we consider a single-edge-notched specimen subjected to monotonic tension, see Fig. 1(a).

The specimen is a square plate with side length 1 mm and thickness 0.1 mm. A sharp, centrally located horizontal crack of length 0.5 mm is pre-inserted. All degrees of freedom along the bottom edge are constrained, while a vertical displacement $\bar{u}(t)$ is prescribed on the top edge (lateral contraction is left free). The lateral edges are traction-free.

The elastic response is characterized by the Lamé constants $\lambda = 121.15 \times 10^3 \text{ N/mm}^2$, $\mu = 80.77 \times 10^3 \text{ N/mm}^2$, and density $\rho = 7.7 \times 10^{-6} \text{ kg/mm}^3$. We employed light damping only in the 2D case (see *Numerical considerations* for the stability comparison between 2D and 3D): Rayleigh coefficients $\alpha_M = 0.5 \text{ s}^{-1}$ and $\beta_K = 0.05 \text{ s}$ in the momentum balance, and a small phase-field viscosity $\eta_s = 10^{-6}$. These values were chosen to filter high-frequency modes without altering the peak load.

Fracture parameters are $G_c = 2.7 \text{ N/mm}$ and internal length $l = 0.02 \text{ mm}$.

A displacement-controlled loading is adopted [59]: the prescribed displacement is advanced by fixed increments of $1 \times 10^{-5} \text{ mm}$ per μs for the first 500 μs and then reduced to $1 \times 10^{-6} \text{ mm}$ to accurately resolve crack initiation and post-peak softening. the corresponding boundary condition for $\mathbf{v}$ is obtained by time differentiation. In particular, the velocity is set to $1 \times 10^{-5} \text{ mm}/\mu\text{s}$ during the first 500 μs , and is subsequently reduced to $1 \times 10^{-6} \text{ mm}/\mu\text{s}$.

We point out that in this test, which is substantially two dimensional, we exploited the generality of the solver implemented to run both 2D and 3D simulations. In particular, 2D simulation result in lower computational cost and are naturally suited to reproduce plane strain conditions. Instead, the 3D simulations are generally more time consuming, but can be used to reproduce in a rather straightforward way plane stress loading conditions.

The results of both 2D (plane strain) and 3D (plane stress) simulations for the single notch test are presented in Figs. 1(b) and 1(c). The contour plot of the phase field variable s obtained with the 3D simulation suggests that the solver is able to reproduce the correct path of the crack. To provide a more quantitative assessment of the solution accuracy, in the diagram the computed stress–elongation response (BDF, 2D/3D) is compared with the quasi-static reference from [59].

As expected the 2D BDF solution closely reproduces the reference solution by Miehe et al. [59]. The plane-strain response obtained in this work appears in fact correct in the elastic range. Also the peak load and post-peak softening are close to the reference values. In contrast, the 3D curve exhibits a lower elastic slope, which is consistent with the plane-stress conditions imposed by means of stress-free top/bottom surfaces over the 0.1 mm thickness of the plate. The 3D curve also attains the peak at a slightly larger strain. Also in this case, this is an expected consequence of the increased compliance under plane stress.

Despite being formulated and solved in a dynamic setting, the proposed approach remains computationally more efficient than a staggered quasi-static strategy, owing to the use of adaptive time stepping; a detailed cost comparison is provided in 5.

4.2. One-hole compression tests

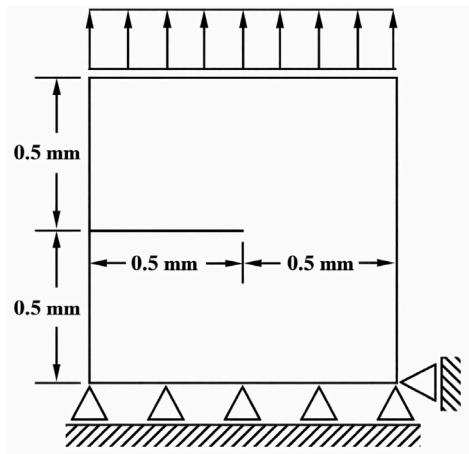
As a second benchmark, we consider a rectangular specimen with a single circular hole subjected to monotonic compression, as depicted in Fig. 2(a).

The plate measures 39 mm $\times$ 48 mm $\times$ 10 mm and contains a single circular hole, see Fig. 2(a). Short sharp pre-notches are introduced at the locations where vertical crack growth is expected, to provide a well-defined initiation site. All degrees of freedom on the bottom edge are fixed in the vertical direction, while a vertical displacement $\bar{u}(t)$ is prescribed on the top edge; the lateral edges are traction-free. Since the prescribed vertical displacement $\bar{u}(t)$ at the top edge is negative, this configuration is designed to produce a uniaxial compression state in the y -direction.

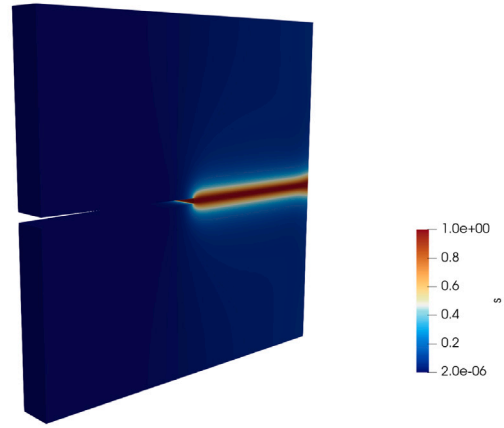
The material parameters are given by the Lamé constants $\lambda = 1188 \text{ N/mm}^2$, $\mu = 1056.54 \text{ N/mm}^2$ (equivalently $E \approx 2944 \text{ MPa}$, $\nu \approx 0.351$), and density $\rho = 1.2 \times 10^{-6} \text{ kg/mm}^3$.

In the present test case, we specified null damping coefficients in both the displacement and phase field equation. In fact, the fracture evolution is here quite slow, which results in a relatively small impact of the damping forces on the solution. As for the remaining fracture parameters, we set $G_c = 0.36 \text{ N/mm}$ and internal length $l = 0.45 \text{ mm}$.

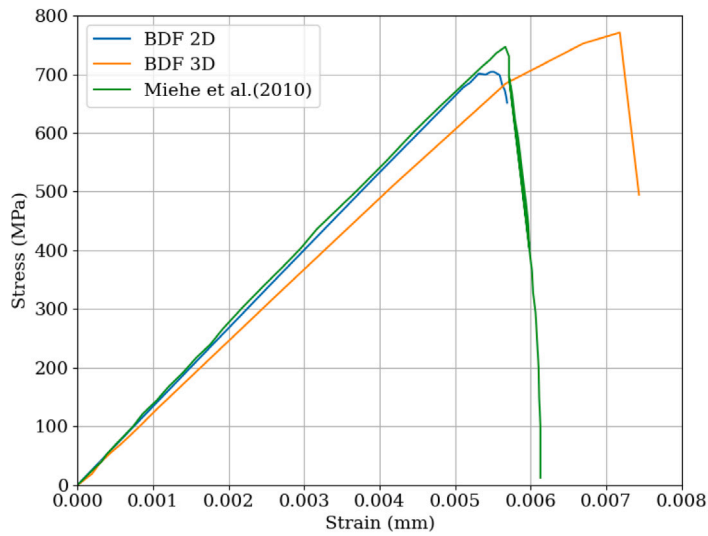
Compression is applied in displacement control, advancing the prescribed displacement by fixed increments $\Delta u = 2.5 \times 10^{-4} \text{ mm}$ per μs , which allows a gradual evolution of the damage field and a stable tracking of crack propagation from the vertical pre-cracks.



a Sketch of the single-edge-notched tension test.



b 3D simulation snapshot.



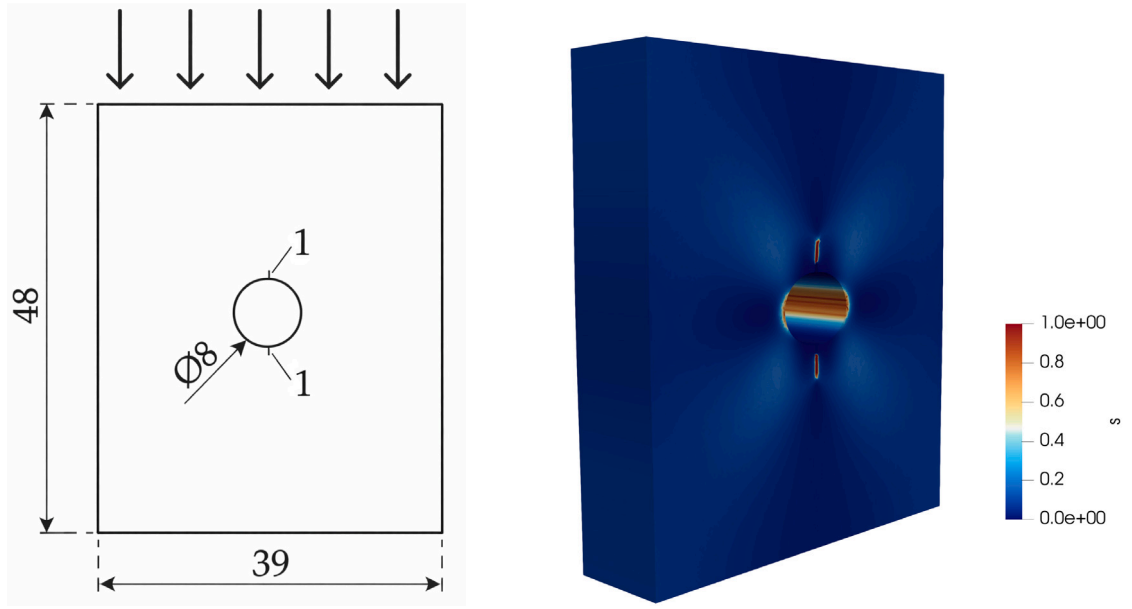
c Stress-strain response.

Fig. 1. Results for the single-edge-notched tension test. (a) Sketch of the notched specimen. (b) Phase-field variable s obtained from the three-dimensional plane-stress simulation. (c) Stress–elongation response. The yellow curve refers to the 3D plane-stress simulation, the blue curve to the 2D plane-strain simulation, and the green curve reports the reference result from Miehe et al. [59].

As for the previous test, the corresponding boundary condition for $\mathbf{v}$ is obtained by time differentiation. In particular, the loading rate is set to 2.5×10^{-4} mm/ μ s.

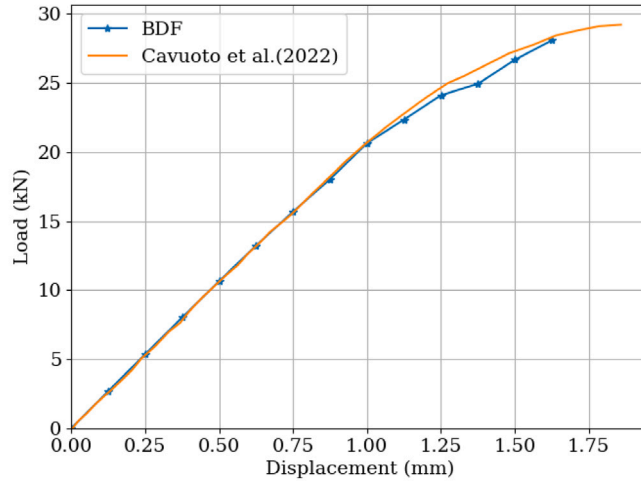
Fig. 2(c) shows a satisfactory agreement between the BDF prediction and the experimental curve by Cavuoto et al. [67]. In particular, the elastic slope is clearly matching the measured one, and the gradual hardening is closely following the experiments, until small discrepancies are observed around the peak values. Nevertheless, the agreement remains overall good, indicating that the proposed formulation reproduces with good accuracy the global load–displacement response up to the onset of the final stage of failure.

As for the previous test, the proposed monolithic dynamic formulation is computationally cheaper than a staggered quasi-static treatment for the present problem 5.



a Geometry of the one-circular-hole specimen.

b Simulation snapshot.

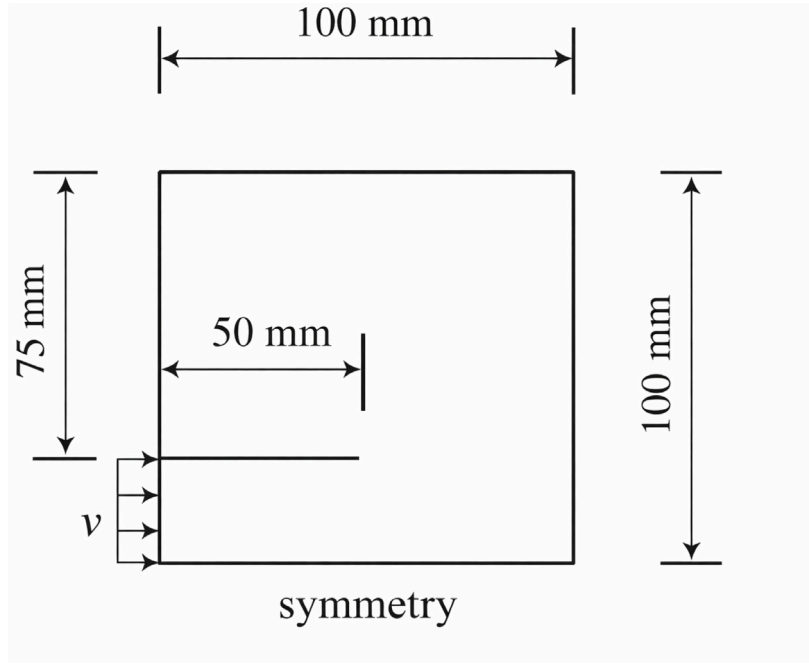


c Nominal load-displacement response.

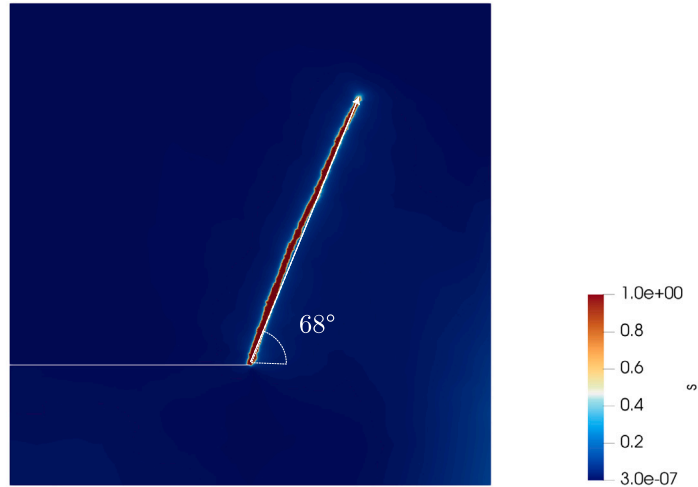
Fig. 2. One-hole compression test. (a) Geometry of the specimen, with dimensions reported in mm. (b) Phase-field variable s near the hole, highlighting crack initiation and vertical propagation. (c) Global nominal load-displacement response under displacement-controlled compression. The BDF solution closely follows the experimental reference of Cavuoto et al. [67] over the loading history.

4.3. Projectile impact on a pre-cracked plate

We adopt the Kalthoff-type impact set-up [6]: a pre-notched plate is modeled using symmetry to halve the numerical domain. Loading is applied kinematically by prescribing a velocity pulse on the impacted edge, while all other boundaries are traction-free; the initial crack is represented as a sharp geometric discontinuity at the notch tip. In the experiment the projectile speed is 33 m/s; in the symmetric model the prescribed velocity is ramped from 0 to $v_0 = 16.5$ m/s in $t_0 = 1$ μ s and then held constant, i.e. $v(t) = \frac{t}{t_0} v_0$ for $t \leq t_0$ and $v(t) = v_0$ thereafter.



a Kalthoff-type impact domain.



b Kalthoff-type impact result.

Fig. 3. Kalthoff-type impact benchmark. (a) Computational domain and initial notch configuration. (b) Phase-field result: the crack initiates at the lower pre-crack tip and kinks upward, forming an oblique damage band with an angle of approximately 68° with respect to the horizontal axis.

We consider a square plate of side 100 mm. An initial vertical crack is pre-inserted along the mid-line. The projectile strikes the lower edge over a 25 mm-wide segment, while the remaining boundaries are traction-free. We report the domain of the simulation in Fig. 3(a).

The elastic solid is characterized by the parameters $\rho = 8000 \text{ kg/m}^3$, $E = 190 \text{ GPa}$, $\nu = 0.3$ and toughness $G_c = 2.213 \times 10^4 \text{ J/m}^2$ is used ($k = 0$).

We imposed a small phase-field viscosity $\eta_s = 10^{-3}$ for numerical stabilization.

The numerical results show a distinctly transient fracture regime in which crack evolution is driven by the coupling between the pre-existing defect, the impact-induced stress waves, and the redistribution of elastic energy during propagation. After a short

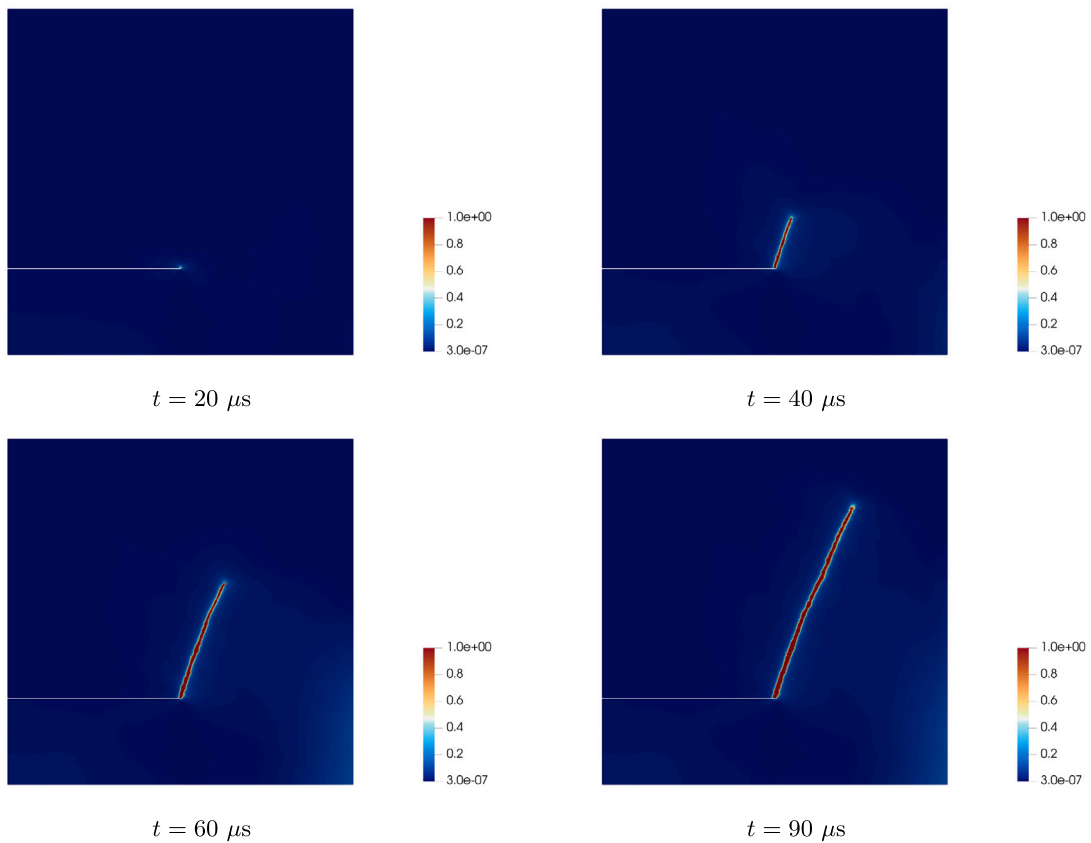


Fig. 4. Evolution of the fracture phase field at representative times under impact loading.

incubation stage, crack growth is activated at the notch tip and rapidly departs from the initial horizontal discontinuity, selecting an oblique path through the specimen. This behavior is indicative of a dynamic mixed-mode propagation mechanism, in which the crack direction is not prescribed by the initial geometry alone, but emerges from the instantaneous near-tip stress state generated by the propagating wave field.

The wave-field plots show that the mechanical disturbance generated by impact is initially localized around the notch, but progressively spreads across the domain as the fracture evolves. At the same time, the propagating crack acts as a secondary source of wave emission and scattering, producing a marked reorganization of the displacement field on both sides of the discontinuity. The observed response is thus governed by a two-way interaction: the incident and reflected waves trigger fracture advance, while the advancing crack alters the subsequent wave pattern by changing the local compliance and by introducing an evolving kinematic discontinuity. This is a characteristic signature of dynamic brittle failure and cannot be reduced to a quasi-static extension mechanism.

Overall, the computed response is consistent with an impact-driven dynamic fracture process. The results indicate that the adopted formulation captures the essential ingredients of the problem and in particular the strong coupling between fracture evolution and transient wave propagation. In this sense, the simulation provides evidence that the model is able to reproduce not only the onset of failure, but also the mechanically relevant structure of the fully dynamic propagation regime (see Figs. 4 and 5).

Additionally, the energetic histories depicted in Fig. 6 provide additional insight into the qualitative differences between the proposed fully coupled BDF formulation and results presented in [38], that have been obtained with a staggered numerical scheme. Both approaches reproduce the same overall scenario, but with some meaningful observations.

A first observation concerns the elastic strain energy. In both cases, the strain energy increases during the early stage of the simulation, which corresponds to the build-up of stresses induced by the impact and to the progressive concentration of energy near the crack tip. However, the BDF solution consistently attains higher energy levels and maintains them for a longer portion of the transient response. By contrast, the compared staggered scheme exhibits a uniformly lower elastic energy profile, both during the loading stage and in the subsequent post-peak regime. This difference is not merely quantitative: it is consistent with the fact that the staggered operator split introduces an additional source of algorithmic dissipation, which is absent, or at least significantly reduced, in the monolithic time integration.

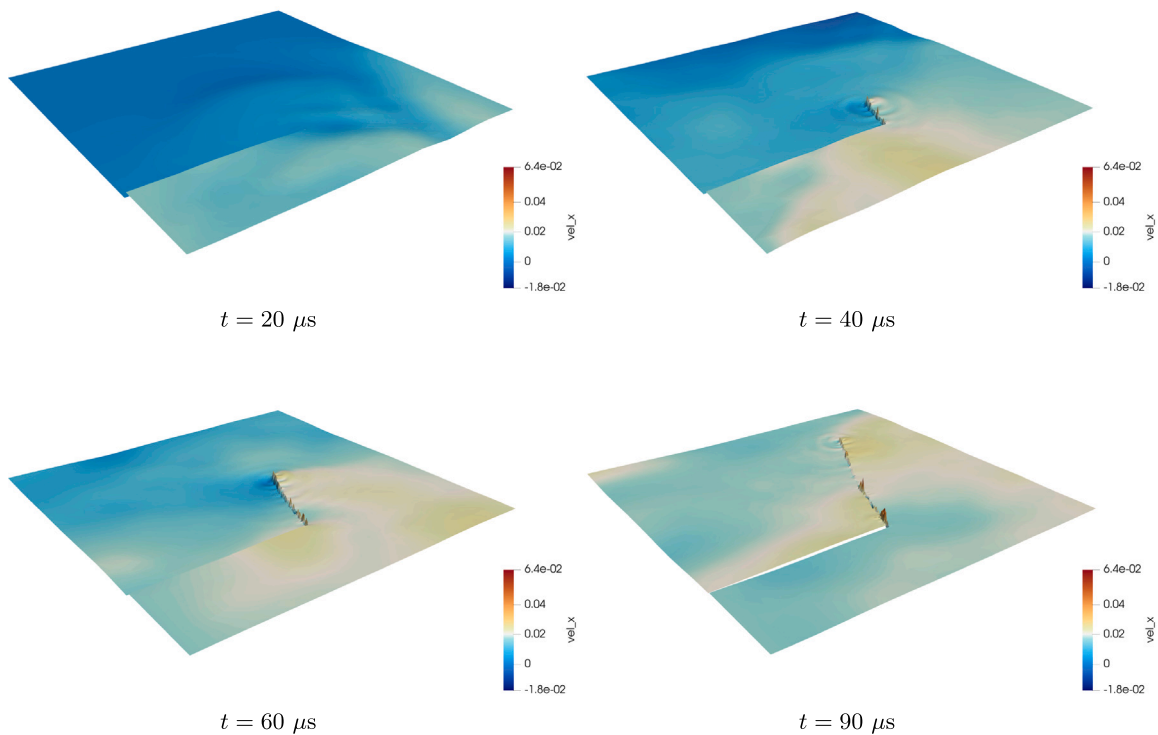


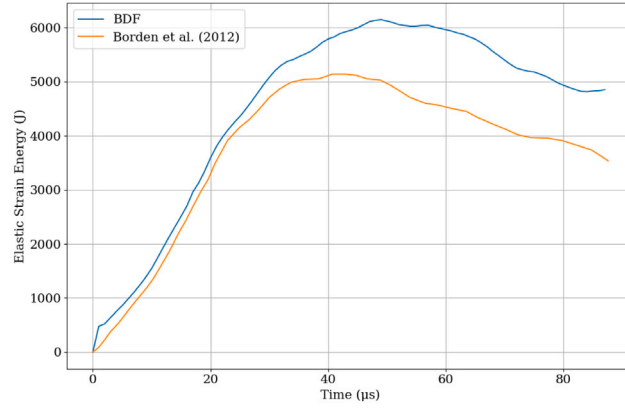
Fig. 5. Corresponding evolution of the deformed velocity field on the x-direction, highlighting the interaction between transient waves and crack propagation. The corresponding animation, showing the evolution of the horizontal velocity field v_x , is available at: <https://youtu.be/CsOjc3OYtoQ>.

From this viewpoint, the larger elastic energy predicted by the BDF formulation should not be interpreted as an overestimation, but rather as the consequence of a more faithful retention of the mechanical energy carried by the transient field. Since the monolithic discretization advances the coupled displacement–phase-field system in a single implicit step, it avoids the artificial energy loss associated with sequential updates of the subproblems. The staggered strategy, on the other hand, solves the mechanics and fracture fields in a partitioned fashion, and this splitting could perturb the discrete energy balance. The lower strain–energy curve observed for the staggered scheme is therefore compatible with a numerical mechanism in which part of the available mechanical energy is dissipated at the algorithmic level rather than being fully represented in the physical response.

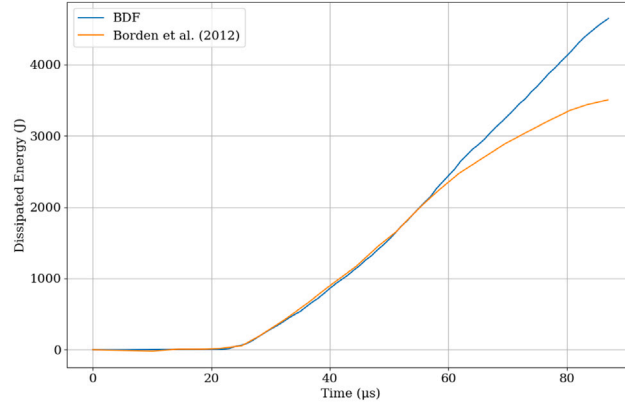
This interpretation is reinforced by the dissipated-energy history. The two curves remain close during the initial stage, when fracture is still weak and the response is dominated by elastic storage. Once crack evolution becomes significant, the BDF formulation produces a larger cumulative dissipated energy, indicating a more pronounced development of the fracture process. In contrast, the staggered scheme yields a smaller dissipation level at late times. Read together with the strain–energy plot, this behavior suggests that the monolithic formulation preserves more of the energy injected into the system and subsequently converts a larger fraction of it into physically meaningful fracture dissipation. The staggered method instead appears to lose part of this energy through the splitting error, thereby reducing both the stored elastic energy and the final fracture-related dissipation.

An important consequence is that the discrepancy between the two formulations should be interpreted in terms of energetic consistency rather than only solution amplitude. The BDF method exhibits a response that is globally more energetic because it better captures the transfer between bulk elastic storage and fracture dissipation. The staggered scheme, by construction, introduces a bias toward a more dissipative discrete dynamics, which tends to smooth the response and lower the energy content available to drive crack growth. In the present impact setting, where wave propagation and fracture evolution are strongly coupled, this distinction is particularly relevant, since even moderate artificial dissipation may significantly alter the transient redistribution of energy and, in turn, the resulting crack evolution.

Overall, the results indicate that the monolithic BDF discretization provides a clearer representation of the energetic structure of the fully dynamic fracture problem. In particular, the higher elastic energy levels and the larger accumulated dissipated energy are consistent with a reduced amount of algorithmic dissipation and with a more direct description of the exchange between reversible storage and irreversible fracture processes. At the same time, the staggered formulation remains a valuable and robust alternative, also because the additional dissipation introduced by the split may have a beneficial stabilizing effect on the computations. From this perspective, reducing numerical dissipation improves the physical interpretability of the solution, but also makes it more important to account explicitly for the dissipative mechanisms that are effectively present in the material response, for instance through suitable damping terms and an appropriate calibration of the corresponding coefficients.



a Elastic strain energy as a function of time.



b Accumulated dissipated energy as a function of time.

Fig. 6. Energetic comparison between the monolithic BDF formulation and the staggered splitting reported in [38]. The monolithic scheme retains a higher elastic energy level and yields a larger cumulative fracture dissipation, consistently with a reduced amount of algorithmic dissipation. Strain and dissipated energies are computed as $E_{\text{strain}} = \int_{\Omega} (g(s)\psi^+(\epsilon) + \psi^-(\epsilon)) d\Omega$ and $E_{\text{diss}} = \int_{\Omega} \left(\frac{G_c}{2l} s^2 + \frac{G_c l}{2} |\nabla s|^2 \right) d\Omega$.

To further assess the energetic behavior of the proposed monolithic DAE–BDF formulation, we monitor the relative error in the global discrete energy balance and compare it with the staggered Newmark scheme (SNS), which is used here as a standard reference strategy for dynamic phase-field fracture. The relative energy-balance error is computed over time by comparing the variation of the mechanical and fracture energies, including the dissipative contributions, with the work of the external loading.

The comparison in Fig. 7 shows that the monolithic DAE–BDF formulation does not provide an exactly energy-conserving discretization, as expected for a first-order BDF scheme. Nevertheless, the energy-balance error remains controlled and of the same order as that obtained with the staggered Newmark scheme. The initial BDF-1 error is interpreted as a transient effect related to the start of the displacement-controlled loading and to the evaluation of the external work in the first accepted time steps. By contrast, the staggered Newmark scheme shows a smaller initial error, but a more monotone accumulation of the balance residual as the fracture process develops. This result supports the reliability of the proposed monolithic DAE–BDF integration, whose energetic behavior is comparable with that of a standard scheme commonly used in the literature, while retaining the advantages of a fully coupled adaptive time-integration framework.

4.4. Dynamic crack propagation from a circular hole

This test consists in a validation with an experiment performed in [68].

The specimen is a 140×70 mm plate containing a circular hole of diameter 30 mm whose right tangent point is connected to a short pre-notch of length 15 mm (Fig. 8(a)). Horizontal displacement is prescribed on the left edge, while the remaining boundaries are traction-free. Boundary conditions are prescribed as reported by experimental data in [68].

The material parameters are the same of the PMMA plate under compression. Although, the value of G_c has been increased to 1100 N/m according to literature.

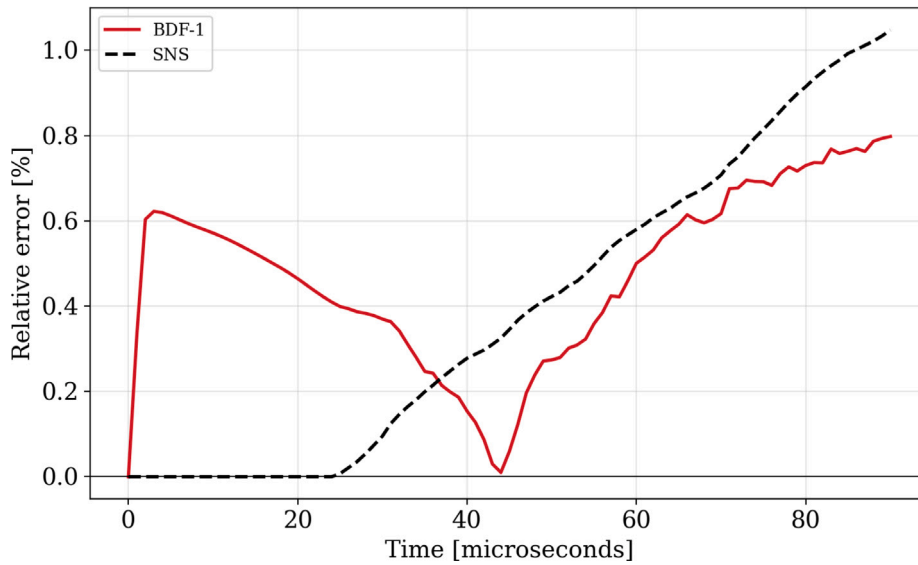


Fig. 7. Relative error in the global discrete energy balance for the monolithic DAE–BDF formulation and the staggered Newmark scheme (SNS). The BDF-1 solution exhibits an initial transient error, which is mainly associated with the onset of the prescribed displacement ramp and with the sensitivity of a relative error measure at early times. After this initial stage, the error decreases and remains controlled during the subsequent dynamic evolution. The SNS solution does not show the same initial transient, but the energy-balance error progressively accumulates during crack evolution. In the considered test, both methods remain within an error of about one percent, with the monolithic DAE–BDF formulation leading to a slightly lower final error than SNS.

A representative snapshot of the phase-field s is shown in Fig. 8(b). After a short transient, the crack initiates at the pre-notch tip and propagates to the right along the ligament. The damage band has thickness consistent with the internal length; small micro-cracks appear around the hole rim due to the mixed-mode stress state, but the dominant path remains aligned with the ligament.

In order to validate the simulation, we performed the computation of the crack tip via Paraview as follows.

The tip position $x_{\text{tip}}(t)$ is obtained by tracking the $s = 0.9$ iso-contour restricted to the connected component intersecting the ligament. The raw speed is computed as $\dot{x}_{\text{tip}}$ and then smoothed with a Butterworth filter to remove frame-to-frame noise.

The crack-tip velocity is obtained by differentiating the post-processed crack-tip position. The latter is extracted from the phase-field solution by identifying a prescribed contour of the damage variable at discrete output times. Therefore, small frame-to-frame fluctuations may arise from the finite mesh resolution, the diffuse nature of the phase-field crack representation, and the threshold-based localization of the crack tip. These fluctuations are numerical post-processing artifacts rather than physical oscillations of the crack path. Moreover, their effect is amplified when the crack-tip velocity is obtained by numerical differentiation of the position signal.

To reduce this high-frequency numerical jitter, a low-pass Butterworth filter is applied to the crack-tip velocity signal in the post-processing stage. The filter does not alter the computed displacement or phase-field solution. It is used only to obtain a smoother estimate of the velocity curve. The Butterworth filter is chosen because of its maximally flat response in the pass band, which attenuates high-frequency fluctuations while preserving the lower-frequency trend associated with crack propagation.

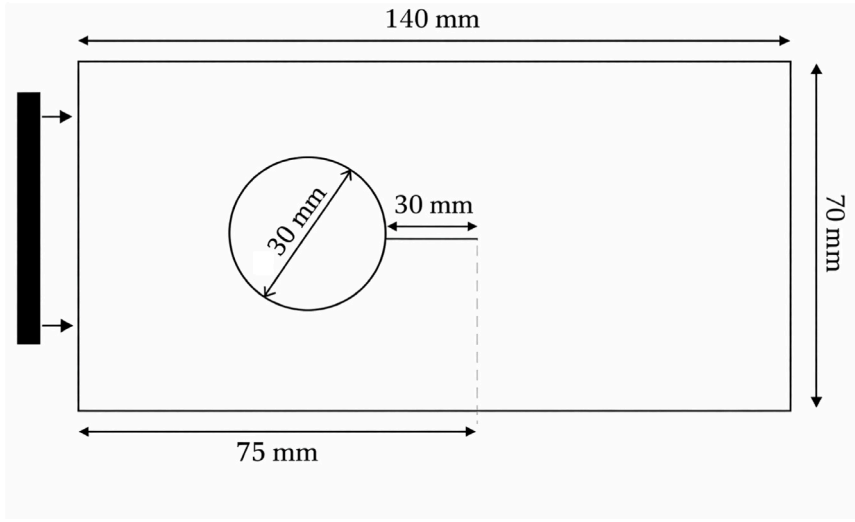
Fig. 9 (orange line) reports the resulting non-dimensional speed $\dot{x}_{\text{tip}}/c_R$ together with the reference envelope [68]. The peak speed reaches about $0.55\text{--}0.60 c_R$ around $80\text{--}95 \mu\text{s}$, followed by a progressive decay to $0.25\text{--}0.35 c_R$ as the crack lengthens and waves reflected by the hole interfere with the energy available at the tip. We indicate with c_R the Rayleigh wave speed that is equal to 1287 m/s .

Overall, the normalized peak around $0.6 c_R$ and the subsequent decay are in line with dynamic crack propagation. Residual differences in the reference and a slightly faster late-time branch can be traced to the displacement-controlled actuation used and the chosen smoothing of the speed signal.

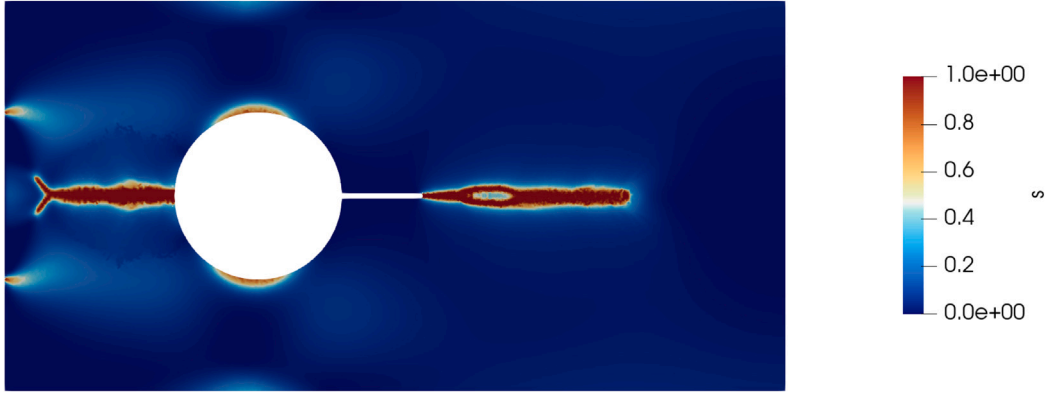
5. Numerical considerations on the method

5.1. BDF order, stiffness and non-convexity

During our simulations, we have noticed that the BDF solver could fail convergence during the Newton iterations or present oscillations in the solution. These features are typical for a stiff numerical problem.



a Specimen geometry.



b Phase-field snapshot.

Fig. 8. Dynamic crack with hole benchmark. (a) Geometry of the 140×70 mm plate, with a circular hole of diameter 30 mm and a pre-notch of length 15 mm starting from the right tangent of the hole. (b) Phase-field snapshot s during propagation: the main crack initiates at the pre-notch tip and advances to the right, while a weak micro-cracking halo is visible around the hole rim.

Our coupled dynamics in (u, s) is stiff for two reasons. The mechanical part supports fast elastic waves (high-frequency oscillatory modes), while the s -equation may be treated either as a *quasi-static elliptic constraint* ($\eta_s = 0$) leading to an index-1 DAE (evolution in u , algebraic solve in s) at each time step, or in *evolutionary viscous form* ($\eta_s > 0$) which adds controlled dissipation and improves Jacobian conditioning. In the viscous case the diffusion operator produces damped modes, while the mechanical branch carries high-frequency oscillations; the coexistence of fast decays and fast oscillations creates widely separated time scales, especially near crack nucleation and softening.

In addition, the total energy is non-convex due to tensile degradation of ψ_e^+ and the irreversibility constraint on s . These features directly impact the choice of BDF order and the robustness of a monolithic solver. In particular, they become relevant in strong dynamical regimes of the fracture.

It is a well-known result that for the test equation $y' = \lambda y$ with $\Re(\lambda) < 0$, linear multistep schemes cannot be both A-stable and of order > 2 (*second Dahlquist barrier*) [69,70]. BDF1 and BDF2 are A-stable; for $k \geq 3$ the methods are only $A(\alpha)$ -stable: the stability region becomes a sector whose angle shrinks rapidly with k (for $k = 3, 4, 5$ the sector is progressively narrower). In practice, when the discrete operator yields very negative eigenvalues, BDF with $k \geq 3$ impose restrictive constraints on h to stay inside the sector; step-size control then reduces h severely, nullifying the benefit of higher order and, worse, triggering oscillations or frequent step rejections.

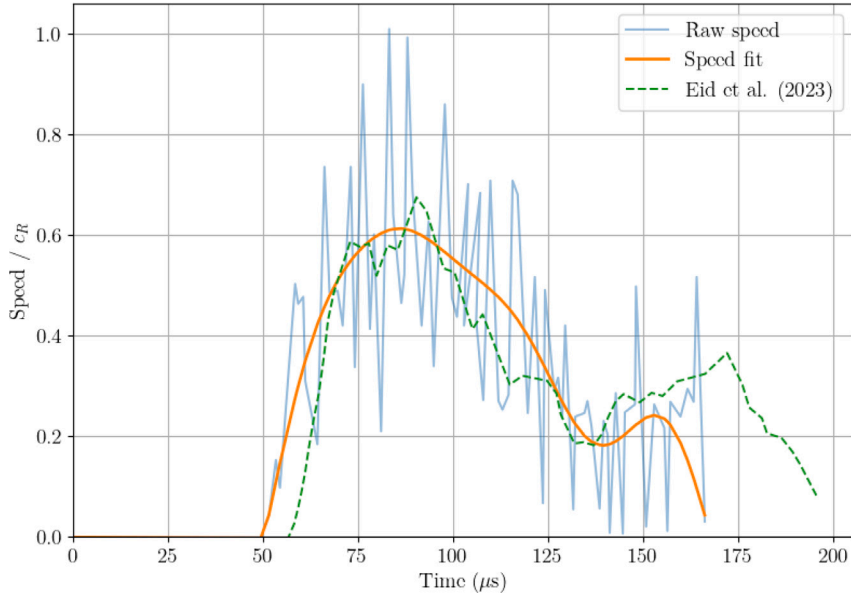


Fig. 9. Tip-speed history normalized by c_R : branch-aware tracking (orange) and raw values (light blue) against a literature envelope (green dashed). Peak speeds near $0.6 c_R$ are followed by a gradual deceleration as the crack length increases.

With variable step sizes (as in IDA) the *effective* stability region of BDF of order k shrinks further and depends on consecutive step ratios. For index-1 DAEs, algebraic constraints add sensitivity: step/order changes amplify the multistep “memory” and may increase constraint projection errors, with a further loss of robustness for $k \geq 3$.

Additionally, BDF1 and, to a large extent, BDF2 admit *G-stability* estimates and discrete energy-dissipation properties for convex operators [71,72]. For $k \geq 3$ such properties are lost. In our non-convex setting (selective ψ_e^+ , nucleation/softening), we have noticed limited energy dissipation like spurious growth of elastic or fracture energy between steps. This was usually associated to Newton and/or time advancing scheme stagnation or divergence.

Limiting to $k \leq 2$ strongly mitigates these effects because the method remains globally dissipative and does not amplify high-frequency modes.

Eventually, we can always use the damping factors to damp eventual numerical limited dissipation, like we have done in the single-notch tension test.

In stiff regimes [73] (fine meshes, small ℓ , large G_c/ℓ , strong nonlinearity), fixing the order to $k = 1, 2$ in IDA consistently yields fewer rejected steps, smoother residuals with effective line searches, and reliable energy accounting. Temporal accuracy is then controlled by the timestep h ; as stiffness grows, stability rather than order dictates the step, so the practical benefit of $k > 2$ disappears. For these reasons we cap the BDF order at $k \leq 2$ in all dynamic phase-field fracture simulations with the monolithic solver and eventually we also limited the maximum time step. This is a common strategy to solve stiff numerical problems.

A brief reminder of the notions of *A-stability*, *A(α)-stability*, and *G-stability* is provided in [Appendix B](#).

Additionally, the damping terms included in the dynamic formulation can also be interpreted, at the model level, as rate-dependent regularizing contributions. In particular, they may improve the numerical robustness of the simulation during rapid crack initiation and propagation, where abrupt changes in the phase-field and mechanical response can otherwise lead to severe nonlinear convergence difficulties. This role is analogous, in a broad sense, to conventional viscous stabilization strategies used in phase-field fracture.

However, in the present formulation, the inclusion of damping terms is not only motivated by numerical regularization. The monolithic DAE structure naturally contains the variables required to represent rate-dependent effects, such as the velocity field and, when needed, the rate of the phase-field variable. Therefore, damping contributions can be incorporated without introducing additional primary unknowns or substantially modifying the algebraic structure of the coupled problem. This provides a convenient framework for treating damping and viscous effects implicitly within the same adaptive time-integration strategy.

We finally remark that, in the benchmark problems considered here, damping parameters are occasionally included to obtain consistency with reference simulations available in the literature [36], where damping is often introduced to regularize high-rate dynamic fracture computations. At the same time, the DAE–BDF framework offers an additional numerical control through the possibility of restricting the integration order. In several simulations, limiting the adaptive scheme to BDF1 or BDF2 was sufficient to obtain a stable evolution, avoiding the use of higher-order formulas that may be less robust during rapid crack initiation and propagation.

Table 1

Wall-clock runtime comparison for the single-notch, one-hole compression, and Kalthoff-type impact benchmarks. Here, NR denotes the reference Newton–Raphson computation, while SNS denotes the staggered Newmark scheme. The reported timings provide an indicative assessment of the computational cost of the proposed monolithic DAE–BDF formulation for the considered implementations and discretizations. In the reported tests, the BDF runs lead to lower runtimes than the corresponding reference computations, and the parallel executions further reduce the overall cost in quasi-static setting. The Kalthoff-type impact benchmark extends the comparison to a high-rate dynamic fracture setting, where it shows a more expensive but still scalable behavior typical of high-rate dynamic fracture simulations for the monolithic approach.

Benchmark	Method	N. of CPUs	Problem size	Total runtime
Single-notch	NR 2D	1	3.3×10^4	3251 s
	BDF 2D	1	3.3×10^4	437.1 s
	BDF 2D	8	3.3×10^4	146.5 s
	BDF 3D	10	2.8×10^5	2251 s
One-hole compression	NR 2D	1	1.4×10^5	57 372 s
	BDF 2D	1	1.4×10^5	8354.7 s
	BDF 2D	8	1.4×10^5	762.3 s
	BDF 3D	15	4.0×10^5	5672.1 s
Kalthoff-type impact	SNS 2D	1	1.5×10^5	3829.4 s
	BDF 2D	1	1.5×10^5	19 296.4 s
	BDF 2D	8	1.5×10^5	4824.1 s
	BDF 3D	15	3.1×10^5	10 275.0 s

5.2. BDF methods vs. classical integration scheme

We used the single notch and one-hole compression test as benchmarks to compare our BDF integrator with a standard monolithic quasi-static Newton–Raphson (NR) and a staggered Newmark scheme (SNS) workflow; the results are reported in Table 1.

All NR and SNS runs were carried out in *Python/FEniCS*, while the BDF runs used our C++/deal.II implementation with SUNDIALS/IDA. Consequently, the wall-clock times reflect both algorithmic and implementation effects: (i) the BDF path employs adaptive time stepping, so step sizes and possible rejections are selected automatically; (ii) an *inexact Newton* strategy with Jacobian lagging is used on the BDF side (the Jacobian/factorization is reused and updated only when necessary), which lowers per-step cost at the price of a few additional nonlinear iterations; (iii) compiled C++ generally executes faster than a Python-driven workflow due to interpreter overhead.

The nonlinear systems arising at each time step are solved by a NR or SNS iteration with relative and absolute convergence tolerances equal to 10^{-4} and 10^{-6} , respectively. When the time integration is performed through IDA, the corresponding nonlinear solver tolerances are set to 10^{-3} and 10^{-5} . This slightly less restrictive choice was adopted because IDA employs a weighted nonlinear convergence test based on the estimated convergence rate of the nonlinear correction. In our computations, these tolerances provided a robust compromise between nonlinear convergence, adaptive time-step selection, and accuracy of the reported quantities of interest.

Fig. 10 summarizes the adaptive time-stepping behavior of the BDF integrator for the one-hole compression and single-notch benchmarks, in both 2D and 3D.

For the one-hole compression test, the time-step history shows a rapid transition from the start-up regime to a plateau at the prescribed upper bound, with only limited local reductions. The corresponding histograms confirm that most accepted steps are concentrated near the maximum admissible value. This indicates that, for this benchmark, the solution remains sufficiently regular over large portions of the simulation and that the adaptive strategy operates close to its most favorable regime, requiring temporal refinement only in a limited number of localized transient stages.

The single-notch test exhibits a qualitatively similar initialization phase, but a markedly different behavior in the subsequent evolution. Although the adaptive scheme again reaches a plateau after the initial transient, the accepted time steps are more broadly distributed and more frequent reductions are observed, especially in the final stage of the 2D simulation. This reflects the stronger temporal localization of the fracture event in this benchmark, where crack initiation and propagation induce sharper variations in the solution and therefore demand a finer temporal resolution.

Taken together, these results show that the proposed BDF strategy remains robust across materially different fracture scenarios while adapting its temporal resolution in a problem-dependent manner. In smoother regimes, such as the one-hole compression benchmark, the method remains close to the imposed upper bound for most of the computation, thereby maximizing efficiency. In more localized fracture configurations, such as the single-notch test, the same adaptive mechanism automatically refines the time discretization near the critical stages of the response. This behavior is consistent with the intended role of variable-order, variable-step BDF integration in fracture dynamics: not to enforce a uniformly fine temporal mesh, but to concentrate computational effort only where the transient evolution of the coupled fields requires it.

In fully dynamic simulations, however, the situation is different: the time step must remain sufficiently small to resolve the transient wave propagation and crack evolution, in accordance with the restrictions commonly adopted in the literature [38]. As a consequence, the overall efficiency is influenced not only by the time integrator itself, but also by solver robustness, implementation efficiency, and parallel scalability. Accordingly, staggered strategies remain computationally competitive in the dynamic setting.

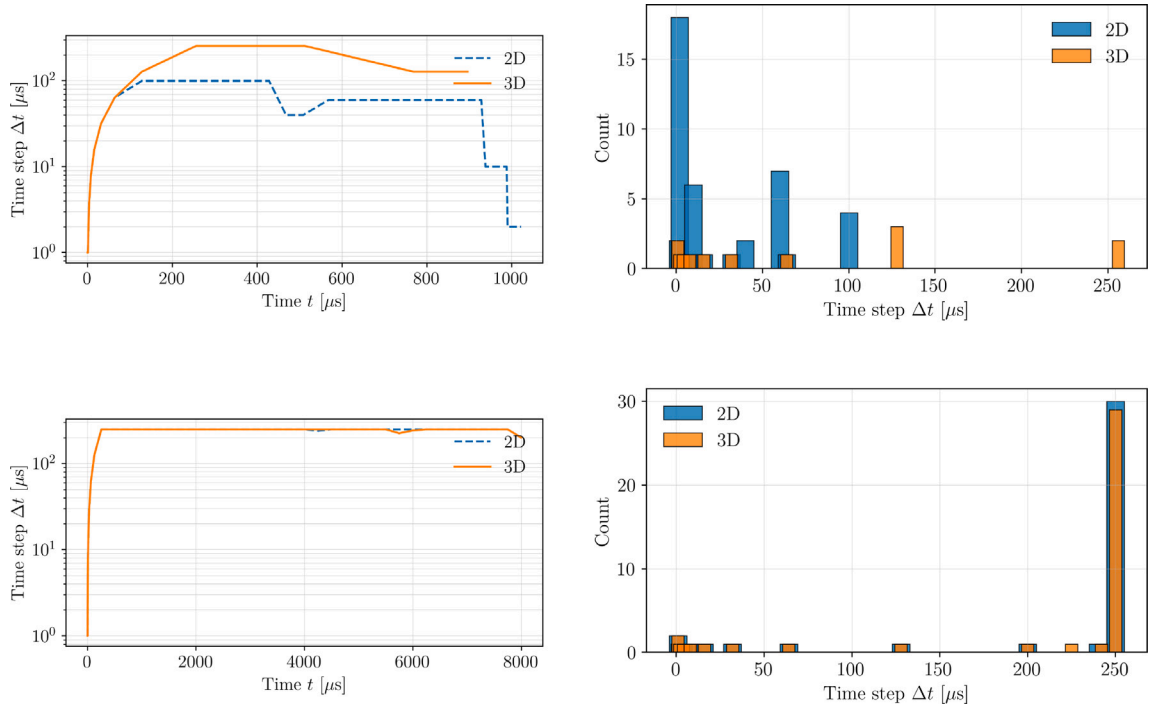


Fig. 10. Adaptive time-step selection for the one-hole compression and single-notch benchmarks, in both 2D (dashed blue) and 3D (solid orange). The left column reports the accepted time-step history, while the right column shows the corresponding distributions. The first row refers to the single-notch test, whereas the second row refers to the one-hole compression test.

This behavior is illustrated by the Kalthoff-type impact benchmark reported in Table 1. In this case, the serial 2D monolithic DAE-BDF simulation is more expensive than the corresponding staggered Newmark scheme (SNS). Nevertheless, the BDF runtime is reduced from 19 296.4 s to 4824.1 s when using eight processors, showing a significant benefit from parallel execution. These results indicate that staggered strategies remain computationally competitive in high-rate dynamic settings.

5.3. Block preconditioning based on an approximate dynamic schur complement

At each Newton step, the monolithic linearized problem is written in block form as

$$\mathcal{J} \begin{bmatrix} \delta \mathbf{u} \\ \delta \mathbf{v} \\ \delta \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u \\ \mathbf{r}_v \\ \mathbf{r}_d \end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix} J_{00} & J_{01} & J_{02} \\ J_{10} & J_{11} & 0 \\ J_{20} & J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} K(d) & \alpha M + C & J_{ud} \\ \alpha I & -I & 0 \\ J_{du} & J_{dv} & J_{dd} \end{bmatrix}, \quad (20)$$

where the first two row blocks represent the dynamic mechanical subsystem, while the third row block corresponds to the phase-field equation. In this setting, an effective preconditioning strategy must account for the strong algebraic interaction between the dynamic unknowns, while preserving a sufficiently cheap action on the phase-field block.

To this end, we employ a lower block-triangular preconditioner based on an approximate Schur reduction of the mechanical subproblem. The starting point is the exact factorization of the 2×2 mechanical block

$$\mathcal{J}_m = \begin{bmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{bmatrix}, \quad (21)$$

whose formal Schur complement with respect to J_{11} is given by

$$S_u = J_{00} - J_{01} J_{11}^{-1} J_{10}. \quad (22)$$

An exact realization of (22) would, however, be too expensive in the present context, since it would require repeated applications of J_{11}^{-1} inside the preconditioner. We therefore replace the inverse of the kinematic block with a purely diagonal approximation,

$$J_{11}^{-1} \approx D_{11}^{-1}, \quad D_{11} := \text{diag}(J_{11}), \quad (23)$$

which yields the approximate dynamic Schur complement

$$\tilde{S}_u = J_{00} - J_{01} D_{11}^{-1} J_{10}. \quad (24)$$

The resulting preconditioner is then constructed as

$$P^{-1} \approx \begin{bmatrix} \tilde{S}_u & 0 & 0 \\ J_{10} & D_{11} & 0 \\ J_{20} & J_{21} & J_{22} \end{bmatrix}^{-1}, \quad (25)$$

where the inverse is not formed explicitly but applied by block forward substitution. More precisely, given a residual vector

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_u \\ \mathbf{r}_v \\ \mathbf{r}_d \end{bmatrix}, \quad (26)$$

the preconditioned vector $\mathbf{z} = P^{-1}\mathbf{r}$ is obtained through the sequence

$$\begin{aligned} \hat{\mathbf{v}} &= D_{11}^{-1} \mathbf{r}_v, \\ \hat{\mathbf{r}}_u &= \mathbf{r}_u - J_{01} \hat{\mathbf{v}}, \\ \mathbf{z}_u &\approx \tilde{S}_u^{-1} \hat{\mathbf{r}}_u, \\ \hat{\mathbf{r}}_v &= \mathbf{r}_v - J_{10} \mathbf{z}_u, \\ \mathbf{z}_v &= D_{11}^{-1} \hat{\mathbf{r}}_v, \\ \hat{\mathbf{r}}_d &= \mathbf{r}_d - J_{20} \mathbf{z}_u - J_{21} \mathbf{z}_v, \\ \mathbf{z}_d &\approx J_{22}^{-1} \hat{\mathbf{r}}_d. \end{aligned}$$

In other words, the mechanical predictor is first filtered through a cheap approximation of the kinematic inverse, then corrected through the dynamic Schur operator, and finally injected into the phase-field block through the lower-triangular couplings.

From a numerical viewpoint, the rationale behind (25) is twofold. First, the preconditioner retains the dominant interaction between displacement-like and velocity-like variables through the Schur term (24), rather than treating the two mechanical blocks independently. This is crucial in dynamic settings, where the off-diagonal couplings J_{01} and J_{10} are responsible for a substantial portion of the spectral stiffness of the monolithic Jacobian. Second, replacing J_{11}^{-1} by the inverse of its diagonal preserves the main scaling and local balance of the kinematic block at negligible cost, thereby yielding an effective compromise between algebraic fidelity and computational efficiency.

The phase-field component is kept in the same lower-triangular framework. Although J_{22} is preconditioned independently, the residual entering the phase-field solve is corrected by the already computed mechanical contributions. Hence, the preconditioner still reflects the one-way coupling induced by the lower-left blocks J_{20} and J_{21} , which is consistent with the block structure of the monolithic Jacobian.

Overall, the proposed construction can be interpreted as an approximate block factorization of the coupled dynamic phase-field system, where the main effort is concentrated on the effective Schur operator associated with the mechanical subsystem.

To assess the effect of the proposed block preconditioner on the linear solver, Fig. 11 compares the number of Krylov iterations with those obtained by using a global ILU+GMRES strategy on the fully coupled linear system. The comparison is performed for the single-notch tension benchmark.

The results indicate that the block preconditioner yields a lower and more stable iteration count throughout the simulation. In particular, the average number of Krylov iterations is reduced from 344.96 for ILU+GMRES to 89.21 for the block preconditioner, while the maximum number of iterations decreases from 397 to 196. This behavior suggests that exploiting the coupled block structure of the DAE system and approximating the associated dynamic Schur complement is more effective, for this problem, than applying a global ILU preconditioner to the full monolithic matrix.

This observation is also relevant from a parallel-computing perspective. A global ILU factorization of the fully coupled matrix may become increasingly difficult to use efficiently in parallel, due to the sequential nature of triangular solves and to the communication costs associated with global coupling. By contrast, the proposed block strategy separates the main mechanical and phase-field contributions and allows one to use solvers and preconditioners tailored to the individual blocks, providing a more flexible preconditioning framework for the monolithic DAE system.

5.4. Single-hole compression: small- vs. large-deformation runs

Fig. 12 shows that the small-strain (SD) simulation matches the reference response in the elastic regime and remains close through peak load, with only minor deviations in softening. The large-deformation (LD) run follows the same trend pre-peak but displays a slightly higher load at equal displacement beyond 1.0–1.2 mm and stops earlier.

Two mechanisms explain the difference:

1. *Finite-kinematics effects.* In LD, geometric nonlinearity (second-order terms and membrane-like action around the hole ligaments) increases the effective tangent stiffness, producing a modest upward bias of the load–displacement curve in the post-yield range relative to SD.

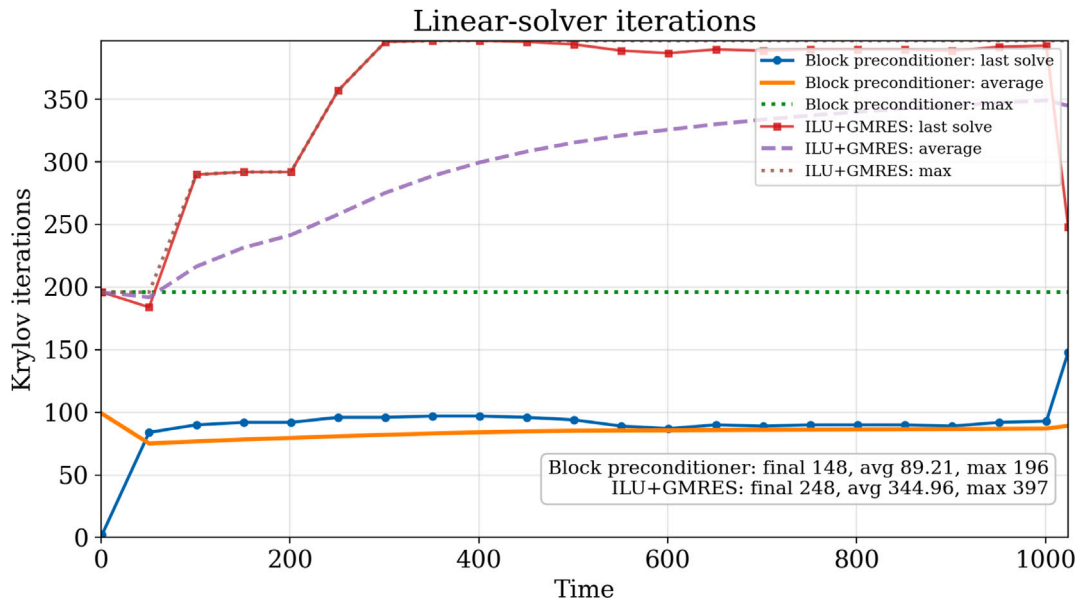


Fig. 11. Linear-solver iterations obtained with the proposed block preconditioner and with a global ILU+GMRES strategy for the single-notch tension benchmark. The curves report the number of Krylov iterations for the last linear solve of each time step, together with the running average and maximum value. For the considered simulation, the block preconditioner keeps the iteration count substantially lower than the global ILU+GMRES approach, with an average of 89.21 iterations compared with 344.96, and a maximum of 196 iterations compared with 397. This comparison provides numerical evidence of the effectiveness of the approximate dynamic Schur-complement construction used in the block preconditioner.

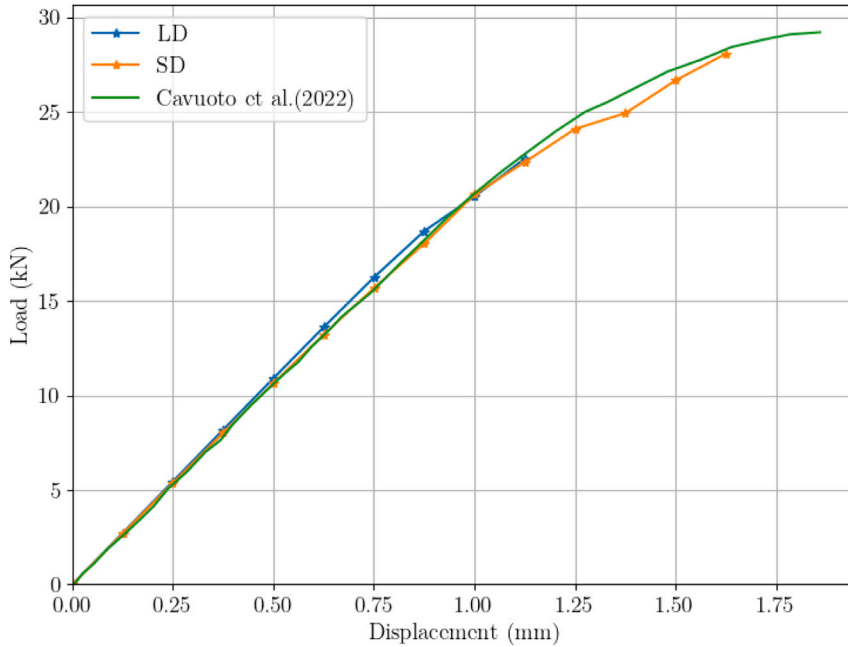


Fig. 12. Single-hole compression: load-displacement curves for large-deformation (LD) and small-deformation (SD) formulations, compared with the Ref. [67].

2. *Stability floor via κ .* To prevent FEM cell degeneration at large strains, the residual stiffness parameter k is increased in LD. Indeed, we used $k = 10^{-6}$ for small deformations and $k = 10^{-2}$ for large deformations. While this improves robustness (avoids singular tangents and element inversion), it leaves nonzero stiffness in fully damaged zones and slightly widens the damage band, further lifting the LD curve during softening.

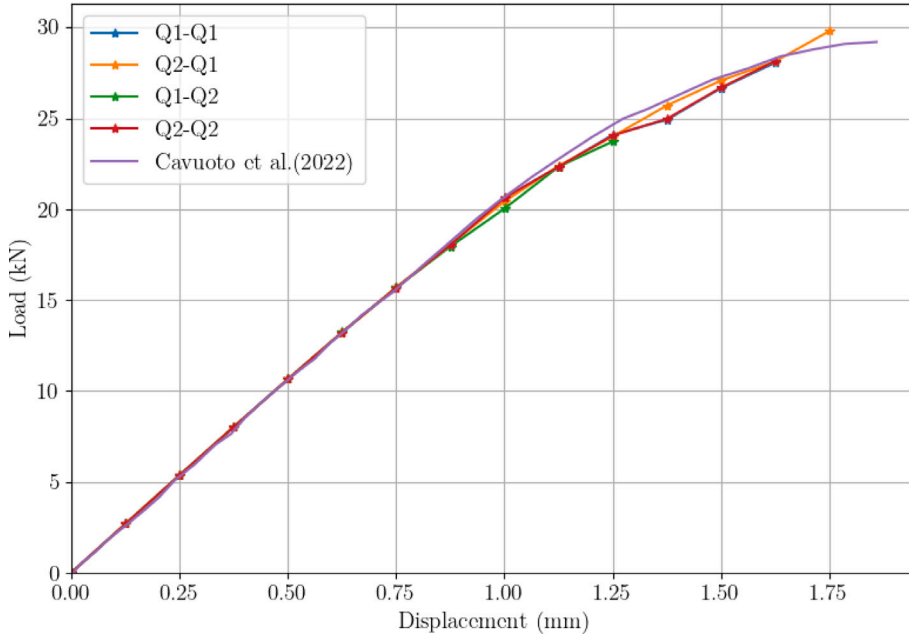


Fig. 13. Single-hole compression: load–displacement response for different polynomial orders of the finite elements, with $(u, s) \in \{Q_1-Q_1, Q_2-Q_1, Q_1-Q_2, Q_2-Q_2\}$, compared against the reference curve [67]. Same mesh, time-stepping, and damping/phase-field parameters are used in all runs.

Despite the larger κ , the LD simulation eventually reaches near-singular configurations as elements within the crack band become highly distorted. This deteriorates the conditioning of the problem and triggers solver breakdown, explaining the earlier termination.

5.5. FEM-degree study

Relying on the deal.II library, we exploit the independence of the formulation with respect to the FEM degree on each variable.

Therefore, we tested the single hole compression test with different degrees of the FEM discretization for the mechanical field and the phase-field one.

Fig. 13 shows that all discretizations collapse onto the reference in the elastic range and up to the vicinity of the peak; differences emerge mainly in the softening branch. The Q_2-Q_1 case (quadratic in (u, v) , linear in s) yields a slightly higher load for the same displacement beyond 1.6 mm, whereas Q_1-Q_2 (linear in (u, v) , quadratic s) is softer between 1.0 and 1.25 mm; Q_2-Q_1 lies between these two and remains closest to the reference over the full range.

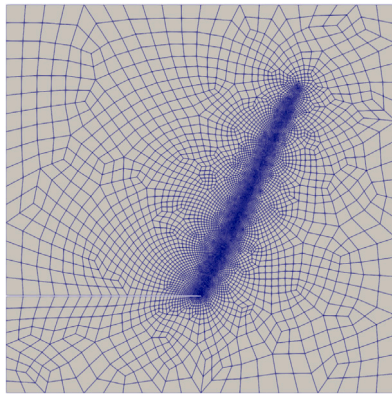
We notice that:

- Increasing the order for the *mechanical* fields improves geometric/kinematic accuracy in the ligaments and adds a small membrane-like contribution, which tends to raise the tangent stiffness post-peak, explaining the mild upward bias of Q_2-Q_1 .
- Increasing the order for the *phase field* variable gives a sharper representation of the diffusive crack and reduces mesh-induced anisotropy; the global response softens earlier (as seen for Q_1-Q_2).

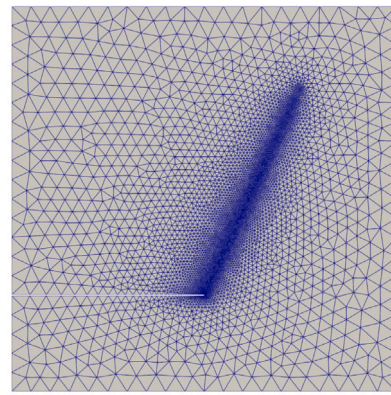
All four pairs reproduce stiffness and initiation robustly. When cost allows, Q_2-Q_1 provides the smoothest response and the best agreement with the reference. Mixed pairs can be used to bias the response slightly toward stiffer (Q_2-Q_1) or softer (Q_1-Q_2) behavior while keeping the same mesh.

5.6. Mesh-type independence

Our implementation supports both quadrilateral and triangular conforming meshes under the same phase-field framework. In 2D we employ isoparametric Q_1 elements for u and s on quads and P_1 elements on triangles; boundary conditions, material constants, and the internal length ℓ are identical across runs. Fig. 14 shows a one-to-one comparison of the phase-field contour: the crack path coincides within discretization error. This behavior is consistent with standard finite-element approximation theory for regular mesh families. Indeed, both Q_1 - and P_1 -based discretizations converge to the same weak solution as $h \rightarrow 0$.



Quadrilateral mesh



Triangular mesh

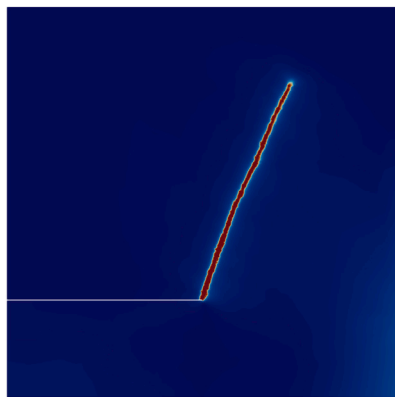
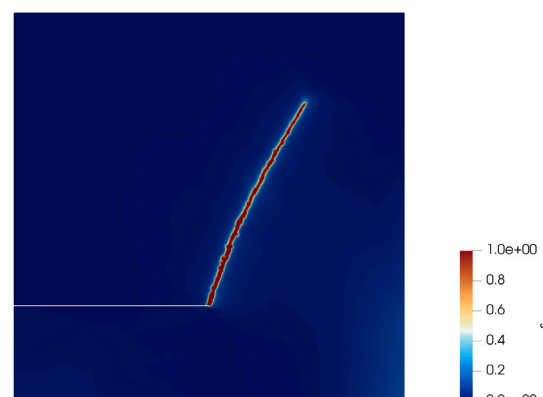
Quadrilateral mesh (Q_1) resultTriangular mesh (P_1) result

Fig. 14. Comparison between quadrilateral and triangular discretizations for the Kalthoff-type impact benchmark. The top row shows the two meshes, generated with comparable $h_{\min}$, while the bottom row reports the corresponding phase-field contours s obtained using identical model and time-stepping parameters.

6. Conclusions

We presented a monolithic DAE formulation for dynamic phase-field fracture with phase field s , tensile split ψ_e^+ , and consistent energy tracking, discretized in time by BDF. The approach was verified on benchmarks spanning 2D and 3D and capable of simulating various scenarios validated via comparison with numerical and experimental data.

From the time-integration viewpoint, stiff regimes motivate capping the BDF order at $k = 2$. Beyond the Dahlquist barrier, higher-order BDF lose $A(\alpha)$ -stability and interact poorly with the non-convex energy landscape during crack nucleation and softening. In practice, $k \leq 2$ yields fewer rejected steps and steadier Newton convergence, increasing the robustness of the proposed approach.

Finally, the performance analysis compares a standard quasi-static Newton–Raphson workflow against our BDF implementation. The reported wall times reflect practical, end-to-end differences like adaptive time stepping on the BDF side and an inexact Newton strategy with Jacobian lagging. Under these conditions, BDF provides reduced runtime while preserving fidelity of the phenomenon across the simulation.

CRediT authorship contribution statement

Gabriel Cassese: Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Conceptualization. **Andrea Mola:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Conceptualization. **Pietro Lenarda:** Writing – review & editing, Validation, Supervision, Methodology, Conceptualization. **Marco Paggi:** Writing – review & editing, Validation, Supervision, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Large displacement formulation at the element level

In this appendix, we address the way we tackle the large displacement setup of our model.

Let $\hat{\Omega} \subset \mathbb{R}^d$ denote the *reference domain*, and let $\Omega(t) \subset \mathbb{R}^d$ denote the corresponding *physical domain* (or real space domain) at time $t \in \mathbb{R}$. We define the mapping χ as follows:

$$\chi : \hat{\Omega} \times \mathbb{R} \rightarrow \Omega, \quad (\hat{\mathbf{x}}, t) \mapsto \chi(\hat{\mathbf{x}}, t) \quad (\text{A.1})$$

This means that each point $\hat{\mathbf{x}} \in \hat{\Omega}$ in the reference configuration is mapped to a point $\mathbf{x} = \chi(\hat{\mathbf{x}}) \in \Omega$ in the physical configuration:

$$\mathbf{x} = \chi(\hat{\mathbf{x}}, t), \quad \text{with} \quad \hat{\mathbf{x}} \in \hat{\Omega}, \mathbf{x} \in \Omega, t \in \mathbb{R}$$

If χ is sufficiently smooth, we may assume:

$$\chi \in C^k(\hat{\Omega} \times \mathbb{R}; \mathbb{R}^d), \quad k \geq 1$$

where d is the spatial dimension.

In the isoparametric framework and a given triangulation, we consider the previous mapping related to the cell that constitutes the triangulation.

We consider the case of a generic n th cell of the triangulation.

Let $N_l(\tilde{x}, \tilde{y}, \tilde{z})$ for $l = 1, \dots, N_L$ denote the shape functions defined in the domain $[0, 1]^3$ and mapped to the n th cell of $\hat{\Omega}$ indicated with Ω^n .

The local parametrization on the n th cell is given by

$$\chi^n(\tilde{x}, \tilde{y}, \tilde{z}, t) = \sum_{l=1}^{N_L} x^{n_l}(t) N_l(\tilde{x}, \tilde{y}, \tilde{z})$$

where $(\tilde{x}, \tilde{y}, \tilde{z}) \in [0, 1]^3$ and $x^{n_l}(t) = \hat{x}^{n_l} + u^{n_l}(t)$ where $\hat{x}^{n_l} \in \hat{\Omega}$ is the n_l th point in the *reference domain* and $u^{n_l}(t)$ is the incremental displacement relative to $\hat{x}^{n_l}$ at time t (see Fig. A.15).

Global base functions ϕ^i on the n th cell are given by

$$\phi_n^i(\chi^n(\tilde{x}, \tilde{y}, \tilde{z})) = \sum_{l=1}^{N_L} \delta_{ik_l} N_l(\tilde{x}, \tilde{y}, \tilde{z})$$

where k_l is the *local to global* indexing and δ_{ij} is the standard Kronecker function.

The local representation of a generic function $\Psi(\chi^n(\tilde{x}, \tilde{y}, \tilde{z}))$ on the n th cell is given by

$$\Psi^n(\tilde{x}, \tilde{y}, \tilde{z}, t) = \sum_{l=1}^{N_L} \phi_{k_l}(t) N_l(\tilde{x}, \tilde{y}, \tilde{z})$$

where $\phi_{k_l}(t)$ are the nodal values at time t . We notice that Φ is a generic function a priori, but in our case it is the solution of our problem.

Given that we use weak formulation for our problems, we deal with integrals. Thus, we need the Jacobian of the transformation χ^n . By denoting it with $\mathbf{J}^n(\tilde{x}, \tilde{y}, \tilde{z}, t)$, we have it holds $d\Omega^n = J^n(\tilde{x}, \tilde{y}, \tilde{z}, t) d\tilde{x} d\tilde{y} d\tilde{z}$ where J^n denotes the absolute value of the determinant of $\mathbf{J}^n(\tilde{x}, \tilde{y}, \tilde{z}, t)$.

This formulation takes into account the actual spatial variation of the geometry. In deal.ii, it is done by using *mapping vectors*. In our case, this is the solution vector, but a priori it could be a different vector also mapped with a different degree mapping with respect to the finite element degree.

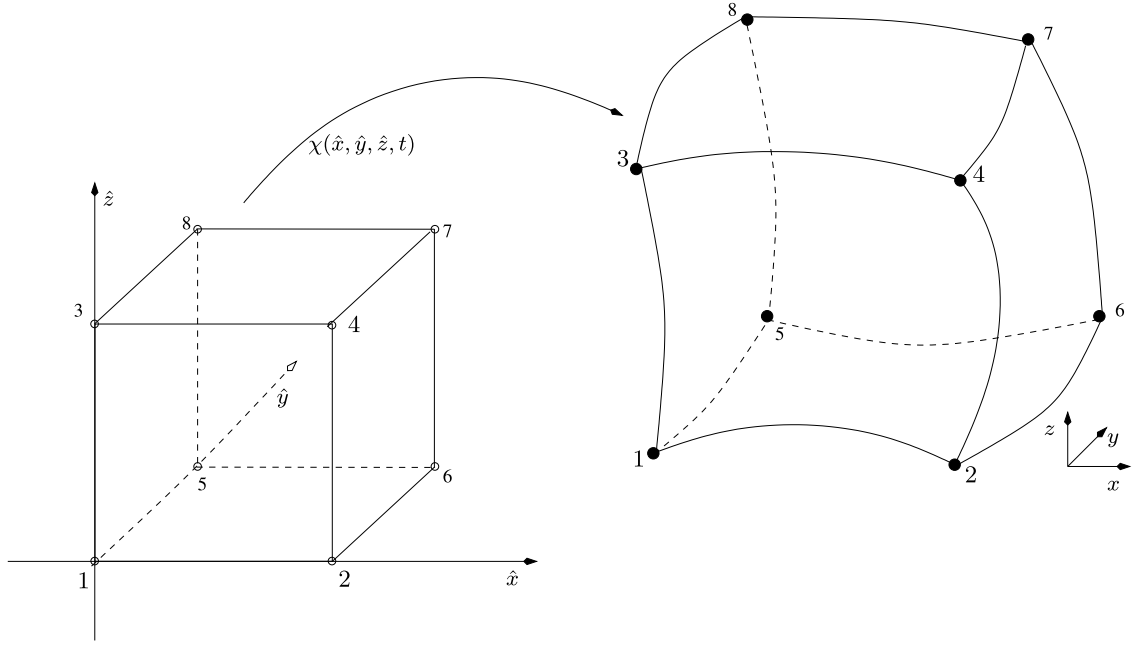


Fig. A.15. The transformation χ maps the linear reference cell to a corresponding physical element in the computational domain.

Appendix B. Stability notions for time-integration methods

For completeness, we briefly recall the stability notions used in the discussion of BDF time-integration methods. We consider an autonomous system of ordinary differential equations

$$\dot{y} = f(y), \quad (B.1)$$

and, in particular, the scalar linear test equation

$$\dot{y} = \lambda y, \quad \lambda \in \mathbb{C}. \quad (B.2)$$

The latter is the standard model problem used to characterize the linear stability properties of time-integration schemes. Let $h > 0$ be the time-step size and set $z = h\lambda$. For a linear multistep method written as

$$\sum_{j=0}^q \alpha_j y_{n+j} = h \sum_{j=0}^q \beta_j f(y_{n+j}), \quad (B.3)$$

application to (B.2) gives the characteristic equation

$$\rho(\xi) - z\sigma(\xi) = 0, \quad \rho(\xi) = \sum_{j=0}^q \alpha_j \xi^j, \quad \sigma(\xi) = \sum_{j=0}^q \beta_j \xi^j. \quad (B.4)$$

A value $z \in \mathbb{C}$ belongs to the region of absolute stability of the method if all roots $\xi_i(z)$ of (B.4) satisfy

$$|\xi_i(z)| \leq 1, \quad (B.5)$$

with the additional requirement that roots on the unit circle are simple.

A method is called *A-stable* if its region of absolute stability contains the whole left half of the complex plane, namely

$$\{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\} \subseteq S, \quad (B.6)$$

where S denotes the stability region. This property is particularly important for stiff dissipative problems, since modes with $\operatorname{Re}(\lambda) < 0$ should not impose restrictive time-step constraints for stability.

The weaker notion of *A*(α)-stability requires the stability region to contain a sector of angle 2α around the negative real axis, i.e.

$$\{z \in \mathbb{C} \setminus \{0\} : |\arg(-z)| \leq \alpha\} \subseteq S, \quad 0 < \alpha \leq \frac{\pi}{2}. \quad (B.7)$$

Thus, *A*-stability corresponds to the limiting case $\alpha = \pi/2$. Higher-order BDF methods are not *A*-stable in general, but they retain *A*(α)-stability for suitable sector angles.

Finally, G -stability is an algebraic stability concept for multistep methods, particularly useful in the analysis of dissipative and nonlinear evolution problems. In broad terms, a multistep method is G -stable if there exists a symmetric positive definite matrix G such that the numerical solution satisfies a contractivity or energy-type estimate in the corresponding G -norm. If

$$Y_n = (y_n, y_{n+1}, \dots, y_{n+q-1}) \quad (\text{B.8})$$

denotes the vector of consecutive time levels, the G -norm is defined by

$$\|Y_n\|_G^2 = Y_n^\top G Y_n. \quad (\text{B.9})$$

For dissipative problems, G -stability provides a discrete analogue of continuous energy decay or contractivity estimates. This is why it is often used to justify the favorable energy behavior of low-order BDF schemes, especially BDF1 and BDF2, in gradient-flow and dissipative evolution problems.

Data availability

Data will be made available on request.

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