Penalty Functions for Handling Large Deviation of Quadrature States in NMPC

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Abstract—**Nonlinear Model Predictive Control for mechanical applications is often used to perform the tracking of time-varying reference trajectories, and is typically implemented using quadratic penalty functions. Controllers for mechanical systems, however, are often required to handle large deviations from the reference trajectory. In such cases, it has been observed that NMPC schemes based on quadratic penalties can have undesirably aggressive behaviours. Heuristics can be developed to tackle these issues, but they require intricate and non-systematic tuning procedures. This paper proposes an NMPC scheme based on a specific class of penalty functions to handle large deviations of quadrature states from their reference, offering an intuitive and easy-to-tune alternative. The behaviour of the proposed NMPC scheme is analysed, and the conditions for its nominal stability are established. The control scheme is illustrated on a simulated quadcopter.**

Index Terms—**Nonlinear model predictive control, Huber penalty function, large deviation from the reference, mechanical systems**

I. Introduction

Nonlinear model predictive control (NMPC) is an effective way of tackling problems with constraints and nonlinear dynamics. NMPC re-calculates at every sampling instant a control policy that minimizes a penalty function defined over a horizon window in the future. The properties of NMPC have been studied for the general class of penalty functions, which are lower bounded by a $\mathcal{K}_\infty$ function [1], [2]. In practice quadratic penalties are preferred because they are straightforward to implement, can be efficiently treated using Gauss-Newton Hessian approximations, and yield controllers having an intuitive behaviour.

NMPC has been extensively used in the process industry [3], where it is often assumed that the error between the system state and its fixed reference is relatively small. However, NMPC is more and more used for mechanical applications. Controllers for mechanical applications are often required to track infeasible trajectories, handle large reference jumps, or perform obstacle avoidance, potentially resulting in large deviations form their reference.

In such situations, it has been observed that NMPC based on quadratic penalties can become very aggressive, i.e. it yields a significant activation of the inputs bounds and state constraints, and taps strongly into the system nonlinearities. The latter often requires an expensive line search to ensure the convergence of the underlying Newton-type scheme. More crucially, whenever a state deviates largely from its reference, the control solution computed by an NMPC scheme based on quadratic penalties is chiefly influenced by the associated penalty, while the competing penalties have a marginal influence. This is especially a problem when such competing penalties must weigh in the cost function regardless of the deviation of quadrature states from their reference. This is e.g. the case for penalties associated to the alleviation of structural fatigue [4], which are momentous in many mechanical applications.

Heuristics such as smoothing and saturation of the regulation error, or a temporary reduction of the quadratic penalties weighting matrices can be used to tackle such issues [5]. However these heuristics can be difficult to set up, and can result in intricate and non-intuitive closed-loop behaviours. Alternatively, ad-hoc input and state bounds can address the problem by limiting the control action of the NMPC scheme when the states are far from their reference. Such bounds, introduced for the purpose of taming the behaviour of the NMPC scheme, are however artificial in the sense that they do not necessarily correspond to actual physical limitations of the system at hand, and can therefore be counterproductive since they arbitrarily limit the control authority of the NMPC scheme.

The undesirable NMPC behaviour previously described is typically due to large deviations from a given reference of a specific type of states, labelled as *quadrature* states in this paper. Quadrature states are states that do not influence the system dynamics and are the result of a pure integration of states and controls. As an example, a car position is purely defined by the integral of its velocity over time, and the position does not impact the other dynamics. We remark that the class of systems of interest having quadrature states is relatively large.

Based on this observation, in this paper, we propose to tackle the aforementioned issues via a specific design of the cost function associated to the quadrature states. We define the class of $\mathcal{P}$-penalty functions, the most known of which is the Huber penalty, displayed in Figure 1. The main feature of $\mathcal{P}$-penalty functions is that, while small deviations from the reference can still be quadratically penalised, large deviations from the reference are penalised less than quadratically. This paper proposes to use a $\mathcal{P}$-penalty function for the quadrature states and a traditional quadratic function for the remaining states and the controls. It will be shown in the following that for this choice of cost function, the feedback law stemming from the NMPC scheme becomes insensitive to the quadrature states when they are far from their reference.

The paper is organized as follows. The class of $\mathcal{P}$-penalty functions is introduced in Section II. Section III details the proposed NMPC scheme. Section IV proposes an analysis of the behaviour of the proposed scheme for large tracking errors of the quadrature state, Section V establishes its nominal stability, Section VI provides insights on the implementation

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of the proposed scheme. Section VII presents an illustrative example.

Contributions of the paper: an extension of the work proposed in [6] to multi-quadrature states is proposed, and the proposed approach based on the Huber penalty function is generalised to a larger class of functions. A smooth and efficient problem formulation is proposed to handle the proposed scheme in a standard QP solver.

II. Preliminaries

A. Quadrature states

The following form of discrete-time system will be studied:

$$x_{i+1} = f(x_i, u_i), \qquad q_{i+1} = q_i + \mathcal{I}(x_i, u_i), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n$ and $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}^{n_q}$ are continuous, possibly nonlinear functions representing the system dynamics, $[x, q] \in \mathbb{R}^{n+n_q}$ is the system state vector, $u \in \mathbb{R}^{n_u}$ the input vector. The states labelled q are those we refer to as *quadrature* states, since they can be construed as an integration of the other states x and the inputs u via the function $\mathcal{I}$, while function f does not depend on the quadrature states q.

We observe that this situation arises for any mechanical system for which the generalised forces are independent of a subset of the generalised coordinates, and for which the Lagrange function is linearly dependent on that subset of generalised coordinates. In such a case, Equation (1) results from the discretisation of the continuous dynamics of the system, regardless of the discretisation method used.

B. $\mathcal{P}$-penalty functions

As a systematic way of dealing with large tracking errors of quadrature states, i.e. states that do not enter the system dynamics, this paper proposes to use a special type of penalty functions, $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}_+$ having the following properties:

1) $\mathcal{P}$ is lower-bounded by a $\mathcal{K}_\infty$ function;
2) $\nabla_z \mathcal{P}(z)$ exists everywhere;
3) $\nabla_z \mathcal{P}(z) \neq 0$, $\forall z \neq 0$ and $\nabla_z \mathcal{P}(0) = 0$;
4) $\nabla^2 \mathcal{P}(z)\, d_j = 0$, $\forall z \notin \mathbb{X}_j$, where $\mathbb{X}_{j=1,\ldots}$ is a collection of non-empty, compact and open sets that contain the origin, and $d_{j=1,\ldots}$ is a collection of unitary constant vectors.

We will further label a quasi $\mathcal{P}$-penalty function a function that fulfils conditions 1-3, and that fulfils condition 4 asymptotically for z large, more precisely:

$$\left\| \nabla^2 \mathcal{P}(z)\, d_j \right\| \leq c \left\| z \right\|_M^{-1}, \quad \forall z \notin \mathbb{X}_j, \tag{2}$$

for some constant $c > 0$ and positive-definite matrix M. The concept of quasi $\mathcal{P}$-penalty function allows us to capture another useful class of penalty functions that yield "well-behaved" responses of the NMPC controller for large errors in the quadrature states.

We provide next an example of a $\mathcal{P}$-penalty function and of a (not strictly) convex $\mathcal{P}$-penalty function.

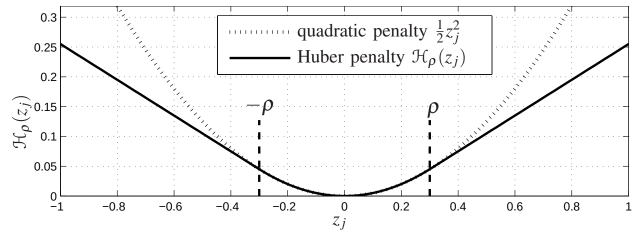

Fig. 1. Huber penalty function $\mathcal{H}_\rho(z)$ for $z \in \mathbb{R}$ and $\rho = 0.3$.

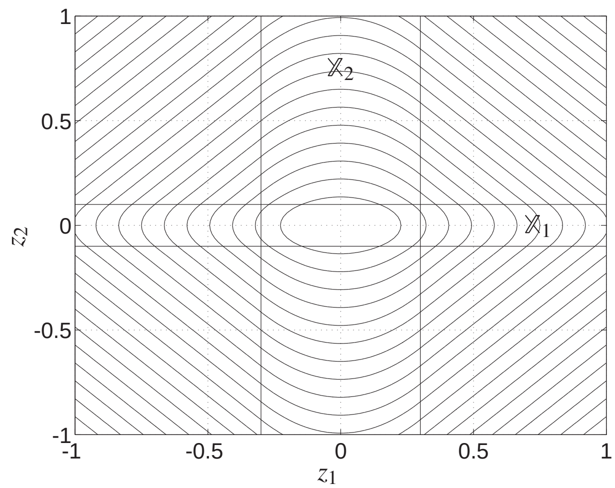

Fig. 2. Huber penalty function $\mathcal{P}(z)$ for $z \in \mathbb{R}^2$, $\rho = [0.3\ \ 0.1]$ and $w = [1\ \ 3]$.

1) Example of $\mathcal{P}$-penalty function: We propose here an example of such penalty function constructed using the Huber penalty function [7] that has arguably very good properties to deploy the proposed idea in practice. The Huber penalty function $\mathcal{H}_\rho(z) : \mathbb{R} \to \mathbb{R}$ reads as: (see Fig. 1):

$$\mathcal{H}_\rho(z) = \begin{cases} \frac{1}{2}z^2, & |z| \leq \rho \\ \rho(|z| - \frac{1}{2}\rho), & |z| > \rho \end{cases}, \tag{3}$$

for the given parameter $\rho \in \mathbb{R}$. It is quasi-convex, equivalent to a quadratic penalty within the region $[-\rho, \rho]$, and similar to an ℓ_1 norm outside. One can then construct from $\mathcal{H}_\rho(z)$ a $\mathcal{P}$-penalty function $\mathcal{P}(z) : \mathbb{R}^{n_q} \to \mathbb{R}$ as follows

$$\mathcal{P}(z) = \sum_{j=1}^{n_q} w_j \mathcal{H}_{\rho_j}(z_j) \tag{4}$$

for the given set of parameters $\rho_{i=1,\ldots,n_q} \in \mathbb{R}$, and the set of weights $w_{1,\ldots,n_q} \in \mathbb{R}$. Here a quadratic penalty $\frac{1}{2}z^\top W z$ is implemented in a neighbourhood of $z = 0$, where W is a diagonal weighting matrix formed by the weights $w_{1,\ldots,n_q}$. It can be easily verified that (4) satisfies the properties of a $\mathcal{P}$-penalty function as:

$$d_j = \mathbf{1}_j, \qquad \mathbb{X}_j = \left\{ z \,\middle|\, \left| z_j \right| \leq \rho_j \right\}, \quad \forall j$$

verifies condition 4, where $\mathbf{1}_j$ is a unit vector with element j equal to 1.

If a quadrature penalty based on a non-diagonal weighting matrix W is desirable, a state-space transformation diagonalising the weighting matrix of the quadrature states can be deployed, so as to recover a function of the form (4). Then all the results presented thereafter are valid in the transformed space. For the sake of brevity, we will not further detail this aspect of the problem.

2) Example of quasi $\mathcal{P}$-penalty function: we provide here an example of quasi $\mathcal{P}$-penalty function based again on the Huber penalty function, reading as:

$$\mathcal{P}(z) = \mathcal{H}_\rho(\|z\|_M) = \begin{cases} \frac{1}{2}\|z\|_M^2 & \text{if} \quad \|z\|_M \leq \rho \\ \rho(\|z\|_M - \frac{1}{2}\rho) & \text{otherwise} \end{cases}.$$

We can verify here that for $\rho < \infty$ condition 4 holds asymptotically for any d and for $\mathbb{X} = \{z \mid \|z\|_M \leq \rho\}$. Indeed, for $z \notin \mathbb{X}$, it can be verified that

$$\left\|\nabla^2 \mathcal{P}(z)\, d_j\right\| = \rho \left\|\nabla^2 \|z\|_M\, d_j\right\| \leq c\, \|z\|_M^{-1} \tag{5}$$

holds for $c = \rho\left(\|M\| + \|M\|^2\right)$ and any unitary vector d_j.

III. NMPC based on $\mathcal{P}$-penalty functions

In the following, the index i is reserved for the current time instants, while the index k is used for the predicted times. In order to clearly distinguish between them, we will use x_i, q_i as the physical states of the system and s_k, I_k as their predictions in the future via the model (1), i.e. we will have:

$$s_0 = x_i, \quad I_0 = q_i \tag{6a}$$
$$s_{k+1} = f(s_k, u_k), \qquad I_{k+1} = I_k + \mathcal{I}(I_k, u_k). \tag{6b}$$

In the following, for referring to a specific state j in q_i and I_k, we will use $q_{i,j}$ and $I_{k,j}$ respectively.

In order to formulate the NMPC problem, let us denote the stage cost related to the non-quadrature states and controls as $\phi(x,u)$. Without loss of generality, we will consider here the problem of driving the system to the origin, therefore $\phi(0,0) = 0$. A classical form of NMPC scheme with terminal equality constraints for system (1) then reads as:

$$\varphi_2(x_i, q_i) = \min_{s,u,I} \quad \sum_{k=0}^{N-1} \phi(s_k, u_k) + \frac{1}{2} I_k^\top W I_k \tag{7a}$$
$$\text{s.t.} \quad s_{k+1} = f(s_k, u_k), \quad s_0 = x_i, \tag{7b}$$
$$I_{k+1} = I_k + \mathcal{I}(s_k, u_k), \quad I_0 = q_i, \tag{7c}$$
$$s_N = 0, \quad I_N = 0, \tag{7d}$$
$$h(s_k, u_k) \leq 0, \quad k = 0, ..., N-1. \tag{7e}$$

where h stands for the set of state and input constraints, and the stage cost is commonly chosen as $\phi(s_k, u_k) = \frac{1}{2} s_k^\top Q s_k + \frac{1}{2} u_k^\top R u_k$ where Q and R are user-defined weighting matrices.

As anticipated in the previous sections, classical NMPC schemes like (7) can result in excessively aggressive control actions for large deviations of the quadrature states from the given reference. We propose next an NMPC scheme which is stabilising but at the same time insensitive to large deviations of q_i from the origin. In order to achieve this goal, we (a) replace the quadratic penalty function $\sum_{k=0}^{N-1} \frac{1}{2} I_k^\top W I_k$ in the NMPC scheme (7) by a $\mathcal{P}$-penalty function and (b) remove the terminal constraint $I_N = 0$. The proposed scheme takes the following form:

$$\varphi(x_i, q_i) = \min_{s,u,I} \quad \sum_{k=0}^{N-1} \phi(s_k, u_k) + \sum_{k=0}^{N} P(I_k, I_N), \tag{8a}$$
$$s.t. \quad s_{k+1} = f(s_k, u_k), \quad s_0 = x_i, \tag{8b}$$
$$I_{k+1} = I_k + \mathcal{I}(s_k, u_k), \quad I_0 = q_i, \tag{8c}$$
$$s_N = 0, \quad h(s_k, u_k) \leq 0, \tag{8d}$$

where

$$P(I_k, I_N) = \max\{\mathcal{P}(I_k),\, \mathcal{P}(I_N)\}. \tag{9}$$

Removing the terminal constraints $I_N = 0$ from the proposed scheme (8) is required in order to establish Theorem 1, which will state that when the system state is far enough from the reference, the NMPC control solution is not affected anymore by further changes in the system states. At a more intuitive level, the need for removing this terminal constraint should come as fairly clear. Indeed, if the NMPC scheme has to steer the system state to the origin in the allotted prediction horizon N, then a very large deviation of the system states from their reference has to trigger an aggressive response from the NMPC scheme, as a very large correction of the system state is required within that horizon. A too large deviation leads typically to the NMPC scheme being infeasible. In order for the NMPC scheme to not deliver an aggressive response to a large deviation of the system state from the reference, this terminal constraint has to be removed.

One can observe the peculiar form of the penalty P applied to the quadrature states $I_{k=0,...,N}$. Since no terminal penalty nor terminal set is used on the terminal quadrature state, this formulation plays the role of a terminal cost, and is instrumental in establishing a proof of stability for the NMPC scheme (8). In the following, we will use the notation

$$\Phi(s,u) = \sum_{k=0}^{N-1} \phi(s_k, u_k), \tag{10}$$

and we will assume throughout this paper that:

$$\overline{\alpha}(s,u) \geq \Phi(s,u) \geq \underline{\alpha}(s,u) \quad \text{and} \quad \nabla\Phi(0,0) = 0, \tag{11}$$

where $\overline{\alpha}$, $\underline{\alpha}$ are K_∞ functions. E.g. positive-definite quadratic cost functions fulfil these criteria.

IV. Insensitivity to large errors of the quadrature states

In this section, we will show that the control input u delivered by the NMPC scheme (8) is insensitive to the initial conditions q_i when they are sufficiently large. The definition of insensitivity to the quadrature state we will use here is

$$\frac{\partial u^\star}{\partial q_i} = 0, \tag{12}$$

where $u^\star$ is the control input computed by the NMPC scheme for the given system states x_i, q_i.

In the following, for the simplicity of the analysis it will be useful to consider problem (8) in the context of single shooting, where the states are eliminated from the problem via simulating the dynamics (8b)-(8c), so that s and I can be viewed as functions of the inputs u and the initial conditions x_i, q_i only. Furthermore, we consider a slack formulation [7] of (9) so as to construct a reformulation of (8) based on differentiable functions only. We then rewrite (8) in the equivalent form:

$$\varphi(x_i,q_i) = \min_{u,w} \quad \Phi\left(s\left(u,x_i\right),u\right) + \sum_{k=0}^{N} w_k, \tag{13a}$$

$$\text{s.t.} \quad s_N\left(u,x_i\right) = 0, \tag{13b}$$

$$h\left(s_k\left(u,x_i\right),u\right) \leq 0, \tag{13c}$$

$$w_k \geq \mathcal{P}\left(I_k\left(u,x_i,q_i\right)\right), \tag{13d}$$

$$w_k \geq w_N. \tag{13e}$$

In this subsection, it will be established that the control policy delivered by the NMPC scheme (13) becomes insensitive to q_i for large values of $|q_i|$. This statement is further discussed at the end of the following theorem.

Theorem 1: if $\mathcal{P}$ is a $\mathcal{P}$-penalty function, for any given initial conditions x_i, the solution to (13) is insensitive to changes in the initial quadrature state q_i in the direction d_j if the predicted quadrature state $I_k\left(u^*,x_i,q_i\right) \notin \mathbb{X}_j$, $\forall k$, where u^* is solution of (13).

Proof: We define the Lagrange function:

$$\begin{aligned} \mathcal{L} = \; & \Phi\left(s\left(u,x_i\right),u\right) + \sum_{k=0}^{N} w_k \\ & + \mu^\top h\left(s_k\left(u,x_i\right),u\right) + \lambda_N^\top s_N\left(u,x_i\right) + \\ & + \sum_{k=0}^{N} \nu_k\left(\mathcal{P}\left(I_k\left(u,x_i,q_i\right)\right) - w_k\right) + \sum_{k=0}^{N-1} \eta_k\left(w_N - w_k\right), \end{aligned} \tag{14}$$

where we omit the arguments in $\mathcal{L}$ for the sake of brevity. The solution to (13) satisfies the stationarity conditions [8]:

$$\nabla_u \mathcal{L} = 0, \quad \nabla_w \mathcal{L} = 0. \tag{15}$$

We will show next that if the current quadrature state q_i is such that $I_k\left(u^*,x_i,q_i\right) \notin \mathbb{X}_j$, $\forall k$, the conditions (15) are insensitive to variations of the initial conditions q_i along the direction d_j, i.e. a small change in the direction $\Delta q_i = d_j$ does not impact the control actions of the proposed NMPC scheme.

First we can observe from (14) that $\nabla_w \mathcal{L}$ is independent of q_i, and therefore insensitive to variations in q_i in any direction. Additionally,

$$\nabla^2_{u,q_i} \mathcal{L} = \sum_{k=0}^{N} \nu_k \nabla^2_{u,q_i} \mathcal{P}\left(I_k\left(u,x_i,q_i\right)\right),$$

and we observe that since the quadrature state trajectory $I_{0,\ldots,N}$ depends linearly on the initial conditions of the quadrature state, i.e. q_i:

$$\frac{\partial I_k\left(u,x_i,q_i\right)}{\partial q_i} = \mathbb{I}, \tag{16}$$

for any time k. It follows that for any input u_j:

$$\frac{\partial}{\partial u_j} \frac{\partial I_k\left(u,x_i,q_i\right)}{\partial q_i} = 0 \tag{17}$$

holds, and consequently:

$$\begin{aligned} & \nabla^2_{u,q_i} \mathcal{P}\left(I_k\left(u,x_i,q_i\right)\right) \\ & \quad = \frac{\partial I_k\left(u,x_i,q_i\right)}{\partial u}^\top \left.\frac{\partial^2 \mathcal{P}(z)}{\partial z^2}\right|_{z=I_k\left(u,x_i,q_i\right)} \frac{\partial I_k\left(u,x_i,q_i\right)}{\partial q_i} \\ & \quad = \frac{\partial I_k\left(u,x_i,q_i\right)}{\partial u}^\top \left.\frac{\partial^2 \mathcal{P}(z)}{\partial z^2}\right|_{z=I_k\left(u,x_i,q_i\right)}. \end{aligned} \tag{18}$$

Using property 4 of the $\mathcal{P}$-penalty functions, it then holds that

$$\nabla^2_{u,q_i} \mathcal{P}\left(I_k\left(u,x_i,q_i\right)\right) d_j = 0, \tag{19}$$

and, in turn,

$$\nabla^2_{u,q_i} \mathcal{L}\left(u,x_i,q_i\right) d_j = 0, \tag{20}$$

such that (15) is insensitive to changes in the initial conditions of the quadrature state q_i in the direction d_j. We additionally observe that the constraints:

$$h\left(s_k\left(u^*,x_i\right),u\right) \leq 0, \qquad s_N\left(u^*,x_i\right) = 0, \tag{21}$$

are insensitive to q_i, such that they remain feasible upon perturbing the initial conditions q_i. ■

Implications of Theorem 1:

1) The NMPC control policy loses its sensitivity to the initial condition $q_{i,j}$ for every value of the quadrature state j far enough from its reference, such that the optimal control trajectory of the prediction corresponding to that state, i.e. $I_{k,j}$, $k = 0,\ldots,N+1$, lies entirely outside the central set $\mathbb{X}_j$.
2) In order to get that effect, one shall design convex central sets $\mathbb{X}_{j=1,\ldots,n_q}$ such that $q_{i,j} \notin \mathbb{X}_j$ for large values of $q_{i,j}$. Additionally, it is reasonable to design the collection of central sets $\mathbb{X}_{1,..,n_q}$ such that $\left(\bigcup_{j=1,\ldots,n_q} \mathbb{X}_j\right)^{\mathrm{c}}$ is non empty, and such that $q_i \notin \bigcup_{j=1,\ldots,n_q} \mathbb{X}_j$ when $\|q_i\|$ is large. One may observe that the example of $\mathcal{P}$-penalty constructed from the Huber penalty function fulfils these criteria, and is arguably a good choice of $\mathcal{P}$-penalty (see Figure 1 for an illustration).
3) In practice, the NMPC control policy enters a gradual insensitivity to the current value $q_{i,j}$ of a quadrature state j as it gets further away from the reference and as more and more elements k of the predicted trajectories $I_{k,j}$ leave the central set $\mathbb{X}_j$.

We now turn to extending Theorem 1 to quasi $\mathcal{P}$-penalty functions. In this case, the insensitivity holds asymptotically, i.e. for q_i large.

Theorem 2: Consider a quasi $\mathcal{P}$-penalty function $\mathcal{P}$. For any given initial conditions x_i, the solution to (13) is asymptotically insensitive to changes in the direction d_j if $I_k\left(u^*,x_i,q_i\right) \notin \mathbb{X}_j$, $\forall k$, where u^* is the solution of (13). We moreover assume that $\frac{\partial I_k\left(u,x_i,q_i\right)}{\partial u}$ is bounded at the optimal solution for any x_i, q_i.

Proof: The proof proceeds along the same lines as Theorem 1, yielding

$$\left\|\nabla^2_{u,q_i} \mathcal{P}\left(I_k\left(u,x_i,q_i\right)\right) d_j\right\| \leq c \left\|\frac{\partial I_k\left(u,x_i,q_i\right)}{\partial u}\right\| \left\|I_k\left(u,x_i,q_i\right)\right\|_Q^{-1},$$

where we used (16), such that

$$\left\|\nabla^2_{u,q_i}\mathcal{L}\left(u,x_i,q_i\right)d_j\right\| \leq c\sum_{k=0}^{N}|\nu_k|\left\|\frac{\partial I_k\left(u,x_i,q_i\right)}{\partial u}\right\|\left\|I_k\left(u,x_i,q_i\right)\right\|_Q^{-1}.$$

■

V. Nominal stability of NMPC based on $\mathcal{P}$-penalties

It shall be observed here that the proposed scheme is a mixture of NMPC with and without terminal constraints. Removing all terminal constraints in the proposed scheme would allow one to use stability results for NMPC without terminal constraints [1]. However, the relatively simple assumptions required here to establish the nominal stability of the proposed scheme would be turned into the fairly convoluted ones resulting from the analysis of stability for NMPC without terminal constraints. Our goal here is to obtain a stability proof that is as similar as possible to the classic proof of nominal stability for NMPC with terminal constraints. The proposed NMPC scheme will be instrumental in achieving that goal.

In this section, the nominal stability of the NMPC scheme (8) is investigated. First we consider the case of a terminal point constraint. Second, we extend the stability theory to the case of a terminal cost and set constraint.

1) Terminal point constraint : The following theorem establishes that, under some conditions, $\varphi\left(x_i,q_i\right)$ is a Lyapunov function of system (1) controlled by the NMPC scheme (8). First, four key assumptions are introduced.

1) $\mathcal{I}(0,0)=0$ and $f(0,0)=0$.
2) The origin is feasible.
3) The min-max problem:

$$\max_{D\in\mathcal{D}}\ \min_{\Delta u,\Delta s,\Delta I,\Delta w}\ \Phi\left(\Delta s,\Delta u\right)+\sum_{k=0}^{N}\Delta w_k \tag{22a}$$

$$\text{s.t.}\quad \Delta s_{k+1}=A\Delta s_k+B\Delta u_k, \tag{22b}$$

$$\Delta I_{k+1}=\Delta I_k+C\Delta s_k, \tag{22c}$$

$$\Delta s_0=0,\quad \Delta s_N=0,\quad \Delta I_0=0, \tag{22d}$$

$$\Delta w_k\geq D\Delta I_k,\quad \Delta w_k\geq\Delta w_N, \tag{22e}$$

where

$$A=\nabla_s f\left(0,0\right)^\top\in\mathbb{R}^{n\times n},\quad B=\nabla_u f\left(0,0\right)^\top\in\mathbb{R}^{n\times m},$$
$$C=\nabla\mathcal{I}\left(0\right)^\top\in\mathbb{R}^{n_q\times n}$$

yields a negative cost. Here $\mathcal{D}=\{\mathcal{P}(z)\,|\,z\in\mathbb{R}^{n_q}\}$.
4) The system is stabilisable and the value function $\varphi(x_i,q_i)$ is upper bounded by a $\mathcal{K}$ function $\tilde{\alpha}_2$ in an arbitrarily small neighbourhood $\mathcal{N}$ of the origin.
5) Function h constrains the states x and controls u to be in a compact set, while the quadrature states q are possibly unconstrained.

Note that Assumptions 4 and 5 are rather standard in the literature on MPC stability, see e.g. [9]. Assumption 3 can be checked offline and, as shown in [6], it is straightforward to check in the case $n_q=1$. However, to the authors best knowledge, checking it in case several quadrature states need to be handled requires solving (22), which can be computationally demanding. From a practical point of view, it appears that Assumption 3 is typically satisfied for N sufficiently long. In order to gain some intuition on this statement, note that Assumption 3 entails that a trajectory $x=0$, $u=0$ and $q\neq0$ is not a solution of the problem. It follows that there exists a feasible perturbation around $x=0$, $u=0$ that allows the quadrature state for approaching the origin. A typical condition in which this fails to be the case is when the horizon is so short that there is not enough control authority to do so.

Theorem 3: Let $\mathbb{X}$ be the set of feasible initial conditions (x_0,q_0) for problem (8). Then, under Assumptions 1-5, the optimal cost function $\varphi\left(x_i,q_i\right)$ is a Lyapunov function for the nominal closed-loop system:

$$x_{i+1}=f(x_i,u^*\left(x_i,q_i\right)),\qquad q_{i+1}=q_i+\mathcal{I}(x_i,u^*\left(x_i,q_i\right))$$

in the set $\mathbb{X}$, where $u^*\left(x_i,q_i\right)$ is the control delivered by the NMPC scheme (8), i.e. the first element of the optimal control input sequence $u^*_{0,\ldots,N}(x_i,q_i)$ delivered by (8).

Proof: By design of the stage cost, the cost function $\varphi\left(x_i,q_i\right)$ is bounded from below, i.e. $\varphi\left(x_i,q_i\right)\geq\alpha_1(\|x_i\|+\|q_i\|)$ with $\alpha_1(\cdot)\in\mathcal{K}_\infty$.

Proving the existence of the upper bound $\varphi\left(x_i,q_i\right)\leq\alpha_2(\|x_i\|+\|q_i\|)$ with $\alpha_2(\cdot)\in\mathcal{K}_\infty$ is in general more difficult and some additional assumptions are needed. Using Assumption 4, i.e. $\varphi\left(x_i,q_i\right)\leq\tilde{\alpha}_2(\|x_i\|+\|q_i\|)$ on $\mathcal{N}$, [9, Proposition 11] and Assumption 5, we can extend the upper bound to any arbitrarily large compact set $\mathcal{N}_2$ on which $\varphi\left(x_i,q_i\right)$ is bounded, i.e. $\varphi\left(x_i,q_i\right)\leq\bar{\alpha}_2(\|x_i\|+\|q_i\|)$, for all $(x_i,q_i)\in\mathcal{N}_2$. As there are no bounds on q_i, we will make use next of Theorem 1 to establish that $\varphi\left(x_i,q_i\right)\leq\alpha_2(\|x_i\|+\|q_i\|)$ on the domain of Problem (8). Let us denote the j-th component of vector q_i as $q_{i,j}$. Choose $\mathcal{N}_2$ large enough such that, for any point (x_i,q_i) on the boundary $\partial\mathcal{N}_2$, $I_k(u^*,x_i,q_i)\notin\mathbb{X}_j,\forall k=0,\ldots,N$. Then, for any $\bar{q}_i$ with one or more components j such that $\bar{q}_{i,j}\geq q_{i,j}$, it holds that $\varphi\left(x_i,\bar{q}_i\right)=\varphi\left(x_i,q_i\right)+N\left(\mathcal{P}(\bar{q}_i)-\mathcal{P}(q_i)\right)\leq\varphi\left(x_i,q_i\right)+N\mathcal{P}(\bar{q}_i)$. Then, by choosing $\alpha_2(\|x_i\|+\|q_i\|)=\bar{\alpha}_2(\|x_i\|+\|q_i\|)+N\mathcal{P}(q_i)$, one finally gets

$$\varphi\left(x_i,q_i\right)\leq\alpha_2(\|x_i\|+\|q_i\|).$$

This proves that the MPC value function is upper and lower bounded by $\mathcal{K}_\infty$ functions. In the following we will prove descent of the MPC value function along all closed loop trajectories.

First an upper bound for $\varphi\left(x_{i+1},q_{i+1}\right)-\varphi\left(x_i,q_i\right)$ is computed. In the absence of perturbation and model error, the initial values at time $i+1$ match the predicted trajectories, i.e.:

$$x_{i+1}=s_1^*(x_i,q_i),\qquad q_{i+1}=I_1^*(x_i,q_i).$$

We then consider the shifted trajectories (where the arguments x_i, q_i are omitted):

$$s_+ = \begin{bmatrix} s_1^* \\ \cdots \\ s_N^* \\ \mathbf{0}_{n\times 1} \end{bmatrix}, \quad I_+ = \begin{bmatrix} I_1^* \\ \cdots \\ I_N^* \\ I_N^* \end{bmatrix}, \quad u_+ = \begin{bmatrix} u_1^* \\ \cdots \\ u_{N-1}^* \\ \mathbf{0}_{n_u\times 1} \end{bmatrix}, \quad (23)$$

which, from Assumptions 1-2, are feasible for Problem (8) with the initial values x_{i+1}, q_{i+1}. This observation ensures the recursive feasibility of (8). The trajectories (23) yield the cost function φ_+ given by:

$$\varphi_+ = \varphi(x_i, q_i) - \phi(x_i, u_0^*) - P(I_0^*, I_N^*) + \mathcal{P}(I_N^*),$$

and by optimality

$$\varphi_+ \geq \varphi(x_{i+1}, q_{i+1})$$

holds. It follows that:

$$\varphi(x_{i+1}, q_{i+1}) - \varphi(x_i, q_i, N) \leq \xi(x_i, q_i),$$

with

$$\xi(x_i, q_i) = -\phi(x_i, u_0^*) - P(q_i, I_N^*) + \mathcal{P}(I_N^*). \quad (24)$$

Finally, from (9)

$$P(q_i, I_N^*) \geq \mathcal{P}(I_N^*), \quad (25)$$

it follows that

$$\varphi(x_{i+1}, q_{i+1}) - \varphi(x_i, q_i) \leq \xi(x_i, q_i) \leq -\phi(x_i, u_0^*) \leq -\bar{\alpha}(\|x_i\|),$$

i.e. $\varphi(x_i, q_i)$ is decreasing with the rate $-\bar{\alpha}(\|x_i\|)$.

Next, it is established by contradiction that $\varphi(x_i, q_i)$ is non-decreasing only at the reference, i.e.

$$\xi(x_i, q_i) = 0 \;\Rightarrow\; x_i = 0, q_i = 0. \quad (26)$$

First it can be verified that the implication:

$$\xi(x_i, q_i) = 0 \quad \Rightarrow \quad x_i = 0, \quad P(I_0^*, I_N^*) = \mathcal{P}(I_N^*), \quad (27)$$

follows from (24)-(25), $I_0^* = q_i$, $\phi(\cdot,\cdot) \geq 0$ and $\mathcal{P}(\cdot) \geq 0$. Using Definition (9), it further yields:

$$\xi(x_i, q_i) = 0 \quad \Rightarrow \quad x_i = 0, \quad \mathcal{P}(I_0^*) \leq \mathcal{P}(I_N^*). \quad (28)$$

We then observe that the trajectory $u = 0$, $s = 0$, $I = q_i$ is feasible for (8) with $x_i = 0$, and provides the upper bound:

$$\varphi(0, q_i) \leq \Phi(0,0) + \sum_{k=0}^{N} P(q_i, q_i) = (N+1)\mathcal{P}(q_i). \quad (29)$$

It follows that, noting s^*, u^*, I^* the optimal trajectory of (8) for $x_i = 0$:

$$\begin{aligned} &(N+1)\mathcal{P}(q_i) \geq \varphi(0, q_i) = \\ &\Phi(s^*, u^*) + \sum_{k=0}^{N} P(I_k^*, I_N^*) \geq \Phi(s^*, u^*) + \sum_{k=0}^{N} \mathcal{P}(I_N^*) \geq \\ &\Phi(s^*, u^*) + \sum_{k=0}^{N} \mathcal{P}(I_0^*) = \Phi(s^*, u^*) + (N+1)\mathcal{P}(q_i), \end{aligned}$$

which implies that $\Phi(s^*, u^*) = 0$. Hence, if $\xi(x_i, q_i) = 0$, then the optimal trajectory is

$$u^* = 0, \quad s^* = 0, \quad I^* = q_i. \quad (30)$$

We now turn to proving that if $\xi(x_i, q_i) = 0$ then $q_i = 0$ necessarily follows. We proceed with showing by contradiction that the trajectories (30) are not a solution of (8) unless $q_i = 0$. We consider the slack formulation of (8):

$$\begin{aligned} \varphi(0, q_i) = \min_{u,s,I,w} \quad & \Phi(s, u) + \sum_{k=0}^{N} w_k && (31a) \\ s.t. \quad & s_{k+1} = f(s_k, u_k), \quad s_0 = 0, s_N = 0, && (31b) \\ & I_{k+1} = I_k + \mathcal{I}(s_k, u_k), \quad I_0 = q_i, && (31c) \\ & w_k \geq \mathcal{P}(I_k), \quad w_k \geq w_N. && (31d) \end{aligned}$$

If (30) is a solution to (31) then since QP (22) is the linearization of NLP (31), it takes the trivial solution $\Delta s = 0, \Delta u = 0, \Delta I = 0, \Delta w = 0$. This latest observation is, however, in contradiction with Assumption 3 whenever $q_i \neq 0$. Since QP (22) cannot have a trivial solution, it follows that the cost function of problem (31) at the solution (30) with $q_i \neq 0$ is not a local minimum, and therefore not a global one. As a result, $\xi(x_i, q_i) = 0$ entails $q_i = 0$.

We finally turn to bounding $\xi(x_i, q_i)$ in both its arguments. First, it can be verified that any trajectory with $x_i = 0$ and $\mathcal{P}(I_N^*) > \mathcal{P}(q_i)$ yields a larger cost in problem (31) than trajectory (30), which is itself not a solution of (31) if $q_i \neq 0$. From this fact and Assumption 3, it follows that at the solution of (31), the inequality $P(q_i, I_N^*) = \mathcal{P}(q_i) > \mathcal{P}(I_N^*)$ must hold for all $q_i \neq 0$. Then function $\beta(q_i) := \mathcal{P}(q_i) - \mathcal{P}(I_N^*)$ is by construction a positive definite function, i.e. $\beta(0) = 0$, $\beta(q_i) > 0$, for all $q_i \neq 0$. This means that

$$\xi(0, q_i) \leq -\beta(q_i).$$

and function $\varphi(x_i, q_i)$ has the following properties on $\mathbb{X}$:

$$\varphi(x_{i+1}, q_{i+1}) - \varphi(x_i, q_i) \leq \xi(x_i, q_i),$$

with

$$\xi(x_i, q_i) \leq -\phi(x_i, u_0^*)$$

and

$$\xi(0, q_i) \leq -\beta(q_i).$$

■

Discussion of Theorem 3 :

- It should be observed that the quadrature cost (9) plays a key role in the stability result established in Theorem 3, since it is needed to ensure that a) $\varphi(x_i, q_i)$ is non-increasing, and that b) $x_i = 0$, $q_i = 0$ is the only point where $\varphi(x_i, q_i)$ is non-decreasing. Since it gives a particular importance to the terminal quadrature state I_N, the penalty on the quadrature state (9) can be in some sense construed as a special form of terminal cost.
- It is unfortunate that Theorem 3 does not allow the construction of a convergence rate for the quadrature states. Indeed, while one can arguably expect an exponential convergence of the regular states x_i when the cost function Φ has the adequate properties, the same cannot generally be done for the quadrature states. In practice, however, a linear convergence rate seems to be usually achieved on the quadrature states.

- We have assumed in the NMPC schemes (7)-(8) that the quadrature states are unconstrained. The introduction of constraints on the quadrature states is however, not a major problem in the context proposed here. In the presence of constraints on the quadrature states, the result of Theorem 1 remains valid as long as the quadrature state constraints are not active. Moreover, it can be verified that the stability analysis presented in Theorem 3 is not affected by the introduction of quadrature state constraints.
- There is no limitation on how short the horizon can be in order for the proposed NMPC scheme to be stable. However, for a too short horizon, one typically observes the failure of Assumption 3, which invalidates the result of Theorem 3

2) Stability using a terminal set constraint: In this section we provide the extension of the nominal stability proof to the case where a terminal set constraint $\mathbb{T}(I_N)$ and terminal cost $T(s_N)$ is used in the NMPC scheme, i.e. we consider the problem:

$$\varphi(x_i,q_i) = \min_{s,u,I} \quad T(s_N) + \Phi(s,u) + \sum_{k=0}^{N} P(I_k, I_N) \tag{32a}$$

$$s.t. \quad s_{k+1} = f(s_k,u_k), \quad s_0 = x_i, \tag{32b}$$

$$I_{k+1} = I_k + \mathcal{I}(s_k,u_k), \quad I_0 = q_i, \tag{32c}$$

$$s_N \in \mathbb{T}(I_N), \quad h(s_k,u_k) \leq 0. \tag{32d}$$

Nominal stability hinges on the following theorem.

Theorem 4: Assume that the assumptions of Theorem 3 hold. Assume moreover that there exist a control law $K(x_i)$ such that:

- $u = K(x_i)$ exists and is feasible $\forall x_i \in \mathbb{T}(I_N)$;
- $f(x_i, K(x_i)) \in \mathbb{T}(I_N)$ holds $\forall x_i \in \mathbb{T}(I_N)$;
- $\mathbb{T}(I_N) \subseteq \mathbb{X}, \ \forall \ I_N$;
- The following inequalities holds $\forall \ x_i \in \mathbb{T}(I_N)$:

$$T(f(x_i, K(x_i))) - T(x_i) \leq -\phi(x_i, K(x_i)), \tag{33a}$$

$$\mathcal{P}(q_i + \mathcal{I}(x_i, K(x_i))) - \mathcal{P}(q_i) \leq 0. \tag{33b}$$

Then, the optimal cost function of the NMPC scheme (32) is a Lyapunov function for the nominal closed-loop system:

$$x_{i+1} = f(x_i, u^*(x_i,q_i)), \qquad q_{i+1} = q_i + \mathcal{I}(x_i, u^*(x_i,q_i)),$$

in the set $\mathbb{X}$, where $u^*(x_i,q_i)$ is the control delivered by the NMPC scheme (32), i.e. the first element of the optimal control input sequence $u^*_{0,\ldots,N}(x_i,q_i)$ delivered by (32).

Proof: First, we observe that Equation (33b) entails that

$$P(I_N^*, I_N^* + \mathcal{I}(s_N^*, u_{\mathrm{T}})) \leq \mathcal{P}(I_N^*), \tag{34}$$

where we use the short notation $u_{\mathrm{T}} = K(s_N^*)$. We then consider the shifted state and control sequence

$$s_+ = \begin{bmatrix} s_1^* \\ \ldots \\ s_N^* \\ f(s_N^*, u_{\mathrm{T}}) \end{bmatrix}, \quad I_+ = \begin{bmatrix} I_1^* \\ \ldots \\ I_N^* \\ I_N^* + \mathcal{I}(s_N^*, u_{\mathrm{T}}) \end{bmatrix},$$

$$u_+ = \begin{bmatrix} u_1^* \\ \ldots \\ u_{N-1}^* \\ u_{\mathrm{T}} \end{bmatrix}, \tag{35}$$

which is feasible for problem (32) for x_{i+1}, q_{i+1}. The shifted trajectories (35) yield the cost

$$\begin{aligned}\varphi_+ =& \varphi(x_i,q_i) + T(f(s_N^*,u_{\mathrm{T}})) - T(s_N^*) \\ &+ \phi(s_N^*,u_{\mathrm{T}}) - \phi(s_0^*,u_0^*) + \sum_{k=1}^{N} P(I_k, I_N^* + \mathcal{I}(S_N^*,u_{\mathrm{T}})) \\ &- \sum_{k=0}^{N} P(I_k, I_N^*) + \mathcal{P}(I_N^* + \mathcal{I}(S_N^*,u_{\mathrm{T}})).\end{aligned}$$

Equation (33a) entails that

$$T(f(s_N^*,u_{\mathrm{T}})) - T(s_N^*) + \phi(s_N^*,u_{\mathrm{T}}) \leq 0.$$

Moreover, from (33b), we obtain

$$P(I_k, I_N^* + \mathcal{I}(S_N^*,u_{\mathrm{T}})) - P(I_k, I_N^*) \leq 0, \tag{36}$$

such that

$$\sum_{k=1}^{N} P(I_k, I_N^* + \mathcal{I}(S_N^*,u_{\mathrm{T}})) - \sum_{k=0}^{N} P(I_k, I_N^*) \leq -P(I_0, I_N^*),$$

and

$$\varphi_+ \leq \varphi(x_i,q_i) - \phi(s_0^*,u_0^*) - P(I_0,I_N^*) + \mathcal{P}(I_N^* + \mathcal{I}(S_N^*,u_{\mathrm{T}})).$$

Using (36) it follows from the properties of the function P that

$$-P(I_0,I_N^*) \leq -P(I_0, I_N^* + \mathcal{I}(S_N^*,u_{\mathrm{T}})) \leq -\mathcal{P}(I_N^* + \mathcal{I}(S_N^*,u_{\mathrm{T}})),$$

and finally

$$\varphi(x_{i+1},q_{i+1}) - \varphi(x_i,q_i) \leq -\phi(s_0^*,u_0^*).$$

The remaining of the proof follows the same arguments as in Theorem 3. ■

Discussion of Theorem 4: The dependence of the terminal set $\mathbb{T}$ on I_N is arguably crucial in order to be able to construct a control law that achieves conditions (33). Indeed, in general, the quadrature states may be very insensitive to the control inputs u, the extreme case being that function $\mathcal{I}$ is not function of u. In such a case, (33b) hinges on having an adequate terminal state s_N such that

$$\mathcal{P}(I_N^* + \mathcal{I}(s_N^*)) \leq \mathcal{P}(I_N^*)$$

always holds. This cannot generally hold if the terminal set $\mathbb{T}$ is independent of I_N^*. Indeed, let us e.g. assume that $\mathcal{I}(s_N^*) = s_N^* \in \mathbb{R}$, such that

$$\mathcal{P}(I_N^* + s_N^*) \leq \mathcal{P}(I_N^*)$$

must always hold. Such a condition clearly enforces I_N^*-dependent restrictions on s_N^*.

VI. Implementation

It this section, we discuss the implementation of (8) using the $\mathcal{P}$-penalty (4) based on the Huber penalty function. A difficulty arising from using Huber penalties is their non-smoothness. Indeed, on can observe that the Hessian of the Huber penalty function is not Lipschitz continuous at $z_i = \rho_i$, i.e.:

$$\lim_{\varepsilon \to 0_+} \left| \nabla^2 \mathcal{H}_{\rho_i}(\rho_i + \varepsilon) - \nabla^2 \mathcal{H}_{\rho_i}(\rho_i) \right| = W_i, \tag{37}$$

and requires extra care in the context of Newton-type techniques. To tackle that issue, non-smooth Newton techniques or first-order techniques can be deployed. In this section, we investigate instead the deployment of a classical reformulation of the Huber penalty function [7] into a smooth problem for tackling numerically the NMPC scheme (8). We observe, however, that some of the constraints proposed in [7] can be removed, as they are unnecessary and counterproductive for a real-time implementation.

We aim here at providing a smooth reformulation of (8) when a Huber penalty is used, which is suitable for Newton-type methods, and minimising the computational demand. Note that here we ought to assume that functions f and $\mathcal{I}$ are everywhere at least twice differentiable. We provide it in the form of the following proposition:

Proposition 1: The NMPC scheme

$$\varphi(x_i, q_i) = \min_{s,u,I,\mu,\nu} \Phi(s,u) + \sum_{k=1}^{N} \rho^\top W \nu_k + \frac{1}{2}\mu_k^\top W \mu_k \tag{38a}$$

$$\begin{aligned} s.t. \quad & s_{k+1} = f(s_k, u_k), \quad I_{k+1} = I_k + \mathcal{I}(s_k, u_k), \\ & I_0 = q_i, \quad s_0 = x_i, \quad s_N = 0, \\ & h(s_k, u_k) \leq 0, \quad \forall k, \\ & \nu_k \geq 0, \quad \forall k > 0, & (38b) \\ & -\mu_k - \nu_k \leq I_k \leq \mu_k + \nu_k, \quad \forall k > 0, & (38c) \\ & \mu_N + \nu_N \leq \mu_k + \nu_k, \quad \forall 0 < k < N, & (38d) \end{aligned}$$

is equivalent to (8). Here the slack variables have the dimension $\mu_k, \nu_k \in \mathbb{R}^{n_q}$, such that all constraints are imposed element-wise.

Proof: We will proceed by showing that the slack variables μ_k, ν_k together with the associated constraints yield the cost function (8). It is helpful to note that:

$$\rho^\top W \nu_k + \frac{1}{2}\mu_k^\top W \mu_k = \sum_{j=1}^{n_q} W_j \left(\frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_{k,j} \right), \tag{39}$$

since W is diagonal, such that in the following we will investigate the terms $\frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_{k,j}$ individually.

First, we observe that because of the ℓ_1 penalty on $\nu_{k=1,\ldots,N}$ and the ℓ_2 penalty on $\mu_{k=1,\ldots,N}$, at every time stage $k = 1, \ldots, N$ in (38), either $|I_k| \leq \mu_k + \nu_k$ or $\mu_k + \nu_k \geq \mu_N + \nu_N$ are necessarily active for every quadrature state. Therefore $\mu_k + \nu_k = \max\{|I_k|, \mu_N + \nu_N\} = \max\{|I_k|, |I_N|\}$.

In the following, in order to prove some properties of the optimisation Problem (38), we will use the following modified slack variables

$$\tilde{\mu}_{k,j} = \mu_{k,j} - \Delta, \qquad \tilde{\nu}_{k,j} = \nu_{k,j} + \Delta,$$

where Δ will be defined differently for each property that we will prove. Note that this choice results in $\tilde{\mu}_{k,j} + \tilde{\nu}_{k,j} = \mu_{k,j} + \nu_{k,j}$, which makes slack variables $\tilde{\mu}_{k,j}$, $\tilde{\nu}_{k,j}$ feasible with respect to Constraint (38c) and (38d). Feasibility w.r.t. Constraint (38b) will be proven for each specific case individually.

We proceed then with showing that any solution to (38) satisfies $0 \leq \mu_k \leq \rho$. First suppose that (38) yields $\mu_{k,j} > \rho_j$ for some time stage k and quadrature state j. Then define $\Delta = \mu_{k,j} - \rho_j > 0$ and consider the modified slack variables:

$$\tilde{\mu}_{k,j} = \mu_{k,j} - \Delta = \rho_j, \quad \tilde{\nu}_{k,j} = \nu_{k,j} + \Delta > 0,$$

so that slack variables $\nu_{k,j}$ are feasible w.r.t. Constraint (38b). The contribution of $\tilde{\mu}_{k,j}$, $\tilde{\nu}_{k,j}$ to the cost function reads:

$$\begin{aligned} \frac{1}{2}\tilde{\mu}_{k,j}^2 + \rho_j \tilde{\nu}_{k,j} &= \frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_{k,j} + \Delta\left(\rho_j - \mu_{k,j} + \frac{1}{2}\Delta\right) \\ &= \frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_k - \frac{1}{2}\Delta^2 \\ &< \frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_{k,j}, \end{aligned}$$

hence a solution to (38) cannot admit a $\mu_{k,j} > \rho$.

Secondly, suppose that (38) yields $\mu_{k,j} < 0$ for some time stage k and quadrature state j. Defining $\Delta = \mu_{k,j} < 0$, the modified slack variables read as:

$$\tilde{\mu}_{k,j} = \mu_{k,j} - \Delta = 0, \qquad \tilde{\nu}_{k,j} = \nu_{k,j} + \Delta = \nu_{k,j} + \mu_{k,j}.$$

Constraint (38c) enforces $\tilde{\nu}_{k,j} = \mu_{k,j} + \nu_{k,j} \geq |I_{k,j}| \geq 0$, so that the modified slack variables $\tilde{\nu}_{k,j}$ are feasible w.r.t. Constraint (38b). The contribution of $\tilde{\mu}_{k,j}$, $\tilde{\nu}_{k,j}$ to the cost function reads:

$$\frac{1}{2}\underbrace{\tilde{\mu}_{k,j}^2}_{=0} + \rho_j \tilde{\nu}_{k,j} = \rho_j \nu_{k,j} + \underbrace{\rho_j \Delta}_{<0} < \frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_{k,j},$$

since $\Delta < 0$ and $\rho_j > 0$. Therefore a solution to (38) cannot admit a $\mu_{k,j} < 0$.

We then turn to proving that Constraint (38d) entails that $\mu_N \leq \mu_k$ and $\nu_N \leq \nu_k$. Suppose that (38) yields $\mu_{N,j} > \mu_{k,j}$ for some time k and quadrature state j. We define $\Delta = \mu_{k,j} - \mu_{N,j} < 0$, and the modified slack variables as:

$$\begin{aligned} &\tilde{\mu}_{k,j} = \mu_{k,j} - \Delta = \mu_{N,j}, \\ &\tilde{\nu}_{k,j} = \nu_{k,j} + \Delta = \nu_{k,j} + \mu_{k,j} - \mu_{N,j} \geq \nu_{N,j} \geq 0, \end{aligned}$$

so that slack variables $\tilde{\nu}_{k,j}$ are feasible w.r.t. Constraint (38b). The contribution of $\tilde{\mu}_{k,j}$, $\tilde{\nu}_{k,j}$ to the cost function reads:

$$\begin{aligned} &\frac{1}{2}\tilde{\mu}_{k,j}^2 + \rho_j \tilde{\nu}_{k,j} = \\ &\qquad = \frac{1}{2}\mu_k^2 + \rho_j \nu_{k,j} + \Delta\left(\rho_j - \mu_{k,j} + \frac{1}{2}\left(\mu_{k,j} - \mu_{N,j}\right)\right) \\ &\qquad = \frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_{k,j} + \underbrace{\Delta\left(\rho_j - \frac{1}{2}\mu_{k,j} - \frac{1}{2}\mu_{N,j}\right)}_{<0} \\ &\qquad < \frac{1}{2}\mu_{k,j}^2 + \rho_j \nu_{k,j}, \end{aligned}$$

since $\Delta < 0$ and $\mu_{k,j} < \mu_{N,j} \leq \rho_j$. Hence a solution to (38) cannot admit $\mu_k < \mu_N$. Suppose now that (38) yields $\nu_{N,j} >$

$\nu_{k,j} \geq 0$ for some time k and quadrature state j. Then defining $\Delta = \rho_j - \mu_{N,j} > 0$, the modified slack variables read as:

$$\tilde{\mu}_{N,j} = \mu_{N,j} - \Delta, \qquad \tilde{\nu}_{N,j} = \nu_{N,j} + \Delta > 0,$$

so that slack variables $\tilde{\nu}_{N,j}$ are feasible w.r.t. Constraint (38b). The contribution of $\tilde{\mu}_{k,j}$, $\tilde{\nu}_{k,j}$ to the cost function reads:

$$\begin{aligned}\frac{1}{2}\tilde{\mu}_{N,j}^2 + \rho\tilde{\nu}_{N,j} &= \frac{1}{2}\mu_{N,j}^2 + \rho\nu_{N,j} + \Delta\left(\rho_j - \mu_{N,j} + \frac{1}{2}\Delta\right) \\ &= \frac{1}{2}\mu_{N,j}^2 + \rho_j\nu_{N,j} - \frac{1}{2}\Delta^2 \\ &< \frac{1}{2}\mu_{N,j}^2 + \rho_j\nu_{N,j},\end{aligned}$$

hence a solution to (38) cannot admit $\nu_{k,j} < \nu_{N,j}$.

We finally turn to prove that if $\mu_{k,j} < \rho_j$, then $\nu_{k,j} = 0$. Suppose that we have $\mu_{k,j} < \rho_j$, and $\nu_{k,j} > 0$ for some time k and some quadrature state j. Then defining $\Delta = \rho_j - \mu_{k,j} > 0$, the modified slack variables read as:

$$\tilde{\mu}_{k,j} = \mu_{k,j} - \Delta = \rho_j, \quad \tilde{\nu}_{k,j} = \nu_{k,j} + \Delta = \mu_{k,j} + \nu_{k,j} - \rho_j \geq 0,$$

The modified cost function reads as:

$$\begin{aligned}\frac{1}{2}\tilde{\mu}_{k,j}^2 + \rho_j\tilde{\nu}_{k,j} &= \rho_j\mu_{k,j} + \rho_j\nu_{k,j} - \frac{1}{2}\rho_j^2 \\ &= -\frac{1}{2}\left(\mu_{k,j} - \rho_j\right)^2 + \frac{1}{2}\mu_{k,j}^2 + \rho_j\nu_{k,j} \\ &< \frac{1}{2}\mu_{k,j}^2 + \rho_j\nu_{k,j},\end{aligned}$$

such that $\mu_{k,j} < \rho_j$, and $\nu_{k,j} > 0$ cannot hold.

We summarise next the previous observations and conclude. At a solution of (38) the following holds $\forall k = 1,\ldots,N$:

$$0 \leq \mu_k \leq \rho, \quad \mu_N \leq \nu_k, \quad \nu_N \leq \nu_k, \tag{40a}$$
$$\mu_{k,j} < \rho_j \quad \Rightarrow \quad \nu_{k,j} = 0, \tag{40b}$$
$$\mu_k + \nu_k = \max\{|I_k|, |I_N|\}. \tag{40c}$$

The above result entails that $\mu_N + \nu_N = |I_N|$. Using the definition of Huber penalty, we obtain

$$\sum_{j=1}^{n_q} w_j\left(\frac{1}{2}\mu_{N,j}^2 + \rho_j\nu_{N,j}\right) = \mathcal{P}(I_N).$$

Moreover, from Equation (40c) we obtain

$$w_j\left(\frac{1}{2}\mu_{k,j}^2 + \rho_j\nu_{k,j}\right) = \max\{\mathcal{P}(I_k), \mathcal{P}(I_N)\}.$$

■

We remark here that reformulation (38) allows one to use state-of-the-art solvers for NMPC also for $\mathcal{P}$-penalty NMPC in an efficient way. However, an increase in the computational time is to be expected, compared to standard NMPC formulations. The development of new tailored algorithms is expected to reduce this gap and is the object of ongoing research.

VII. Illustrative Example

The problem of managing large deviations of the quadrature states from a given reference is crucial in the context of airborne applications, where the reference position provided to the NMPC scheme can be arbitrarily far away from the current position of the machine, resulting in very aggressive manoeuvres when a classical NMPC scheme is used. Such difficulties have been e.g. observed in the context of NMPC-based aircraft control [5]. In this section, we offer a comparison of the proposed approach on a simulated quadcopter, for which we want to be allowed to provide an arbitrary position reference.

A. Quadcopter Model

We use here a quaternion-based model, which avoids the singularity of a classical approach based on Euler angles [5]. The quadcopter dynamics read as:

$$\dot{p} = v, \tag{41a}$$
$$\dot{v} = m^{-1}RF - g\mathbf{1}_z, \tag{41b}$$
$$\dot{\omega} = J^{-1}\left(T + \omega \times J\omega\right), \tag{41c}$$
$$\dot{q} = \frac{1}{2}E^\top\omega, \tag{41d}$$
$$\dot{\Omega}_k = J_{\mathrm{p}}^{-1}\left(T_k + (-1)^k\frac{1}{2}\rho A C_{\mathrm{D}}\Omega_k^2\right), \tag{41e}$$

where $p \in \mathbb{R}^3$ is the quadcopter position, which coincides with the only three quadrature states of the model, $v \in \mathbb{R}^3$ its linear velocity, $\omega \in \mathbb{R}^3$ its angular velocity, and $q \in \mathbb{R}^4$ is the quaternion vector defining its orientation. Vectors $\Omega_k \in \mathbb{R}^3$, $k = 1,\ldots,4$ are the propellers angular velocities, and $T_k \in \mathbb{R}$, $k = 1,\ldots,4$ are the motor torques. Matrix $R \in \mathbb{R}^{3\times3}$ is the rotation matrix between the inertial reference frame and the quadcopter reference frame, and is given by:

$$R = EG^\top, \tag{42a}$$
$$G = \begin{bmatrix} -q_2 & q_1 & q_4 & -q_3 \\ -q_3 & -q_4 & q_1 & q_2 \\ -q_4 & q_3 & -q_2 & q_1 \end{bmatrix}, \tag{42b}$$
$$E = \begin{bmatrix} -q_2 & q_1 & -q_4 & q_3 \\ -q_3 & q_4 & q_1 & -q_2 \\ -q_4 & -q_3 & q_2 & q_1 \end{bmatrix}. \tag{42c}$$

The vectors $T, F \in \mathbb{R}^3$ are the torque and force acting on the quadcopter in its own reference frame, they read as:

$$T = \sum_{k=1}^{4} Le_k \times \mathbf{1}_z\frac{1}{2}\rho A C_{\mathrm{L}}\Omega_k^2 - \mathbf{1}_z(-1)^k\frac{1}{2}\rho A C_{\mathrm{D}}\Omega_k^2, \tag{43a}$$
$$F = \sum_{k=1}^{4}\mathbf{1}_z\frac{1}{2}\rho A C_{\mathrm{L}}\Omega_k^2. \tag{43b}$$

Here $\mathbf{1}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ and

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

The model and control parameters are summarized in Table I.

TABLE I
MODEL PARAMETERS

Parameter	Value	Unit
m	10	(kg)
J	$\begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$(\text{kg}\cdot\text{m}^2)$
J_P	10^{-2}	$(\text{kg}\cdot\text{m}^2)$
g	9.81	(ms^{-2})
ρ	1.23	$(\text{kg}\cdot\text{m}^{-3})$
A	0.1	(m^2)
C_D	0.075	$(\text{Nm}^2\text{s}^2\cdot\text{kg}^{-1})$
C_L	0.25	$(\text{Nms}^2\cdot\text{kg}^{-1})$
L	0.5	(m)

B. Control Problem and Simulation Results

Without loss of generality, we consider the problem of driving the quadcopter to the position $p=0$ from an arbitrary starting point. The states and inputs read as:

$$s=\begin{bmatrix} v \\ \omega \\ q \\ \Omega_1 \\ \dots \\ \Omega_4 \end{bmatrix}, \qquad I=p, \qquad u=\begin{bmatrix} T_1 \\ \dots \\ T_4 \end{bmatrix}.$$

We discretise the dynamics (41) using a multiple-shooting approach [10], with a sampling time $\Delta t = 0.2\,\text{s}$. The position dynamics then take the form

$$p_{k+1} = p_k + \int_{t_k}^{t_{k+1}} v(\tau)\,d\tau,$$

such that we look at the function $\mathcal{I}(s_k,u_k)$ as:

$$\mathcal{I}(s_k,u_k) = \int_{t_k}^{t_{k+1}} v(\tau)\,d\tau,$$

where $v(\tau)$, $\tau\in[t_k,\,t_{k+1}]$ is given by the position-independent dynamics (41b)-(41e). The problem reads as:

$$\min_{s,u,I} \ \frac{1}{2}\sum_{k=0}^{N-1}\left(s_k^\top Q s_k + u_k^\top R u_k + I_k^\top W I_k\right) + \sum_{k=0}^{N} P(I_k,I_N) \quad (44a)$$

$$\text{s.t.} \quad s_{k+1} = f(s_k,u_k), \quad s_0 = x_i, \quad (44b)$$

$$I_{k+1} = I_k + \mathcal{I}(s_k,u_k), \quad I_0 = q_i, \quad (44c)$$

$$s_N = 0, \quad (44d)$$

$$h(s_k,u_k) \le 0, \quad k=0,\dots,N-1, \quad (44e)$$

where function h enforces the following constraints:

$$T\in[-8.83, 8.83]^4, \qquad \dot{T}\in[-10,10]^4. \quad (45)$$

We considered 4 different controllers:

(a) standard NMPC formulation, with $P(I_k,I_N)=0$ and $W=\text{diag}(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix})$;
(b) $\mathcal{P}$-penalty NMPC formulation, with $\rho=0.15$, $P(I_k,I_N)=\max(\sum_{i=1}^3 \mathcal{H}_\rho(I_{k,i}), \sum_{i=1}^3 \mathcal{H}_\rho(I_{N,i}))$ and $W=0$;
(c) ℓ_2 quasi $\mathcal{P}$-penalty formulation, with $\rho = 0.25$, $P(I_k,I_N)=\max(\mathcal{H}_\rho(\|I_k\|_2), \mathcal{H}_\rho(\|I_N\|_2))$ and $W=0$;
(d) ℓ_∞ quasi $\mathcal{P}$-penalty formulation, with $\rho = 0.25$, $P(I_k,I_N)=\max(\mathcal{H}_\rho(\|I_k\|_\infty), \mathcal{H}_\rho(\|I_N\|_\infty))$ and $W=0$.

TABLE II
CONTROL PARAMETERS

Parameter	Value	Unit
Δt	0.2 [s]	Sampling time
T_h	10 [s]	Time horizon
ρ_x, ρ_y	0.15	Quad. zone for x and y
ρ_z	0.25	Quad. zone for z
$w_{x,y,z}$	1	Huber weights
R	diag ([5 ... 5])	Input weights
Q	diag ([0.1 ... 0.1])	ℓ_2 penalty weights

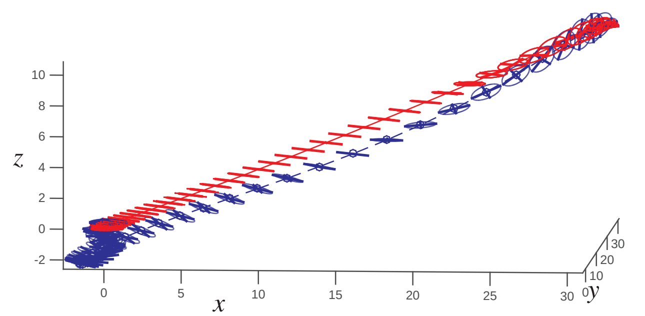

Fig. 3. Three-dimensional trajectory of the quadcopter using the $\mathcal{P}$-penalty approach (b), displayed in red, and the classical NMPC scheme (a), displayed in blue. The classical NMPC scheme overshoots the reference position.

All simulations were performed by solving the NMPC problems to full convergence. The $\mathcal{P}$-penalty problem was formulated in a smooth way, i.e. as Problem (38). The quasi-$\mathcal{P}$-penalty problems were formulated by introducing the slack variable π_k to be used in the Huber penalty, i.e. as $\mathcal{H}_\rho(\pi_k)$, with the following additional constraints:

$$\pi_k \ge \|p_k\|_2, \quad (46)$$

for the ℓ_2 quasi-$\mathcal{P}$-penalty; and

$$\pi_k \ge \pm x_k, \qquad \pi_k \ge \pm y_k, \qquad \pi_k \ge \pm z_k, \quad (47)$$

for the ℓ_∞ quasi-$\mathcal{P}$-penalty. Note that constraint (46) is a (convex) conic constraint, while constraints (47) are linear.

We implemented each controller both using the endpoint constraint (44d) and no terminal constraint. Note that, when having no terminal constraint for $\mathcal{P}$-penalty and quasi $\mathcal{P}$-penalty formulations we replaced $P(I_k,I_N)$ with $P(I_k,I_k)$, i.e. we removed constraint (38d). The first formulation illustrates the theoretical developments of this paper. The second formulation is very common in practical applications and a proof of stability can be found in [1], under the main assumption of having a long enough horizon.

A prediction horizon of $N=51$ was used, corresponding to a time horizon of $T_\text{h}=(N-1)\,T_\text{S}=10\,[\text{s}]$. Trajectories were simulated, using the initial position $[x,\,y,\,z]=[30,\,30,\,10]$, all other states being initialised at steady-state. The 3D trajectory of the quadcopter is reported in Figure 3 and 4, where it can be seen that the standard NMPC scheme is much more aggressive than the newly proposed formulations. The evolution of the inputs, torques, velocity and position are displayed in Figure 7, 6 and 5 respectively. It can be seen that the control inputs $\dot{T}_{1,\dots,4}$ are very aggressive and motor torques $T_{1,\dots,4}$ undergo

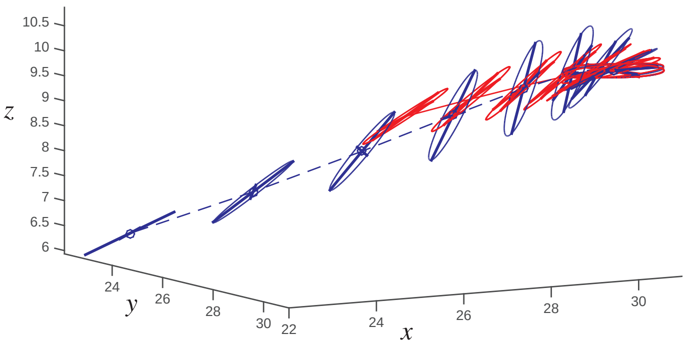

Fig. 4. First 1.5 s of the three-dimensional trajectory of the quadcopter using the $\mathcal{P}$-penalty approach (b), displayed in red, and the classical NMPC scheme (a), displayed in blue.Because of the classical ℓ_2-penalty on the position, the classical NMPC scheme adopts much more extreme attitudes than the proposed NMPC scheme.

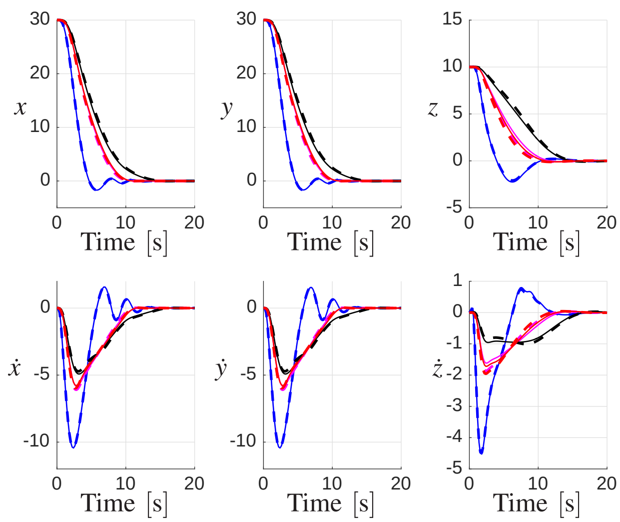

Fig. 5. Position and velocity of the quadcopter. Classical NMPC formulation (a) in blue, $\mathcal{P}$-penalty formulation (b) in red, ℓ_2 quasi $\mathcal{P}$-penalty formulation in magenta, ℓ_∞ quasi $\mathcal{P}$-penalty formulation in black; schemes with terminal point constraint in thick dashed line, schemes without terminal constraint in continuous line. Note that the magenta lines are almost undistinguishable from the red ones by eye inspection. The classical NMPC scheme adopts significantly higher velocities and yields more oscillations than the NMPC schemes using the newly proposed formulations.

significant saturation only for the classic NMPC scheme, while they have a much less aggressive behaviour for the $\mathcal{P}$-penalty and quasi $\mathcal{P}$-penalty formulations. Moreover, the proposed schemes deliver a control move that requires less banking than the classic NMPC scheme. Finally, it can be observed in Figure 5 that the proposed schemes yield less overshoot while reaching the reference steady-state within a time similar to that of the standard NMPC formulation. Note that, while the ℓ_∞ quasi $\mathcal{P}$-penalty formulation takes longer to reach the steady state, by choosing larger values of ρ, e.g. $\rho = 0.5$, the steady state is reached in a time similar to that of the other formulations.

The much more desirable behaviour of the $\mathcal{P}$-penalty and quasi $\mathcal{P}$-penalty formulations is explained by Theorem 1: those penalties penalise large deviations of the position from its reference less than the corresponding quadratic penalty.

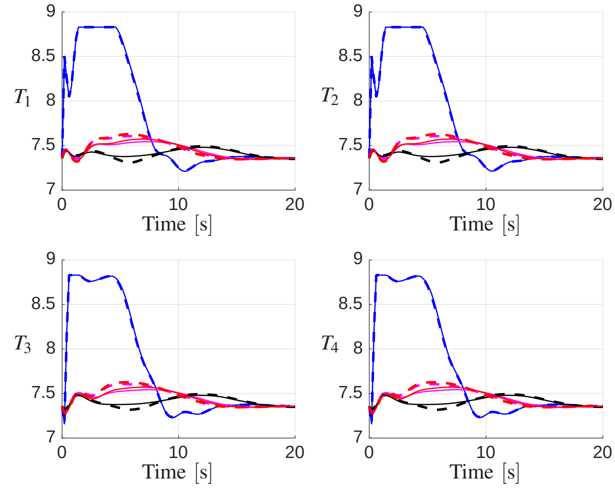

Fig. 6. Propeller torques of the quadcopter. Classical NMPC formulation (a) in blue, $\mathcal{P}$-penalty formulation (b) in red, ℓ_2 quasi $\mathcal{P}$-penalty formulation in magenta, ℓ_∞ quasi $\mathcal{P}$-penalty formulation in black; schemes with terminal point constraint in thick dashed line, schemes without terminal constraint in continuous line. Note that the magenta lines are almost undistinguishable from the red ones by eye inspection. The classical NMPC scheme uses much higher propeller torques, reaching the saturation of the actuators than the NMPC schemes using the newly proposed formulations.

Therefore, less aggressive control actions are preferred, the non-quadrature states are kept closer to their reference and the position is initially stabilised at a slower rate and the reference is attained in a less oscillatory and abrupt way.

C. Real-Time Feasible Implementation

The previous simulation results were obtained by using a generic implementation of the optimal control solver. In this subsection, we investigate using an algorithm tailored for fast NMPC implementations, known as the Real Time Iteration (RTI) scheme [11]. For all details on the method and its theoretical justification, we refer to [11] and references therein. In short, the method consists in taking a single full Newton step per time instant. For the RTI simulations, we used a code-generated RK4 integrator, and the QP solver FORCES [12]. The computational times obtained for the quadratic NMPC scheme and the $\mathcal{P}$-penalty scheme are reported in Figure 8. It can be observed that, though the implementation of the $\mathcal{P}$-penalty scheme increases significantly the computational burden of the NMPC scheme, it does not jeopardize the real-time feasibility of the proposed example. We did not report the control and state trajectories since they are undistinguishable by eye inspection from the ones obtained in the previous subsection.

Note that the ℓ_∞ quasi $\mathcal{P}$-penalty formulation can be implemented in a similar way to the $\mathcal{P}$-penalty formulation, i.e. using a QP solver. It requires less slack variables, but the same amount of constraints as the $\mathcal{P}$-penalty formulation.

The ℓ_2 quasi $\mathcal{P}$-penalty formulation, requires the same slack variables as the ℓ_∞ quasi $\mathcal{P}$-penalty formulation. However, it also requires a conic solver, due to constraint (46).

Finally, because of the specific structure of $\mathcal{P}$-penalty and quasi $\mathcal{P}$-penalty formulations, tailored algorithms should be investigated in order to increase the computational efficiency.

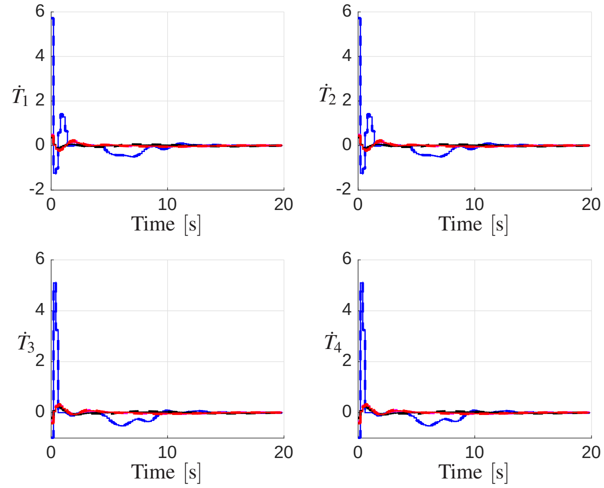

Fig. 7. Control inputs of the NMPC schemes, i.e. time derivatives of the propeller torques of the quadcopter. Classical NMPC formulation (a) in blue, $\mathcal{P}$-penalty formulation (b) in red, ℓ_2 quasi $\mathcal{P}$-penalty formulation in magenta, ℓ_∞ quasi $\mathcal{P}$-penalty formulation in black; schemes with terminal point constraint in thick dashed line, schemes without terminal constraint in continuous line. Note that the magenta lines are almost undistinguishable from the red ones by eye inspection. The classical NMPC scheme uses much more aggressive control inputs than the NMPC schemes using the newly proposed formulations.

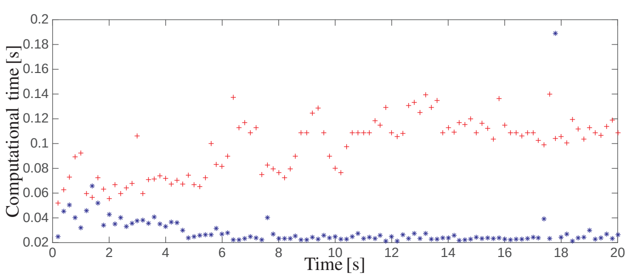

Fig. 8. Computational time for the classic NMPC scheme (a), displayed as blue star signs, and for the $\mathcal{P}$-penalty NMPC scheme (b), using the formulation (38), displayed as red plus signs. Both schemes were implemented without terminal constraints.

VIII. Conclusion & Future work

This paper proposes a class of NMPC schemes based on a specific type of penalty function that addresses the shortcoming of more classical quadratic-penalty based NMPC schemes when dealing with large deviations of quadrature states from their reference. The proposed scheme behaves as a standard NMPC scheme when the system is close to its reference, but yields significantly less aggressive control action when far from its reference. The tuning is intuitive, and based on a single parameter. The properties of the $\mathcal{P}$-penalty NMPC are established formally, including its nominal stability. The closed-loop behaviour of the proposed scheme is illustrated using a simulated quadcopter.

The smooth problem formulation introduces extra computational burden in the NMPC scheme. This computational burden can arguably be alleviated by using tailored approaches to solve the underlying QP problems. This question is the object of current research.

References

[1] L. Grüne, J. Pannek, Nonlinear Model Predictive Control, Springer, London, 2011.

[2] J. B. Rawlings, R. Amrit, Optimizing Process Economic Performance using Model Predictive Control, in: Proceedings of NMPC 08 Pavia, 2009, pp. 119–138.

[3] F. Allgöwer, Z. Nagy, R. Findeisen, Nonlinear model predictive control: From theory to applications, in: Proc. Int. Symp. Design, Operation and Control of Chemical Plants (PSE), 2002.

[4] D. Schlipf, D. Schlipf, M. Kuehn, Nonlinear Model Predictive Control of Wind Turbines Using LIDAR, Wind Energy.

[5] S. Gros, R. Quirynen, M. Diehl, Aircraft Control Based on Fast Nonlinear MPC & Multiple-shooting, in: Conference on Decision and Control, 2012.

[6] S. Gros, M. Diehl, NMPC based on Huber Penalty Functions to Handle Large Deviations of Quadrature States, in: American Control Conference, 2013.

[7] S. Boyd, L. Vandenberghe, Convex Optimization, University Press, Cambridge, 2004.

[8] J. Nocedal, S. Wright, Numerical Optimization, 2nd Edition, Springer Series in Operations Research and Financial Engineering, Springer, 2006.

[9] J. B. Rawlings, D. Q. Mayne, Model Predictive Control: Theory and Design, 2012, Ch. Postface to model predictive control: theory and design.

[10] H. Bock, K. Plitt, A multiple shooting algorithm for direct solution of optimal control problems, in: Proceedings 9th IFAC World Congress Budapest, Pergamon Press, 1984, pp. 242–247.

[11] M. Diehl, R. Findeisen, S. Schwarzkopf, I. Uslu, F. Allgöwer, H. Bock, E. Gilles, J. Schlöder, An Efficient Algorithm for Nonlinear Model Predictive Control of Large-Scale Systems. Part I: Description of the Method, Automatisierungstechnik 50 (12) (2002) 557–567.

[12] A. Domahidi, A. Zgraggen, M. Zeilinger, M. Morari, C. Jones, Efficient Interior Point Methods for Multistage Problems Arising in Receding Horizon Control, in: IEEE Conference on Decision and Control (CDC), Maui, HI, USA, 2012, pp. 668 – 674.

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