Tracking MPC Tuning in Continuous Time: a First-Order Approximation of Economic MPC

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***Abstract*— Economic MPC (EMPC) optimizes closed-loop performance by directly minimizing a given objective function, as opposed to Tracking MPC (TMPC) which instead penalizes deviations from a precalculated optimal reference. The main difference between the two approaches can be observed during transients, as the former always acts optimally, while the latter is only optimal when the reference is accurately tracked. Unfortunately, stability for EMPC is in general difficult to prove, as opposed to TMPC which builds on a rich theory. Additionally, many efficient algorithms are available for TMPC, while solving the EMPC problem can be much harder. In prior works [1], [2], a family of discrete-time TMPC schemes that provide approximate economic optimality has been developed in order to partially overcome these issues. In this paper, we aim at extending such a family of TMPC schemes to the continuous time case. Similarly to the discrete-time case, also in continuous-time we obtain a first-order approximation of the EMPC control law. We demonstrate the theory with a numerical example that confirms the first-order approximation and we show that our continuous-time formulation can be made equivalent to the discrete-time one.**

***Index Terms*— Optimal control, Predictive control for linear systems, Predictive control for nonlinear systems.**

I. Introduction

MODEL Predictive Control (MPC) consists in repeatedly solving an optimal control problem online in order to define a closed-loop control policy. The benefits of MPC include the ability to handle nonlinear constrained dynamics and multiple inputs. Traditionally, MPC is based on a tracking approach, where a positive-definite (typically quadratic) cost function is minimized. This makes it possible to provide stability guarantees. Because in many cases the control performance cannot be easily captured by a positive-definite function, Tracking MPC (TMPC) misses the opportunity to exploit the optimization procedure in order to maximize the desired performance during transients. MPC schemes that directly optimize the performance criterion are usually called Economic MPC (EMPC). The weak points of EMPC are: (a) the difficulty of establishing stability guarantees [3]–[5]; and (b) the difficulty in developing computationally efficient algorithms [6]–[10].

Stability guarantees have been first obtained in [11], and then further analyzed using a strict dissipativity condition in [12]–[15] in discrete time. In [16]–[18] the convergence is studied using the turnpike property of the underlying optimal control problem in continuous time. A stability proof in the absence of terminal constraints is given in [19]. The stability guarantees of discrete-time systems with periodic constraints have been analyzed in, among others, [20]–[22]. In [23]–[25] the relation between dissipativity condition and turnpike properties has been analyzed. We refer to [5] for a complete overview on stability and performance of EMPC.

Stability-enforcing approaches that do not alter the performance criterion are based on strict dissipativity, which is, however, very hard to check in practical applications. This observation motivates the development of TMPC schemes tuned so as to approximately optimize the given economic criterion. Such schemes have been proposed in discrete time in [1], [2], [22]. Moreover, the need for a tracking cost with nonzero gradient in order to correctly approximate the EMPC control law has been discussed in [26], [27]. In this paper, we propose a strategy to compute a positive definite tracking cost function for a nonlinear MPC (NMPC) scheme formulated in continuous time. We will show that our formulation of the tracking positive definite NMPC (PD NMPC) delivers a feedback law that is first-order equivalent [2, Definition 1 (iii)] to that of EMPC. Finally, since the obtained PD NMPC scheme has a positive-definite quadratic cost, the efficient optimization algorithms for real-time TMPC can be directly exploited.

The remainder of the paper is structured as follows. In Section II we formulate the problem; we detail the cost-tuning procedure in Section III. We provide simulations in Section IV and conclude the paper in Section V.

II. Problem Formulation

We consider a time-invariant nonlinear continuous-time systems, described by time $t \in \mathbb{R}$, states $x \in \mathbb{R}^{n_x}$, controls $u \in \mathbb{R}^{n_u}$, and ordinary differential equations

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

subject to the inequality constraints

$$h(x(t), u(t)) \geq 0. \tag{2}$$

The cost to be minimized is, ideally, the infinite-horizon performance

$$J(x(t), u(t)) = \int_{t=0}^{\infty} l(x(t), u(t)) \, \mathrm{d}t. \tag{3}$$

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MPC approximately solves the problem above by truncating the infinite prediction horizon to a finite one $T < \infty$, such that the problem to be solved online reads

$$\min_{x(\cdot),u(\cdot)} \quad \int_{t=0}^{T} l(x(t),u(t)) \ \mathrm{d}t + V_\mathrm{f}(x(T)) \tag{4a}$$
$$\text{s.t.} \quad x(0) - \hat{x}_0 = 0, \tag{4b}$$
$$\dot{x}(t) - f(x(t),u(t)) = 0, \quad t \in [0,T], \tag{4c}$$
$$h(x(t),u(t)) \geq 0, \qquad t \in [0,T], \tag{4d}$$
$$x(T) \in \mathbb{X}_\mathrm{f}, \tag{4e}$$

where V_f defines a terminal cost which, in order to obtain the best performance, should approximate the cost-to-go of the infinite-horizon problem; $\mathbb{X}_\mathrm{f}$ defines a terminal set; $\hat{x}_0$ is the initial state. The terminal cost and constraint are usually introduced in order to obtain closed-loop stability guarantees, which are obtained in case the former is a Lyapunov function on $\mathbb{X}_\mathrm{f}$ and the latter is positive invariant. However, this is not yet sufficient to guarantee asymptotic stability, as the cost further needs to satisfy the conditions

$$l(x,u) \geq \alpha(\|x - x_\mathrm{s}\|), \ \forall\, u; \quad l(x_\mathrm{s},u_\mathrm{s}) = 0, \tag{5}$$

for some steady-state pair $(x_\mathrm{s},u_\mathrm{s})$, i.e., $f(x_\mathrm{s},u_\mathrm{s}) = 0$ [28]; for the sake of simplicity and without loss of generality, we will assume throughout the paper that $(x_\mathrm{s},u_\mathrm{s}) = (0,0)$. In (5), $\alpha : \mathbb{R} \to \mathbb{R}$ is a class-$\mathcal{K}$ function. The running cost l, when satisfying (5), is commonly referred to as *tracking cost*, and we will denote it by l^t. Running costs that do not satisfy such property are commonly referred to as *economic costs* and we will denote them by l^e.

Because the choice of terminal stabilizing conditions is beyond the scope of this paper, for the sake of simplicity we will consider a terminal point constraint, i.e., $\mathbb{X}_\mathrm{f} = \{x_\mathrm{s}\}$, for some suitable x_s. The extension to the general case is straightforward and, therefore, omitted here.

In this paper we focus on the case of optimal steady-state operation. Clearly, there do exist notable cases in which the optimal operation is not stationary, but rather, e.g., periodic [21], [22], [29]. We ought to stress that, in fact, the results of this paper will be the starting point to also cover the periodic case, which is the subject of ongoing research.

In order to characterize the optimal steady state, we introduce the following problem:

$$(x_\mathrm{s},u_\mathrm{s}) = \arg\min_{x,u} \ l(x,u) \tag{6a}$$
$$\text{s.t.} \quad f(x,u) = 0, \quad h(x,u) \geq 0, \tag{6b}$$

which yields the optimal steady-state $x_\mathrm{s}, u_\mathrm{s}$ and the optimal Lagrange multipliers λ_s, μ_s associated respectively with the steady-state and inequality constraints.

For the case of an economic cost, the asymptotic stability of the optimal steady-state has been recently proved by using arguments from dissipativity theory. The necessary condition on the running cost to obtain stability is then called strict dissipativity and requires the existence of a so-called storage function $\xi : \mathbb{R}^{n_x} \to \mathbb{R}$ such that

$$L(x,u) = l(x,u) + \nabla_x \xi^\top f(x,u) \geq \alpha(\|x - x_\mathrm{s}\|). \tag{7}$$

For more details on the topic, we refer to [5], [25] and references therein. Note that here we formulate the dissipation inequality in continuous time, while the vast majority of the results are formulated in discrete time, where strict dissipativity takes a slightly different form.

While dissipativity theory for economic MPC is sound and well-developed, the main issue associated with it is the difficulty of proving the existence of a storage function ξ satisfying (7). Indeed, this is often an insurmountable challenge in practice with the notable exception of linear systems with quadratic costs, for which the storage function is quadratic. To address this issue, tracking MPC which approximately optimizes the economic criterion while delivering stability guarantees have been proposed in [1], [2], [22]. In these works, the MPC problem is formulated in discrete time, such that, for continuous-time systems, the MPC sampling time must be fixed a priori, i.e., *before* computing the positive-definite quadratic cost. In this paper, we extend these ideas to continuous time, such that the positive-definite quadratic cost can be computed once and independently of the sampling time, which can be therefore treated as a tuning parameter to be selected afterwards.

III. Cost Tuning

In this section we propose a procedure that yields a PD NMPC scheme with quadratic cost, hence with easy-to-establish stability guarantees, whose closed-loop control law $\nu^\mathrm{PDN}(x)$ approximates the closed-loop control law $\nu^\mathrm{EN}(x)$ of the nonlinear economic MPC scheme up to the first order, i.e., $\nu^\mathrm{PDN}(x) = \nu^\mathrm{EN}(x) + O(\|x - x_\mathrm{s}\|_2^2)$. The procedure to derive the PD NMPC starting from an economic cost can be subdivided into three intermediate steps. The first one consists in linearizing the economic NMPC problem around the optimal steady-state given by (6), in order to obtain an Economic Linear MPC (ELMPC) scheme; the second step computes a positive definite running cost used to define a PD LMPC scheme; finally, the last step yields the desired PD NMPC formulation which is built such that its linearization is the PD LMPC scheme obtained at the previous step. The whole procedure can be schematized as follows:

$$\text{ENMPC} \leftrightarrow \text{ELMPC} \leftrightarrow \text{PD LMPC} \leftrightarrow \text{PD NMPC},$$

where we use the symbol $\leftrightarrow$ to denote that the MPC schemes to its left and right yield closed-loop control laws that coincide up to first order.

This procedure succeeds in finding a positive definite cost function if the original ENMPC is asymptotically stable in a neighborhood of the optimal steady-state, and, if it fails, then the ENMPC is unstable.

In order to be able to prove the desired result, we need the following assumption.

Assumption 1: MPC Problem (4) is feasible for a nonempty set of initial conditions, has a unique solution satisfying linear independence constraint qualification (LICQ) and second order sufficient conditions (SOSC), and the obtained closed-loop system is asymptotically stable. Moreover, functions f and l are twice continuously differentiable and the MPC Problem (4) is regular positive at the optimal steady-state from (6) in the sense of [26, Definition 1].

A. First-order equivalence between ENMPC and ELMPC

We define the following ELMPC problem:

$$\min_{x(\cdot),u(\cdot)} \int_{t=0}^{T} \frac{1}{2} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^\top W \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + q^\top \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t \tag{8a}$$

$$\text{s.t.} \quad x(0) - \hat{x}_0 = 0, \tag{8b}$$

$$\dot{x}(t) - Ax(t) - Bu(t) = 0, \quad t \in [0, T], \tag{8c}$$

$$Cx(t) + Du(t) + e \geq 0, \quad t \in [0, T], \tag{8d}$$

$$x(T) = 0. \tag{8e}$$

The matrices above are defined as follows:

$$W := \nabla_w^2 \mathcal{H}(w, \lambda, \mu), \qquad q := \nabla_w l^{\mathrm{e}}(x, u), \tag{9a}$$

$$A := \nabla_x f(x, u)^\top, \qquad B := \nabla_u f(x, u)^\top, \tag{9b}$$

$$C := \nabla_x h(x, u)^\top, \qquad D := \nabla_u h(x, u)^\top, \tag{9c}$$

where $w := [x, u]^\top$ and $\mathcal{H}(w, \lambda, \mu)$ is the Hamiltonian evaluated at time t of the economic NMPC defined as:

$$\mathcal{H}(x, u, \lambda, \mu) = l^{\mathrm{e}}(x, u) + \langle \lambda, f(x, u) \rangle + \langle \mu, h(x, u) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product and we omit the dependence on time for the sake of readability. All expressions in Equation (9) are evaluated at the optimal primal-dual steady state from (6). By construction, Problems (8) and (4) satisfy the first-order equivalence condition, as proven in the next lemma [26].

Lemma 1 ([26, Lemma 2]): Consider an MPC Problem formulated as in (4), with optimal steady-state given by (6). If Assumption 1 holds, then Problem (8) yields a first-order approximation of Problem (4), i.e.,

$$a^{\mathrm{EL}}(x) = a^{\mathrm{EN}}(x) + O(\|x - x_{\mathrm{s}}\|^2), \text{ with } a \in \{x, u, \lambda, \mu\}$$

B. Positive-Definite Linear MPC

We analyze next the second first-order equivalence, i.e., ELMPC $\leftrightarrow$ PD LMPC, where we define PD LMPC as:

$$\min_{x(\cdot),u(\cdot)} \int_{t=0}^{T} \frac{1}{2} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^\top \tilde{W} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + q^\top \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t \tag{10a}$$

$$\text{s.t.} \quad (8b) - (8e), \tag{10b}$$

with $\tilde{W} \succ 0$. Note that the only difference between PD LMPC Scheme (10) and ELMPC Scheme (8) is in the Hessian of the running cost, as in general $W \nsucc 0$.

Theorem 1 below establishes that there exists a Hessian matrix $\tilde{W}$ which makes the cost positive definite, without changing the feedback law of Problem (8).

Theorem 1: Let us consider a region $\mathbb{X}_0$ of initial states $\hat{x}_0$ for which the set of active constraints at all prediction times coincides with the active set of the steady-state Problem (8). Let Assumption 1 hold. Then Problem (10) yields a first-order approximation of Problem (8). Moreover, there exist matrices δP and F such that

$$\tilde{W} = W + \mathcal{W}(\delta P) + J_{\mathbf{A}_{\mathrm{s}}}^\top F J_{\mathbf{A}_{\mathrm{s}}} \succ 0, \tag{11}$$

where we define

$$\mathcal{W}(\delta P) := \begin{bmatrix} A^\top \delta P + \delta P A & \delta P B \\ B^\top \delta P & 0 \end{bmatrix}, \tag{12}$$

and $J_{\mathbf{A}_{\mathrm{s}}} = \nabla_w h(x, u)$ is the Jacobian of the constraints that are strictly active at the optimal steady state.

Proof: The proof follows, *mutatis mutandis*, the ones given in [2, Theorem 9] and [1, Theorem 1] and is therefore omitted. ■

In order to provide intuition while avoiding the technicalities of a full proof, we prove that an LQR with running cost matrix W yields the same feedback law as an LQR with running cost matrix $W + \mathcal{W}(\delta P)$ in the next lemma.

Lemma 2: Consider a stabilizing LQR with system matrices A and B, and weighting matrices Q, R and S. Given any real symmetric matrix δP, an LQR with system matrices A and B, and weighting matrices $Q_{\delta P} = Q + A^\top \delta P + \delta P A$, $S_{\delta P} = S + B^\top \delta P$ and $R_{\delta P} = R$ provides the same feedback matrix as the original LQR.

The proof of Lemma 2 is provided in Appendix I. Note that this result implicitly requires $R \succ 0$, as otherwise the LQR cannot be stabilizing. This lemma (the continuous-time counterpart of [1, Lemma 2], instrumental in proving [1, Theorem 1]) is fundamental in establishing that the cost-modifying operator defined in (12) does not alter the optimal feedback law. Clearly, this result is necessary but not sufficient. However, it shows how one can translate the proofs of [1], [2] to continuous time.

In order to compute $\tilde{W}$ we formulate the following Semidefinite Program (SDP):

$$\min_{\delta P, F, \alpha, \beta} \quad \beta + \rho \|F\|_2 \tag{13a}$$

$$\text{s.t.} \quad \beta I \succeq \alpha W + \mathcal{W}(\delta P) + \eta J_{\mathbf{A}_{\mathrm{s}}}^\top F J_{\mathbf{A}_{\mathrm{s}}} \succeq I. \tag{13b}$$

Problem (13) is formulated as such by following two guidelines: (a) the condition number of the running cost should be small in order to avoid numerical difficulties when solving the MPC Problem (10) online; and (b) whenever it is possible to solve (13), the obtained first-order equivalence is independent of the active set. In case one needs $\eta = 1$ to find a solution, then the equivalence only holds for the initial states for which all constraints that are active at the optimal steady state remain active through the whole prediction horizon. If, instead, a solution is obtained for $\eta = 0$, the first-order equivalence holds regardless of the active set. Furthermore, the condition number of matrix $\tilde{W}$ is minimized in this case, while for $\eta = 1$, parameter ρ governs a trade-off between minimizing the condition number and not adding too much regularization through matrix F.

Note that the cost modification given by (11) is a cost rotation with storage function $\xi(x) = x^\top \delta P x$. As we will discuss in the next subsection, a linear term typically needs to be added to ξ. We will discuss the linear term as a separate cost rotation next.

C. Positive-Definite Nonlinear MPC

We now analyze the last first-order equivalence PD LMPC $\leftrightarrow$ PD NMPC. Similarly to the first equivalence ENMPC $\leftrightarrow$ ELMPC we aim at defining the MPC schemes such that Lemma 1 applies.

Differently from the previous case, however, ensuring positive-definiteness of the NMPC Hessian matrix while retaining the equivalence is non-trivial. If we use the same cost for PD LMPC and PD NMPC, by Lemma 1 the LMPC scheme satisfying the equivalence with the PD NMPC scheme would have a running cost with Hessian

$$\bar{W} = \tilde{W} + \sum_{i=0}^{n_x} \lambda_{\mathrm{s}} \nabla^2 f_i(x_{\mathrm{s}}, u_{\mathrm{s}}) - \sum_{j=0}^{n_\mu} \mu_{\mathrm{s}} \nabla^2 h_j(x_{\mathrm{s}}, u_{\mathrm{s}}) \not\succ 0.$$

Since in general both $\sum_{i=0}^{n_x} \lambda_{\mathrm{s}} \nabla^2 f_i(x_{\mathrm{s}}, u_{\mathrm{s}}) \neq 0$ and $\sum_{j=0}^{n_\mu} \mu_{\mathrm{s}} \nabla^2 h_j(x_{\mathrm{s}}, u_{\mathrm{s}}) \neq 0$, then $\bar{W} \neq \tilde{W}$ and hence one cannot apply Lemma 1 to prove the equivalence PD LMPC $\leftrightarrow$ PD NMPC. In order to tackle this issue, let us consider the contribution stemming from the system dynamics and the path constraints separately.

In order to eliminate the first term, one can use Equation (7) with $\xi(x) = \lambda_s^\top x$ to operate a linear rotation on the running cost of the economic NMPC. Since by using this running cost one does not change the primal solution of the economic MPC problem (4), nor that of the optimal steady-state problem (6), we will assume without loss of generality that the economic cost l is linearly rotated such that $\lambda_{\mathrm{s}} = 0$.

In order to eliminate the second term, one can unfortunately not follow the same path, unless a smart reformulation is used, as doing so would render strongly active constraints weakly active, as discussed in [2, Lemma 6]. Such an issue can be circumvented by introducing a vector of time-dependent slacks $s(t)$ and replacing the original inequality constraints with the equality constraints $h(x(t), u(t)) - s(t) = 0$ and the inequality constraint $s(t) \geq 0$. In this way, the new equality constraint can be rotated in the same way as for the constraints relative to the system dynamics. Since the constraint $s(t) \geq 0$ is linear, the term $\sum_{j=0}^{n_\mu} \mu_{\mathrm{s}} \nabla^2 h_j(x_s, u_s)$ is zero even in case $\mu_{\mathrm{s}} \neq 0$.

IV. Example

In this section we provide a numerical example to present the theoretical concepts explained in the previous sections. Let us consider an evaporation process in which a volatile species is removed from a nonvolatile solvent, thus concentrating the solution. All the details regarding this model and the parameter values can be found in [30]. The MPC strategies have been simulated in MATLAB using the CasADi [31] open-source tool. The state equations of the model are:

$$M\dot{X}_2 = F_1 X_1 - F_2 X_2, \quad C\dot{P}_2 = F_4 - F_5, \tag{14}$$

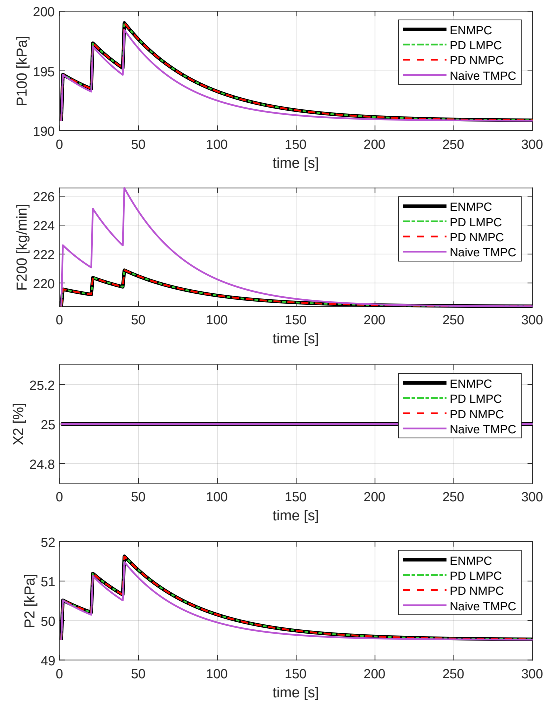

Fig. 1: MPC closed-loop trajectories.

where we have also the following dependencies among system variables:

$$\begin{aligned}
T_2 &= aP_2 + bX_2 + c, & T_{100} &= fP_{100} + g,\\
\lambda F_4 &= Q_{100} - F_1 C_p (T_2 - T_1), & T_3 &= dP_2 + e,\\
Q_{100} &= UA_1(T_{100} - T_2), & UA_1 &= h(F_1 + F_3),\\
Q_{200} &= \frac{UA_2(T_3 - T_{200})}{1 + UA_2/(2C_p F_{200})}, & F_{100} &= \frac{Q_{100}}{\lambda_s},\\
\lambda F_5 &= Q_{200}, & F_2 &= F_1 - F_4,
\end{aligned}$$

The states are $x = [X_2 \;\; P_2]^\top$, the control inputs are $u = [P_{100} \;\; F_{200}]^\top$. The economic cost function is:

$$l(x,u) = 10.09(F_2 + F_3) + 600F_{100} + 0.6F_{200} + 10^{-4}P_{100}^2 \tag{16}$$

The considered chemical system is subject to the following constraints:

$$\begin{aligned}
X_2 &\geq 25\%, & 40\mathrm{kPa} \leq P_2 &\leq 80\mathrm{kPa},\\
P_{100} &\leq 400\mathrm{kPa}, & F_{200} &\leq 400\mathrm{kg/min},
\end{aligned}$$

The steady-state values obtained by (6) are:

$$x_s = \begin{bmatrix} 25 \\ 49.514 \end{bmatrix}, \quad u_s = \begin{bmatrix} 190.815 \\ 218.378 \end{bmatrix}. \tag{18}$$

Figure 1 shows the behaviors of different MPC schemes in a simulation of 300 seconds, with a sampling time $T_{\mathrm{s}} = 1$ s and prediction horizon $T = 200$ steps. We use an explicit Runge-Kutta integrator of order 4 with 10 steps per control interval to simulate the dynamical system and evaluate the cost. The control signal is parameterized as

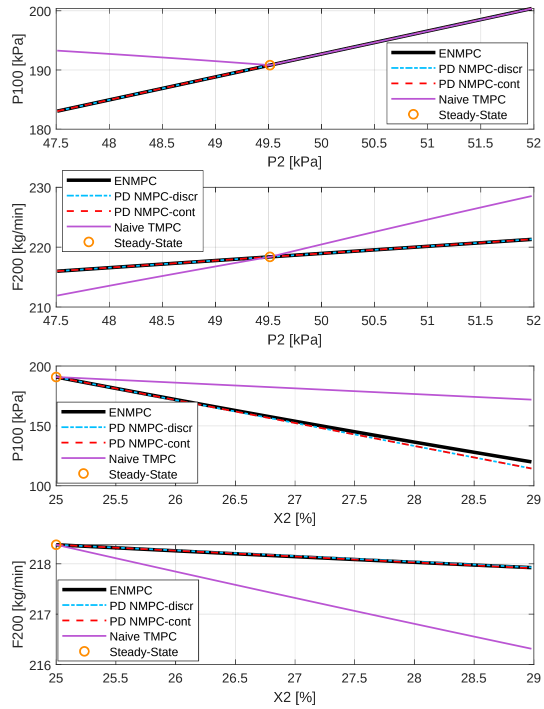

Fig. 2: MPC control law with initial conditions in a neighborhood of the optimal steady-state with $T_s = 1$ s.

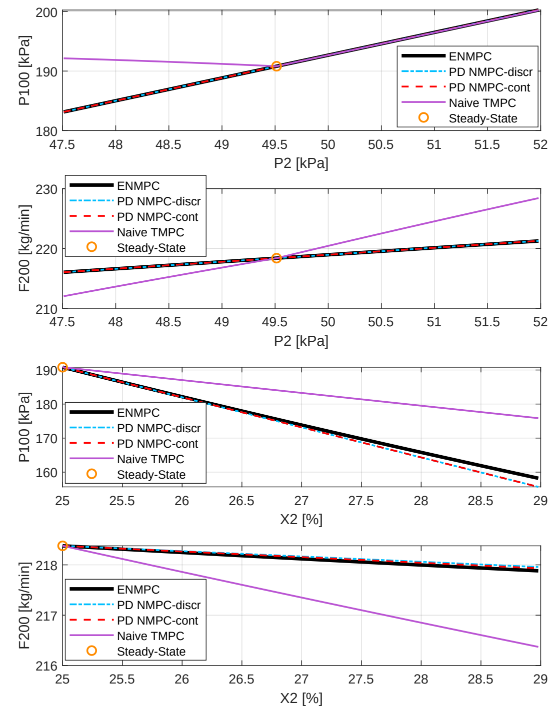

Fig. 3: MPC control law with initial conditions in a neighborhood of the optimal steady-state with $T_s = 2$ s.

a piecewise-constant function. The initial states at the first time-instant are equal to the optimal steady-state values, but along the simulation a pressure disturbance $\Delta P_2 = 1$kPa is applied at the time instants 0, 20 and 40 s. In the simulations, the MPC scheme does not have any information about future disturbances. The plots show the behavior of ENMPC, PD NMPC, PD LMPC, and a Naive TMPC with Hessian matrix equal to $H_{\text{track}} = \text{diag}(10, 10, 0.1, 0.1)$, and without the gradient term in the running cost, are drawn. We can notice that the behaviour of ENMPC is indistinguishable from that of the PD formulations. Finally, we observe that the loss in terms of closed-loop cost with respect to ENMPC is ≈ 9 times smaller for PD NMPC than for Naive TNMPC. Moreover, PD LMPC delivers essentially the same performance as PD NMPC.

Figure 2 displays the control law yielded by different MPC strategies when perturbing the initial states in a neighborhood of the optimal steady-state, using the sampling time $T_s = 1$ s. These simulations confirm the theoretical results, i.e., that the ENMPC and the PD NMPC formulations deliver the same control law up to first order. Figure 2 also displays the behavior of PD NMPC tuned in discrete time as per [2], confirming that the discrete-time procedure provides the same results as the continuous-time one.

In Figure 3 the results are displayed for sampling time $T_s = 2$ s, showing that the first-order approximation obtained with the proposed continuous-time procedure is still correct even if the sampling time is selected after tuning the cost, while the discrete-time procedure has to be run once more with the correct sampling time.

V. Conclusions

In this paper we have proposed a procedure to compute a PD tracking formulation starting from an ENMPC formulation for continuous-time systems. The obtained TMPC approximates the control law of the ENMPC up to first-order, even in case some inequality constraints are active at the optimal steady-state. The reported numerical examples have shown the effectiveness of the procedure and the equivalence with the same procedure applied in discrete time. Further research will consider extending the defined procedure to the case of optimal periodic operation.

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Appendix I

Proof: [Proof of Lemma 2] Let us consider the Continuous-time Algebraic Riccati Equation (CARE):

$$A^\top P + PA + Q - (S^\top + PB)R^{-1}(S + B^\top P) = 0. \tag{19}$$

with feedback matrix $K = R^{-1}(S + B^\top P)$.

Consider a new weighting matrix $\tilde{W}$ defined as:

$$\begin{aligned}\tilde{W} &= W + \mathcal{W}(\delta P) \\ &= \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} + \begin{bmatrix} A^\top \delta P + \delta P A & \delta P B \\ B^\top \delta P & 0 \end{bmatrix}.\end{aligned} \tag{20}$$

The CARE associated with matrix $\tilde{W}$ is

$$A^\top \tilde{P} + \tilde{P}A + \tilde{Q} - (\tilde{S}^\top + \tilde{P}B)\tilde{R}^{-1}(\tilde{S} + B^\top \tilde{P}) = 0, \tag{21}$$

where, from (20) we have:

$$\tilde{Q} = Q + A^\top \delta P + \delta P A, \quad \tilde{S} \;= S + B^\top \delta P, \quad \tilde{R} = R. \tag{22a}$$

Substituting (22) into (21), we obtain:

$$\begin{aligned} & A^\top \tilde{P} + \tilde{P}A + Q + A^\top \delta P + \delta P A \\ & \quad - (S^\top + \delta P B + \tilde{P}B)R^{-1}(S + B^\top \delta P + B^\top \tilde{P}) \\ &= A^\top(\tilde{P} + \delta P) + (\tilde{P} + \delta P)A + Q \\ & \quad - (S^\top + (\tilde{P} + \delta P)B)R^{-1}(S + B^\top(\tilde{P} + \delta P)) = 0. \end{aligned} \tag{23}$$

From the above equivalence we have that the feedback matrix $\tilde{K}$ is equal to:

$$\tilde{K} = R^{-1}(S + B^\top(\tilde{P} + \delta P)). \tag{24}$$

Since we know that, for the cost function defined by Q,S and R, the matrix P solves the CARE defined in (19), and since the stabilizing solution is unique [1], we can state that Equations (19) and (23) are equal, hence:

$$\tilde{P} + \delta P = P, \quad \text{and} \quad \tilde{K} = K. \tag{25}$$

■