Mathematical Structures in Computer Science (2026), **36**, e35, pp. 1–31
doi:10.1017/S0960129526100747

CAMBRIDGE UNIVERSITY PRESS

PAPER

Proofs, programs and arrows for epistemic logics

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(Received 31 March 2025 UTC; revised 17 June 2026 UTC; accepted 18 September 2026 UTC)

Abstract

We address the proof-theoretic, categorical and computational aspects of intuitionistic epistemic logics, extending prior work that mainly focused on axiomatic calculi and possible world semantics. We introduce a modular approach to designing natural deduction systems, extending a previously defined calculus for intuitionistic belief (IEL^-) to a new natural deduction calculus for intuitionistic knowledge (IEL). A natural translation of these calculi into modal type theories is given via the proofs-as-programs paradigm and used to prove in a uniform way normalisation results with respect to detours and permutation elimination. The strong normalisation proofs given here employ, in a systematic way, the method of CPS translations, and allow the derivation of further properties of these intuitionistic epistemic logics as corollaries of normalisation of formal derivations in the calculi. The correspondence between deductions and typed terms is further extended to a categorical interpretation of the epistemic operator as a monoidal functor satisfying specific properties identified by the rewriting systems imposed on deductions in IEL^- and IEL.

Keywords: intuitionistic epistemic logics; normalisation theorems; proofs-as-programs; categorical interpretation; proof-theoretic semantics

1. Introduction

When does a mathematician believe in a mathematical statement? In general, she believes that a statement is true when it has been proven. However, this does not mean that the same mathematician has proven that statement, nor is she capable of producing a proof of it herself.

Indeed, beliefs about mathematics are generated by the *existence* of proofs and, in many cases, this existence does not presuppose an explicit witness for provability. In more practical terms, it is common to believe that a mathematical statement is a theorem of a given theory, even though we do not have direct access to an explicit proof. Sometimes, we rely on the literature to verify claims. In other cases, for example, when new to a field, we trust an authority – lecturer, supervisor or expert – regarding the validity of a statement. It is not unusual to trust a computer regarding the truth of a mathematical claim.

A famous episode is enlightening on this point. Peter Scholze and Dustin Clausen are leading advanced research in a 'grand unification project' called condensed mathematics. The field is highly specialised, and it is unsurprising that only a few mathematicians can work on these concepts to prove meaningful claims. However, it is astonishing that Scholze and Clausen *themselves* were unsure that they had correctly proven a theorem – the liquid tensor experiment in Scholze (2021) – until their self-questioned proof was formalised in the proof assistant Lean (de Moura and Ullrich 2021). This formalisation required efforts from researchers across Europe, and the

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resulting formal proof is arguably less comprehensible to humans than the original. It was this verification that convinced them to trust the liquid tensor experiment.

As the constructive account of truth requires that a statement be considered true only if proven, the liquid tensor experiment episode thus supports this account and highlights the relevance of verificationist epistemology. According to this view, a rational agent should believe in a statement when it has been verified, *at least*.

Verification and belief in mathematics. Artemov and Protopopescu (2016) extends that constructive-intuitionistic analysis of truth to epistemic states. In short, they claim that if proving statement A assures that A is true, we might *believe* A is true without a direct proof: a correct verification suffices. Thus, belief results from a verification process.

At the same time, a mathematical proof is the strictest type of verification. Their intuitionistic account validates the principle of 'constructivity of truth':

$$A \to \Box A,$$

where $\Box$ denotes belief modality.

Even in the informal setting, that principle is clearly validated by contemporary mathematical practice: When a statement A is proven, its very proof can be checked by another mathematician, both by 'pencil and paper', and by relying on computer proof assistants. It is that very checking that lets the mathematical community believe that A holds indeed.

Now, the standard intuitionistic account for truth is embodied by the Brouwer–Heyting–Kolmogorov (BHK) interpretation of intuitionistic logic. This interpretation is based on a semantic reading of propositional variables as problems (or tasks) and logical connectives as operations on *proofs*. In this way, it provides a semantics of mathematical statements in which the computational aspects of proving and refuting are highlighted, as explained by Troelstra (1969).

Despite its name, the BHK interpretation diverges from Brouwer's philosophical intuitionism, which roots mathematics in a mental process of temporal object construction. In BHK interpretation, reasoning intuitionistically is similar to a safe mode of program execution which always terminates; on the contrary, according to the founders of intuitionism, at the heart of mathematical activity there is a continuous mental process of construction of objects starting with the flow of time underlying the chain of natural numbers, and intuitionistic reasoning is what structures that process.[1] On the contrary, the relational semantics by Kripke (1965) formalises this epistemic growth of knowledge over time which characterises the mental life of the mathematician.

Artemov and Protopopescu (2016) bridges BHK's proof-centric semantics with Kripke's epistemic structures, arguing that proof construction provides sufficient reason for belief. In the BHK interpretation, we have an implicit notion of proof whose epistemic aspects are modelled by Kripke structures, and the construction of a proof for a specific proposition that we carried out as a cumulative mental process gives us sufficient reason for (at least) believing in that proposition. It is then possible to also cover traditional epistemic states of belief and knowledge within such a framework, once we recognise the correct clauses for corresponding modal operators: Proving A assures that A is true, proving $\Box A$ is weaker, for we may believe A even though we do not have a direct proof of A itself.

From belief to knowledge. Consider the following scenario.

> Your favourite top-rated mathematician has posted on her blog a proof-sketch of a conjecture A in a very specialised mathematical field. Her informal argument is convincing, and her expertise justifies your trust in the conjecture. You are not an expert in that field, but she is; you do not have direct access to a detailed proof of A, but it is reasonable for you to believe that A does hold.

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This is completely in line with the verificationist account of *belief* that we have just recalled. But how could you safely claim that you *know* that A is indeed a theorem?

In philosophy, knowledge is usually defined as *justified true belief* (JTB). These conditions are considered, in general, but not universally (Ichikawa and Steup (2018)) – to be individually necessary and jointly sufficient for defining knowledge: one cannot know something without believing it (condition B), and one does not know a thing unless she also has reasons for believing it (condition J). Condition B allows us to overload the formal epistemic notation, so that we can use the same $\Box$-modality to capture both epistemic states. Condition J implies that co-reflection $A \to \Box A$ is still valid when dealing with knowledge. What about condition T?

Even at the very intuitive level, the standard method of capturing condition T – that is, adding the schema $\Box A \to A$ to the modal principles of our theory – proves disastrous in the present setting: it causes the modal collapse of the whole language.

It is then appropriate to ask how to formalise the factivity of knowledge required by condition T according to Artemov and Protopopescu's verificationist epistemology. In order to answer that question, let us expand a little bit our scenario about the maths' blog:

> Some time has passed since the mathematician's post on conjecture A. You are again surfing the Net looking for new results in the field A belongs to, and come across another post about that conjecture. This time, it is a famous top-rated computer scientist's blog, announcing that A has been refuted. And there's more: Her refutation has been computerised, and the formal proof is freely available on the Net. If you know the programming language at the base of her computer program to refute A – a proof assistant, or a highly specialised theorem prover – you can read the proof and see that $\neg A$ holds, so that your original trust in A is misplaced.

Artemov and Protopopescu (2016) propose five different schemas to capture this idea:

1. $\neg A \to \neg\Box A$;
2. $\neg(\Box A \wedge \neg A)$;
3. $\neg\Box\bot$;
4. $\neg\neg(\Box A \to A)$;
5. $\Box A \to \neg\neg A$.

Principle 3, despite being the weakest among 1–5, is arguably the closest to a naive intuitionistic version of condition T – 'you cannot know something false' – while principle 4 is the double negation of the classical schema for condition T, and principle 1 is its contrapositive. Principle 5 is chosen by Artemov and Protopopescu to extend intuitionistic *belief* to the logic of intuitionistic *knowledge* $\mathbb{IEL}$; however, as they notice, all of those principles are intuitionistically equivalent over the base calculus $\mathbb{IEL}^-$ for belief, and each of them is intuitionistically weaker than the classical schema for condition T.

This verificationist version of factivity of knowledge allows a rational agent to know that a proposition A holds even if she does not have a proof of A. This is not surprising at all, if we consider empirical knowledge, and the previous scenario about conjecture A suggests that the same is valid for mathematical knowledge: even if the proof of $\neg A$ were classified, or written in a computer program completely unknown to you, you would still know that A does not hold, and your original belief in A would have been false.

What Artemov and Protopopescu (2016) propose is, in a sense, a formal BHK-based version of *knowledge as belief tracking the truth of the matter under discussion* proposed by Nozick (1981).[2]

In simpler terms, their verificationist account of epistemic states is inspired by the intuitionistic interpretation of mathematical rationality, but its potential applications are not limited to

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mathematics. Instead of discussing conjectures, proof sketches and computer-checked proofs, one could consider the following ordinary scenario:

> Any stopped clock indicates the correct time twice per day. This statement has become a platitude. However, looking at such a clock at the right moment, and according to the *JTB* theory, you should *know* the time, since your belief is (uselessly) justified and true.

From the verificationist viewpoint, this example fails to be considered knowledge-producing: if my belief about the time is actual knowledge, then, if it had not been that very time, I would not have believed it was. The intuitionistic version of this argument is captured by any of the modal schemas 1–5 above, but principle 2 is, to the best of our knowledge, the one that better reveals the pattern of reasoning shared by the clock and math blog scenarios.

We notice that, within this verificationist framework, the adoption of intuitionistic rather than classical logic is not merely a philosophical preference, but quite a rigorous necessity. Were classical logic to be employed, the pairing of the constructivity principle ($A \to \Box A$) with the classical reasoning would precipitate the law of excluded middle ($A \vee \neg A$) into an immediate epistemic disjunction $\Box A \vee \Box\neg A$ which would impose a form of universal mathematical omniscience, clearly refuted by known meta-mathematical results (Franzén 2004).

Our contribution. The insight of the BHK interpretation of standard non-modal operators is formally captured by the rules of natural deduction for intuitionistic logic and, even more precisely, by the associated λ-calculus with types of the proofs-as-programs paradigm (Sørensen and Urzyczyn 2006). In spite of this, Artemov and Protopopescu (2016) only discuss axiomatic calculi and possible world semantics for intuitionistic epistemic logics so that the stimulating question of considering the proof-theoretic/computational aspects of epistemic states remained at an informal level and underdeveloped. This is particularly surprising, as the axiomatisation they proposed for intuitionistic belief was already identified by McBride and Paterson (2008) to reason about applicative functors in functional programming languages.

In the present paper, we develop a **modular** approach to the **design of natural deduction systems** for intuitionistic belief *and* knowledge – IEL^- and IEL, respectively – which are naturally translated into functional calculi for deductions, that is, **modal type theories**. In a sense, that move defines a formal counterpart to Artemov and Protopopescu's reading of the epistemic operators by extending the proofs-as-programs paradigm established between natural deduction for intuitionistic logic and simple type theories to handle those modal operators in a principled way along with their corresponding constructors in the proof-term calculi.

Such a proofs-as-programs correspondence is then naturally extended to the **categorical interpretation of the modal operator** for belief as a monoidal functor with a monoidal point which is also dense when modelling the knowledge modality; that is, it punctually preserves the monoidal structure of conjunctions (belief) *and* the initiality of $\bot$ (knowledge, as captured by principle 3, $\neg\Box\bot$), respectively.

The design choices for natural deduction calculi focusing on modularity play a significant role in the proof development of the main results presented here. The original proof of **strong normalisation** for the modal λ-calculus associated with IEL^-, which we presented first in Perini Brogi (2021), is *modularly extended* here to the normalisation proof for the type theory associated with IEL, which extends IEL^- by adding an elimination rule for the modal operator. Both proofs are indeed based on a *CPS-translation* into simple type theory, and, from a logical viewpoint, are applications to modal logics of the well-known Friedman–Dragalin's A-translation (Leivant 1985). Further **properties of the logics** are then derived **as corollaries** of these normalisation results in a uniform manner.

Similarly, we show that the properties of the functor associated with belief modality in Perini Brogi (2021) are preserved by the categorical interpretation of deductions in our extended calculus IEL for intuitionistic knowledge.

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In this way, we provide a more precise proof-theoretic and categorical analysis of the intuitionistic epistemic operators under the lens of the Curry–Howard–Lambek correspondence, extending our previous findings about intuitionistic modalities in Perini Brogi (2019, 2021, 2022, 2025).

Paper structure. The paper is organised as follows: Section 2 collects preliminary materials, including the original axiomatisation and Kripke semantics for intuitionistic epistemic logic as by Artemov and Protopopescu (2016), a presentation of the standard natural deduction system for intuitionistic propositional logic and detour elimination, followed by the main notions at the basis of the Curry–Howard–Lambek correspondence for that logic. In Section 3, the natural deduction systems IEL^- and IEL – for intuitionistic belief and knowledge, resp. – are introduced; proof-rewritings for these calculi are defined by using corresponding modal λ-calculi, modularly showing strong normalisation of the systems with respect to detours and permutation conversions. Furthermore, purely proof-theoretic proofs of known properties of intuitionistic epistemic logics are given as results of the previous proof analysis of our natural deduction calculi. In Section 4, we discuss a categorical interpretation of the epistemic modalities in our calculi as a proof-theoretic semantics based on the Curry–Howard–Lambek correspondence. Finally, Section 5 closes the paper with a collection of related work on intuitionistic epistemic logics and modal type theory.

2. Preliminaries

To make the paper as self-contained as possible, in this section, we recall the preliminary notions and results we use in subsequent sections. We begin with a summary of the presentation by Artemov and Protopopescu (2016), followed by the basics of natural deduction for propositional intuitionistic logic and the proof-as-programs paradigm, which was later extended to the categorical interpretation of proof terms. For a more in-depth analysis of these concepts, we refer the reader to the main sources cited in the following pages.

2.1 Axiomatic calculi for intuitionistic epistemic logics

Let us begin by recalling the syntax and relational semantics for intuitionistic epistemic logic, as by Artemov and Protopopescu (2016). Here, we assume a standard mono-modal propositional language.

Definition 1. *$\mathbb{IEL}^-$ is the axiomatic calculus given by:*

- *Axiom schemas for intuitionistic propositional logic;*
- *Axiom schema* $\mathsf{K} : \Box(A \to B) \to \Box A \to \Box B$;
- *Axiom schema of co-reflection:* $A \to \Box A$;
- *Modus Ponens* $\frac{A \to B \quad A}{B}\,MP$ *as the only inference rule.*

$\mathbb{IEL}$ is the axiomatic calculus extending $\mathbb{IEL}^-$ by the axiom schema corresponding to the principle of intuitionistic factivity of knowledge

$$\Box A \to \neg\neg A.$$

We write $\Gamma \vdash_{\mathbb{L}} A$ when A can be derived in $\mathbb{L}$ for $\mathbb{L} \in \{\mathbb{IEL}^-, \mathbb{IEL}\}$ assuming a set of hypotheses Γ, and we write $\mathbb{L} \vdash A$ when $\Gamma = \varnothing$.

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We immediately have

Proposition 2. *The following properties hold:*

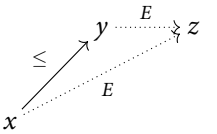
(i) *Necessitation rule* $\frac{A}{\Box A}$ *is derivable in* $\mathbb{L}$;
(ii) *The deduction theorem holds in* $\mathbb{L}$;
(iii) $\mathbb{L}$ *is a normal intuitionistic modal system, that is: it extends the intuitionistic propositional base with a* $\Box$*-modality distributing over implications (axiom* K*), and it is closed under both modus ponens and necessitation.*

Proof. See Artemov and Protopopescu (2016). □

Turning to relational semantics, we have the following

Definition 3. *A model for* $\mathbb{IEL}^-$ *is a quadruple* $\langle W, \leq, v, E\rangle$ *where*

- $\langle W, \leq, v\rangle$ *is a standard model for intuitionistic propositional logic;*
- *E is a binary 'knowledge' relation on W such that:*
  - if xEy, then $x \leq y$; and
  - if $x \leq y$ and yEz, then xEz; graphically, we have



- *v extends to a forcing relation* $\Vdash$ *between worlds* $x \in W$*, the model* $\langle W, \leq, v, E\rangle$*, and a formula A, inductively defined on the structure of A as follows:*
  - $x \Vdash \bot$ for no $x \in W$
  - $x \Vdash \top$ for any $x \in W$
  - $x \Vdash a \iff V(a, x) = 1$
  - $x \Vdash \neg A \iff x \nVdash A$
  - $x \Vdash A \wedge B \iff x \Vdash A$ and $x \Vdash B$
  - $x \Vdash A \vee B \iff x \Vdash A$ or $x \Vdash B$
  - $x \Vdash A \to B \iff (x \Vdash A \implies x \Vdash B)$
  - $x \Vdash A \leftrightarrow B \iff (x \Vdash A \iff w \Vdash B)$
  - $x \Vdash \Box A$ iff $y \Vdash A$ for all y such that xEy.

A model for $\mathbb{IEL}$ *is a model for* $\mathbb{IEL}^-$ $\langle W, \leq, v, E\rangle$ *where the relation E satisfies the seriality condition*

- *for any* $x \in W$*, there exists a* $y \in W$ *such that* xEy.

A formula A is true in a model iff it is forced by each world of that model; we write $\mathbb{L} \vDash A$ *iff A is true in each model for* $\mathbb{L}$.

Note that this semantics assumes Kripke's original interpretation of intuitionistic reasoning as growing knowledge, or discovery process, for an epistemic agent: the relation E defines an audit of 'cognitively' $\leq$-accessible states in which the agent can commit a verification. The seriality condition on E is what assures the consistency of audits, since $w \nVdash \Box\bot$ for any world w.

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This semantics is adequate to the calculi:

Theorem 4. *The following hold:*

- ***Soundness:*** *If* $\mathbb{L} \vdash A$, *then* $\mathbb{L} \vDash A$.
- ***Completeness:*** *If* $\mathbb{L} \vDash A$, *then* $\mathbb{L} \vdash A$.

Proof. Soundness is proven by induction on the derivation of A.
Completeness is proven by a standard construction of a canonical model.
See Artemov and Protopopescu (2016) for further details. □

2.2 Natural deduction for propositional logic

Among the proof systems currently available, natural deduction calculi are the closest to the informal reasoning of mathematical practice. Introduced first by Gentzen (1935), natural deduction systems for classical and intuitionistic logic, usually denoted by NK and NJ, respectively, provide a formal model for *deduction under hypotheses*.

The key feature of these systems is the *absence of logical axioms* from the calculus. The entire deductive apparatus relies only on inference rules that encode the atomic deductive steps involved in any logical argument and reflect the operational meaning of the logical connectives of a given base language.

The starting point of any formal proof is an assumption: any formula A is per se a proof whose premise and conclusion are A itself. These starting points are then developed according to the rules in Figure 1, defining the propositional fragment of NJ, here denoted by NJp.

In those rules,[3] Γ denotes a set of (occurrences of) formulas, which are called active assumptions of the deduction. Note that in $\rightarrow \mathscr{I}$ and $\vee\mathscr{E}$, some assumptions are bracketed and labelled by numbers that are repeated on the side of the rule, which denotes the discharge of the assumptions. In NJp, several as well as no occurrences of the same assumption can be labelled by the same number, and so discharged by a single application of the rule.

The informal notion of proof acquires thus a precise and natural mathematical structure, namely that of a derivation tree having active assumptions as leaves, the conclusion of the proof as root and each intermediate node labelled by a formula occurrence derived from the nodes above it by a correct application of the inference rules.

One of the key features of deductions in Gentzen's paradigm is that branches can be grafted in correspondence of a formula occurring as the root of a given deduction and as an active assumption of another one. This is a highly practical move that captures the mathematician's common way of proving complex theorems: She achieves the goal by proving intermediate lemmas.

Nevertheless, when a deduction f_1 is grafted onto another one, say f_2, a structural *detour* may occur, and it is possible that the conclusion of f_1 has been obtained by an $\mathscr{I}$-rule, whereas the corresponding assumption of f_2 is the major premise of the $\mathscr{E}$-rule. The resulting deduction lacks a certain directedness, and therefore its structure is unnecessarily complex.

A canonical form theorem holds for deductions in NJp: It was established by Prawitz (1965) in terms of a *normalisation process* of deductions in those systems. The main aim of normalisation is to eliminate that kind – as well as more general forms – of useless complexities in the structure of deductions, by manipulating them until simplest and most direct deductions are obtained. This manipulation is made precise by the simplification steps – also called conversions or proof-rewritings – of NJp-deductions summarised in Figure 2.[4]

If no simplification step needs to be performed in a deduction, then we will call that deduction **normal.**

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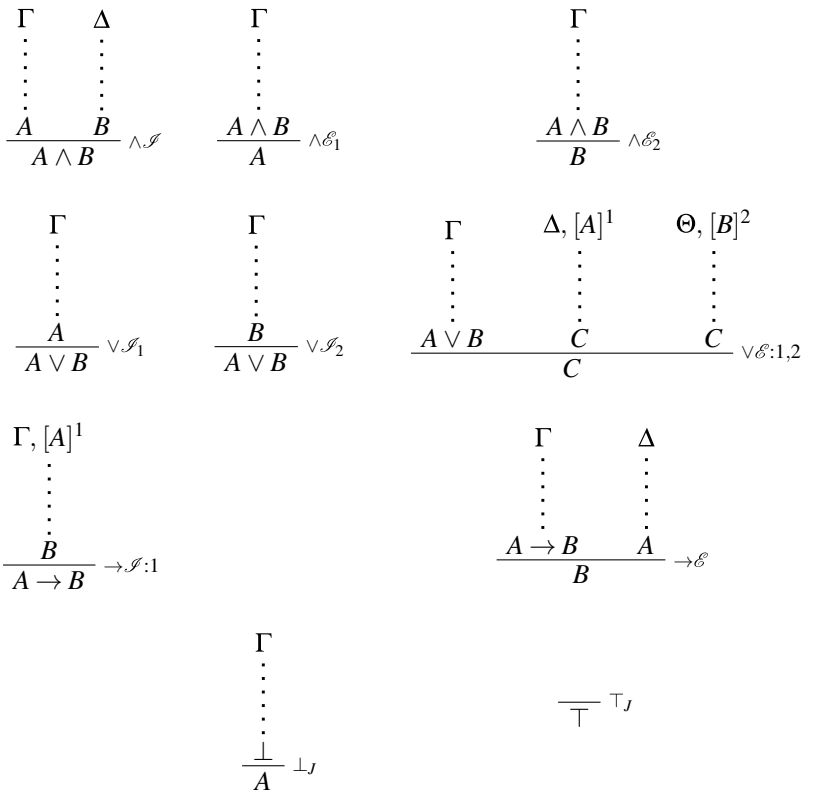


Figure 1. Rules for *NJp*.

Thus, we know that the following holds:

Theorem 5 (Strong normalisation, Prawitz 1965, 1971). *Given a deduction of A from Γ in* NJp *every simplification sequence based on the conversions in Figure 2 terminates in a normal deduction.*

Proof. Refer to Prawitz (1971). □

2.3 Proofs as programs

The partial glossary in Figure 3 summarises the proofs-as-programs viewpoint, that is, the Curry–Howard correspondence.

Such a correspondence allows us to overload our notation so that we can inductively define the class of types T as the formulas of our propositional language:

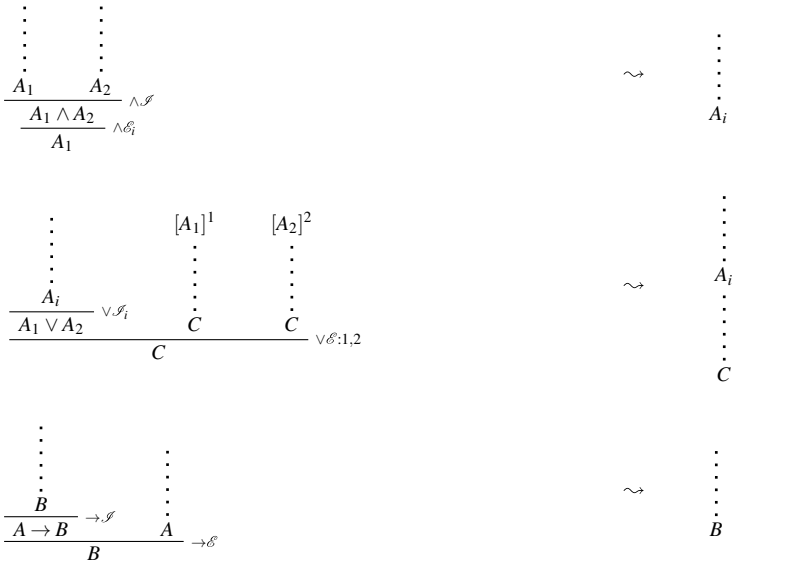
$$A, B \in \mathtt{T} := p \mid A \wedge B \mid A \vee B \mid A \to B \mid \bot \mid \top,$$

where p belongs to a denumerable set of type variables.

In the present setting, $A \wedge B$ is called a **product type**, $A \vee B$ a **(weak) sum type**, $A \to B$ a **function type**, $\bot$ is the **empty type** and $\top$ is the **unit type**.

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Detours:



Permutations:

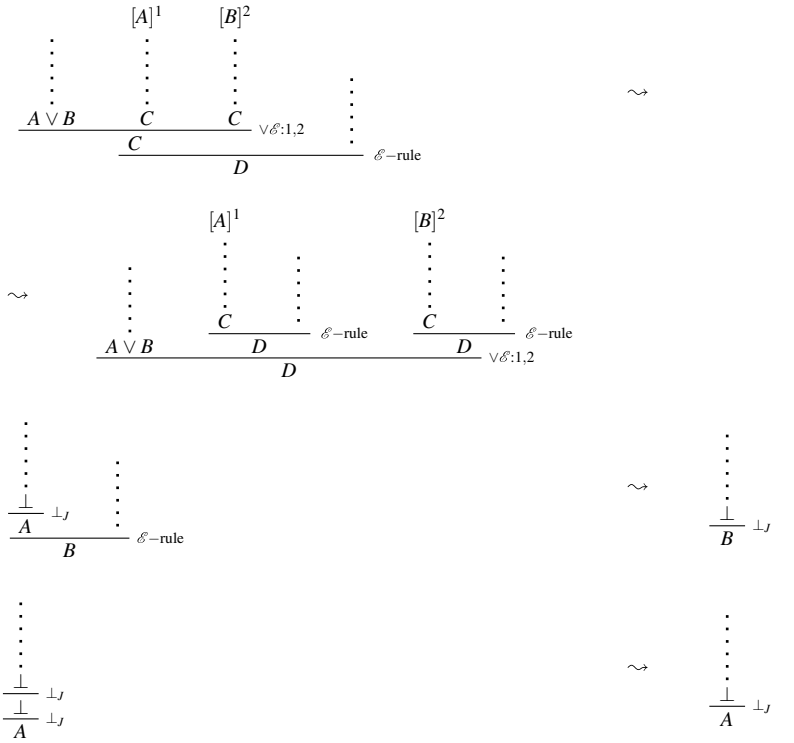


Figure 2. Simplifications for *NJp*-deductions.

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TYPE THEORY	LOGIC
type	proposition
term	proof
type constructor	logic connective
constructor	introduction rule
destructor	elimination rule
redex	proof detour
reduction	normalization
normal form	normal proof
inhabitation	provability

Figure 3. Proofs-as-programs paradigm, in brief.

Definition 6. *The set of typed terms Λ^T of the basic simple type theory is inductively defined by the following grammar:*

$$\begin{aligned} f : A \in \Lambda^T := {} & x : A \mid ((f_1 : A \to B)(f_2 : A)) : B \mid (\lambda x : A.\, f_1 : B) : A \to B \mid \\ & (f_1 : A, f_2 : B) : A \wedge B \mid (\pi^1.(f : A \wedge B)) : A \mid (\pi^2.(f : A \wedge B)) : B \mid \\ & (\mathsf{in}_1(f : A)) : A \vee B \mid (\mathsf{in}_2(f : B)) : A \vee B \mid \\ & (\mathsf{C}((f : A \vee B), x : A.(f_1 : C), y : B.(f_2 : C))) : C \mid \\ & (\bot_J(f : \bot)) : A \mid * : \top \qquad . \end{aligned}$$

*Any expression of the form $f : A$ is said to be a **type-assignment** of subject f to type A.*

*A finite set of type-assignments $\Gamma = \{x_1 : A_1, \ldots, x_m : A_m\}$ whose subjects are term variables and which is consistent – that is, no variable is subject of more than one assignment – is said to be a (type-) **context**.*

It is easy to see that any type assignment corresponds to a labelled deduction in NJp based on the mapping shown in Figure 4. Thus, one could rephrase any proof in NJp in type-theoretic terms, which can be considered proof names for deductions. This justifies the notation overload for deductions and terms of the type theory. In addition, note that in the term constructor for $\to$-types, the discharging of hypothesis A is denoted by the bounding of the variable $x : A$ in λ-abstraction. Similarly, the term destructor for $\vee$-types bounds the variables $x : A$ and $y : B$, as expected.

A sequent-style formalisation of the theory of NJp deductions can then be defined by decorating the sequent-style version of NJp given by Negri and von Plato (2008, Section 1.3) with proof names. The sequents of that type theory will be called **judgements**. For instance, the judgements corresponding to the rules for implication in NJp are[5]

$$\frac{\Gamma_1 \Rightarrow f : A \to B \qquad \Gamma_2 \Rightarrow g : A}{\Gamma_1 \cup \Gamma_2 \Rightarrow (fg) : B} \quad \text{with } \Gamma_1 \cup \Gamma_2 \text{ consistent.}$$

$$\frac{\Gamma \Rightarrow f : A}{\Gamma - (x : A) \Rightarrow (\lambda x.\, f) : A \to B} \quad \text{when } \Gamma \text{ is consistent with } x : A.$$

When a judgement $\Gamma \Rightarrow f : A$ is provable in the type theory, the term f is said to **inhabit** type A in context Γ.

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A	$\longmapsto$	x_i^A,	where i is the parcel of the hypothesis A
$\dfrac{\begin{array}{cc}\vdots f_1 & \vdots f_2\\ A & B\end{array}}{A \wedge B}\ \wedge\mathscr{I}$	$\longmapsto$	$(t : A, s : B)$,	where $t : A$, $s : B$ correspond to f_1 and f_2 resp.
$\dfrac{\begin{array}{c}\vdots f'\\ A \wedge B\end{array}}{A}\ \wedge\mathscr{E}_1$	$\longmapsto$	$\pi_1.(t : A \wedge B)$,	where $t : A \wedge B$ corresponds to f'
$\dfrac{\begin{array}{c}\vdots f'\\ A \wedge B\end{array}}{B}\ \wedge\mathscr{E}_2$	$\longmapsto$	$\pi_2.(t : A \wedge B)$,	where $t : A \wedge B$ corresponds to f'
$\dfrac{\begin{array}{c}\vdots f'\\ B\end{array}}{A \to B}\ \to\mathscr{I}$	$\longmapsto$	$\lambda x_i : A.t : B$,	where $t : B$ corresponds to f' and i is the parcel of discharged hypotheses A
$\dfrac{\begin{array}{cc}\vdots f_1 & \vdots f_2\\ A \to B & A\end{array}}{B}\ \to\mathscr{E}$	$\longmapsto$	$t : A \to B s : A$,	where $t : A \to B$, $s : A$ correspond to f_1 and f_2 resp.
$\dfrac{\begin{array}{c}\vdots f'\\ A\end{array}}{A \vee B}\ \vee\mathscr{I}_1$	$\longmapsto$	$\mathsf{in}_1.t : A$,	where $t : A$ corresponds to f'
$\dfrac{\begin{array}{c}\vdots f'\\ B\end{array}}{A \vee B}\ \vee\mathscr{I}_2$	$\longmapsto$	$\mathsf{in}_2.t : B$,	where $t : B$ corresponds to f'
$\dfrac{\begin{array}{ccc}\vdots f' & [A]\ \vdots f_1 & [B]\ \vdots f_2\\ A \vee B & C & C\end{array}}{C}\ \vee\mathscr{E}$	$\longmapsto$	$\mathsf{C}(t, (x : A.t_1), (y : B.t_2))$	where C bounds all occurrences of x in t_1 and all occurrences of y in t_2, and t, t_1, t_2 correspond to f', f_1, f_2, resp.
$\dfrac{\begin{array}{c}\vdots f'\\ \bot\end{array}}{A}\ \bot_J$	$\longmapsto$	$(\bot_J t) : A$	where t corresponds to f'
$\dfrac{}{\top}\ \top_J$	$\longmapsto$	$* : \top$	

Figure 4. The mapping from deductions to typed terms.

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The normalisation theorem for deductions in NJp acquires a precise procedural meaning: eliminating a detour or performing a permutation corresponds to executing a process for computing values, and normalisation ensures that a computation can always be continued to a final result. This is made precise by rephrasing the detour eliminations and permutations in Figure 2 in a type-theoretic manner in terms of the following rewriting system:

Definition 7. *Let* Ξ *denote the reflexive and transitive closure of the relation* $>_\Xi \subseteq \Lambda^{\mathsf{T}} \times \Lambda^{\mathsf{T}}$ *defined as follows:*[6]

Detours:
- $(\lambda x.f)g >_\Xi f[x := g]$
- $\pi_i(f_1, f_2) >_\Xi f_i$ *for* $i = 1, 2$
- $\mathsf{C}(\mathsf{in}_i f, x.f_1, y.f_2) >_\Xi f_i[x_i := f]$ *for* $i = 1, 2$

Permutations:
- $\mathsf{C}(f, x.f_1, y.f_2)g >_\Xi \mathsf{C}(f, x.f_1 g, y.f_2 g)$
- $\pi_i \mathsf{C}(f, x.f_1, y.f_2) >_\Xi \mathsf{C}(f, x.\pi_i f_1, y.\pi_i f_2)$ *for* $i = 1, 2$
- $\mathsf{C}(\mathsf{C}(f, x.f_1, y.f_2), u.g_1, v.g_2) >_\Xi \mathsf{C}(f, x.\mathsf{C}(f_1, u.g_1, v.g_2), y.\mathsf{C}(f_2, u.g_1, v.g_2))$
- $\bot_J(f)g >_\Xi \bot_J(f)$
- $\pi_i \bot_J(f) >_\Xi \bot_J(f)$ *for* $i = 1, 2$
- $\mathsf{C}(\bot_J(f), x.f_1, y.f_2) >_\Xi \bot_J(f)$
- $\bot_J(\bot_J(f)) >_\Xi \bot_J(f)$.

where $g[x_i := f]$ *denotes the substitution of* f *for all free occurrences of* x_i *in* g.

A Ξ-normal form is just a term of Λ^{T} that cannot be rewritten with respect to Ξ. For this system of rewritings, it is possible to prove the properties that a type-theorist would expect.[7] In particular, the confluence property of Ξ implies that the denotational calculus for the reductions is consistent, whereas the analogous of Theorem 5 guarantees strong normalisation of these computations.

2.4 Computational trinitarianism

The correspondence between type theory and natural deduction is further enlightened by extending the perspective to cover the most algebraic foundations of mathematics currently available, namely category theory. Slightly dramatising, Harper (2011) writes

> The central dogma of computational trinitarianism holds that Logic, Languages, and Categories are but three manifestations of one divine notion of computation. There is no preferred route to enlightenment: each aspect provides insights that comprise the experience of computation in our lives.

In less emphatic terms, the correspondence between proofs and programs is extended to consider arrows in categories which have sufficient structure to capture the behaviour of logical operators. The key ideas behind the so-called computational trinitarianism, or Curry–Howard–Lambek correspondence, are summarised in the partial glossary of Figure 5.

We omit the standard definition of the notions of category, functors and natural transformation, to notice that, here, we implicitly work with *locally small categories* only; that is, we assume that for any category $\mathscr{C}$ and any objects $X, Y : \mathscr{C}_0$, the collection of arrows $\mathsf{hom}(X, Y)$ is indeed a set. Moreover, we overload the notation and write $\mathscr{C}$ for $\mathscr{C}_0$ whenever the context makes it clear that we are dealing with the objects of $\mathscr{C}$. We refer the reader to classic textbooks by Lambek and Scott (1988) and MacLane (2013) for more details.

The categorical notions required by computational trinitarianism are minimal and well established. To ensure the paper remains as self-contained as possible without overburdening the expert reader, we have consolidated them in the following

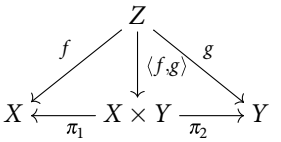
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LOGIC	TYPE THEORY	CATEGORY THEORY
proposition	type	object
proof	term	arrow
theorem	inhabitant	element-arrow
conjunction	product type	product
true	unit type	terminal object
implication	function type	exponential
disjunction	sum type	(weak) coproduct
false	empty type	(weak) initial object

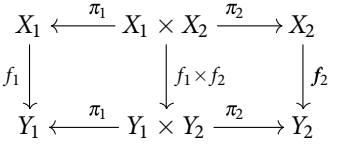
Figure 5. Computational trinitarianism.

Definition 8. *Given a category $\mathscr{C}$, we define the following objects*

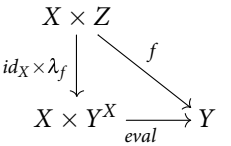
- ***Product of objects*** X, Y**:** *Object $X \times Y$ of $\mathscr{C}$ together with projection arrows $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ such that any diagram*



  commutes for a unique $\langle f, g \rangle : Z \to X \times Y$.
- ***Terminal object:*** *Object 1 of $\mathscr{C}$ such that for any other $X : \mathscr{C}$, there exists a unique arrow $!_X : X \to 1$.*
- ***Product of arrows*** $f_1 : X_1 \to Y_1$**,** $f_2 : X_2 \to Y_2$**:** *Unique arrow $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ making the following diagram commute*

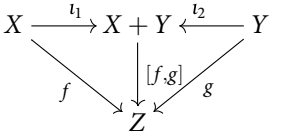


- ***Exponential of objects*** Y ***and*** X**:** *An object Y^X of $\mathscr{C}$ together with an arrow $eval : X \times Y^X \to Y$ such that any diagram*



  commutes for a unique $\lambda_f : Z \to Y^X$.
- ***Element of an object*** X**:** *An arrow $x : 1 \to X$.*
- ***Coproduct of objects*** X, Y**:** *Object $X + Y$ of $\mathscr{C}$ together with injection arrows $\iota_1 : X \to X + Y$, $\iota_2 : Y \to X + Y$ such that any diagram*

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commutes for a unique $[f, g] : X + Y \to Z$.

- ***Initial object:*** *Object* 0 *of* $\mathscr{C}$ *such that for any other* $X : \mathscr{C}$*, there exists a unique arrow* $0_X : 0 \to X$.

The defining properties of these categorical constructions correspond to rewritings of deductions in NJp, as clearly documented in Lambek and Scott (1988, Ch. I.8). Note, however, that the system of rewritings imposed by the universal properties of the initial object and coproducts *is a proper extension* of Ξ discussed in the previous sections. Therefore, we refer to the categorical counterparts of $\bot$ and $\vee$ with respect to system Ξ as the **weak initial object** and **weak coproduct**, respectively.

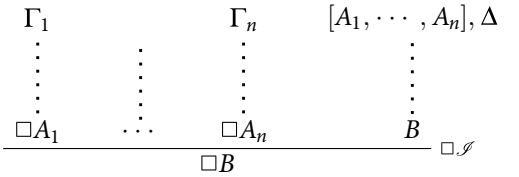
3. Proof-as-Programs for Intuitionistic Epistemic Logics

We now introduce our natural deduction systems IEL^- and IEL which are logically equivalent to $\mathbb{IEL}^-$ and $\mathbb{IEL}$, respectively. In contrast to axiomatic calculi, they are also capable of a computational reading of the epistemic operator and proofs involving this kind of modalities.

The minimal starting point is proving that our natural deduction calculi L, for $\mathsf{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$, are sound and complete with respect to $\mathbb{L}$. Subsequently, we prove that proofs in L can be named by means of λ-terms as stated in the proofs-as-programs paradigm for intuitionistic logic. By using this formalism, we modularly prove a strong normalisation theorem for IEL^- and IEL stating that detours and permutation conversions can be eliminated from all deductions in our calculi.

3.1 Design of the calculi

Definition 9 (From Perini Brogi (2021)). *Let* IEL^- *be the calculus extending* NJp *by the following rule:*



where Γ *and* Δ *are* multisets of formulas, *and* $A_1, \cdots, A_n$ *are* all *discharged.*[8]

We say that B *is the major premise of the rule, and each* $\Box A_i$ *is a minor premise, whose corresponding discharged assumption is* A_i.

The rule can be informally read as follows: Suppose that one can prove B by assuming, among others, $A_1, \cdots, A_n$. Then, if all $A_1, \cdots, A_n$ have been verified, then there is sufficient evidence to state that B is verified, without assuming $A_1, \cdots, A_n$.

The natural deduction system for intuitionistic knowledge we introduce here is obtained by *extending* IEL^- with an elimination rule for the epistemic modality. This rule is the operational counterpart of intuitionistic reflection and captures the intuitionistic principle of the factivity of knowledge described in the introduction.

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Definition 10. *Let* IEL *be the system extending the natural deduction calculus* IEL$^-$ *of Definition 9 by the following elimination rule:*

$$\dfrac{\begin{array}{cc}\Gamma & [A],\Delta \\ \vdots & \vdots \\ \Box A & \bot\end{array}}{\bot}\ \Box\mathscr{E}$$

where Γ *and* Δ *are* multisets of formulas *and* A *is discharged by* $\Box\mathscr{E}$.

We say that $\Box A$ *is the major premise of the rule and* $\bot$ *is its minor premise; the discharged assumption of the rule is* A.

The rule closely corresponds to the truth-tracking definition of knowledge: Suppose that one can prove $\Box A$, that is, she is allowed to believe A. However, if it is possible to prove that A is false, then some of the overall assumptions must be reviewed.

As for the axiomatic counterpart, we write $\Gamma \Rightarrow_{\mathsf{L}} A$ when A is derivable in L for $\mathsf{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$ assuming the set of hypotheses Γ, and we write $\mathsf{L} \vdash A$ when $\varnothing \Rightarrow_{\mathsf{L}} A$.

Let us immediately check that we are dealing with the same logics introduced by Artemov and Protopopescu (2016):

Lemma 11. $\Gamma \Rightarrow_{\mathsf{L}} A$ *iff* $\Gamma \vdash_{\mathbb{L}} A$.

Proof. Assume $\Gamma \vdash_{\mathbb{L}} A$. We proceed by induction on the derivation. We only need to check that the axiom schemas can be derived in the natural deduction calculus:

- Intuitionistic schemas are dealt with NJp rules;
- K: $\dfrac{\Box(A \to B) \qquad \Box A \qquad \begin{array}{c}[A \to B, A] \\ \vdots \\ B\end{array}}{\Box B}\ \Box\mathscr{J}$;
- co-reflection: $\dfrac{\dfrac{[A]^1}{\Box A}\ \Box\mathscr{J}}{A \to \Box A}\ {\to}\mathscr{J}{:}1$;
- finally, for the schema $\Box A \to \neg\neg A$, we have in IEL $\dfrac{\dfrac{\dfrac{[\Box A]^3 \qquad \dfrac{[A]^1 \qquad [\neg A]^2}{\bot}\ {\to}\mathscr{E}}{\bot}\ \Box\mathscr{E}{:}1}{\neg\neg A}\ {\to}\mathscr{J}{:}2}{\Box A \to \neg\neg A}\ {\to}\mathscr{J}{:}3$

Conversely, assume $\Gamma \Rightarrow_{\mathsf{L}} A$.

We need to consider first the $\Box\mathscr{J}$ rule: By induction hypothesis, we have $\Gamma_1 \vdash_{\mathbb{L}} \Box A_1, \cdots, \Gamma_n \vdash_{\mathbb{IEL}^-} \Box A_n$, and $A_1, \cdots, A_n, \Delta \vdash_{\mathbb{L}} B$. Then we have $\Gamma_1, \cdots, \Gamma_n \vdash_{\mathbb{L}} \Box A_1 \wedge \cdots \wedge \Box A_n$, and, by the deduction theorem for $\mathbb{L}$ and ordinary logic, $\Delta \vdash_{\mathbb{L}} A_1 \wedge \cdots \wedge A_n \to B$.

By co-reflection $\mathbb{IEL}^- \vdash (A_1 \wedge \cdots \wedge A_n \to B) \to \Box(A_1 \wedge \cdots \wedge A_n \to B)$, and by K-schema $\mathbb{IEL}^- \vdash \Box(A_1 \wedge \cdots \wedge A_n \to B) \to \Box(A_1 \wedge \cdots \wedge A_n) \to \Box B$. Hence, we have $\Delta \vdash_{\mathbb{IEL}^-} \Box(A_1 \wedge \cdots \wedge A_n) \to \Box B$, whence, by modal logic, we obtain $\Delta \vdash_{\mathbb{IEL}^-} \Box A_1 \wedge \cdots \wedge \Box A_n \to \Box B$, which gives $\Gamma_1, \cdots, \Gamma_n, \Delta \vdash_{\mathbb{IEL}^-} \Box B$, as desired.

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For the $\Box\mathscr{E}$ rule, we have, by inductive hypothesis, $\Gamma \vdash_{\mathbb{IEL}} \Box A$ and $A, \Delta \vdash_{\mathbb{IEL}} \bot$. Then by *MP* with factivity of knowledge, we have $\Gamma \vdash_{\mathbb{IEL}} \neg\neg A$; moreover, by the deduction theorem, $\Delta \vdash_{\mathbb{IEL}} \neg A$. An application of *MP* gives the result. □

3.2 Normalisation

As for NJp, a modal λ-calculus is obtained by decorating L-deductions with proof names.

The λ-term corresponding to $\Box\mathscr{I}$ is indeed ruled by the single constructor:

$$\frac{\Gamma_1 \Rightarrow f_1 : \Box A_1 \qquad \cdots \qquad \Gamma_n \Rightarrow f_n : \Box A_n \qquad x_1 : A_1, \cdots, x_n : A_n, \Delta \Rightarrow g : B}{\Gamma_1, \cdots, \Gamma_n, \Delta \Rightarrow (\mathsf{box}[x_1, \cdots, x_n].\, g \;\mathsf{with}\, f_1, \cdots, f_n) \;:\; \Box B}$$

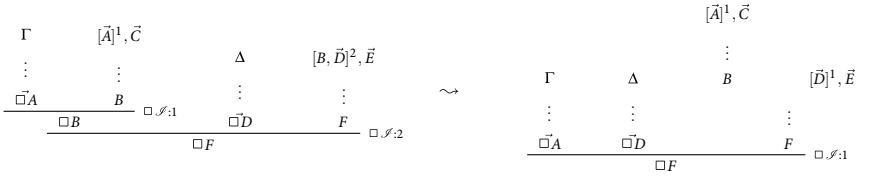
The λ-term corresponding to $\Box\mathscr{E}$ provides the eliminator for modal terms, as expected:

$$\frac{\Gamma \Rightarrow f : \Box A \qquad x : A, \Delta \Rightarrow g : \bot}{\Gamma, \Delta \Rightarrow \; (\mathsf{unbox}\, f \;\mathsf{with}\, x.g) \;:\; \bot}$$

We present next normalisation results for these modal λ-calculi with respect to rewriting systems having precise proof-theoretic relevance.

3.2.1 Normal IEL⁻-deductions

In order to eliminate potential modal detours from IEL⁻-deductions, we introduce the following proof-rewriting:



This rewriting collapses two applications of $\Box\mathscr{I}$ into a single one, making the deduction much simpler, so that can be correctly called a detour elimination. Let's denote it by $\rho_{\Box}$.

It is not hard to prove that IEL⁻-deductions strongly normalise with respect to the rewriting system obtained by adding this detour elimination to the rewriting system Ξ for NJp recalled in Section 2.2:

Lemma 12 *Deductions in* IEL⁻ *strongly normalise with respect to the rewriting system obtained by adding* $\rho_{\Box}$ *to* Ξ.

Proof. From Perini Brogi (2021): We define a translation[9] $|-|$ from the λ-calculus of IEL⁻-deductions to typed λ-calculus with products, sums, unit and empty types:

$$\begin{aligned}
|p| &:= p\\
|\bot| &:= \bot\\
|\top| &:= \top\\
|A \to B| &:= |A| \to |B|\\
|A \wedge B| &:= |A| \wedge |B|\\
|A \vee B| &:= |A| \vee |B|\\
|\Box A| &:= (|A| \to q) \to q
\end{aligned}$$

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$$|x| := x$$
$$|\bot_J(f)| := \bot_J(|f|)$$
$$|*| := *$$
$$|\lambda x.f| := \lambda x.|f|$$
$$|fg| := |f||g|$$
$$|(f,g)| := (|f|,|g|)$$
$$|\pi_i(f)| := \pi_i(|f|)$$
$$|\mathsf{C}(f, x.f_1, y.f_2)| := \mathsf{C}(|f|, x.|f_1|, y.|f_2|)$$
$$|\mathsf{in}_i(f)| := \mathsf{in}_i(|f|)$$
$$|\mathsf{box}[x_1,\cdots,x_n].g \text{ with } f_1,\cdots,f_n| := \lambda k.|f_1|(\lambda x_1.\cdots|f_n|(\lambda x_n.\, k|g|)\cdots)$$

where q is an arbitrary atom.

For the sake of readability, it is convenient to rephrase the translation of $\Box.\mathscr{I}$ in the natural deduction formalism – we consider w.l.o.g. the case with a single minor premise:

$$\dfrac{\begin{array}{cc}\Gamma & [A],\Delta\\ \vdots & \vdots\\ f & g\\ \Box A & B\end{array}}{\Box B}\,\Box\mathscr{I} \quad\longmapsto\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}|\Gamma|\\ \vdots\\ |f|\end{array}}{(|A|\to q)\to q} \qquad \dfrac{\dfrac{[|B|\to q]^2 \qquad \begin{array}{c}[|A|]^1,|\Delta|\\ \vdots\\ |g|\end{array}}{|B|}\,{\to}\mathscr{E}}{|A|\to q}\,{\to}\mathscr{I}{:}1}{q}\,{\to}\mathscr{E}}{(|B|\to q)\to q}\,{\to}\mathscr{I}{:}2$$

To prove strong normalisation, it suffices now to prove that $\rho_\Box$ is preserved by this translation, which is almost straightforward. Thus, if IEL^- was not normalising, then we would have an infinite chain of rewritings starting from, say, $f : A$. But this would lead to an infinite chain of reductions of NJp-deductions with respect to Ξ starting from $|f| : |A|$, contradicting strong normalisation for NJp with respect to Ξ (Theorem 5). □

The system of rewritings $\Xi + \rho_\Box$, however, does not suffice to establish analyticity of IEL^- in terms of a subformula principle, as discussed in Section 3.2.3. To see that, consider, for instance, the following IEL^--deduction f:

$$\dfrac{\dfrac{\Box(A\to(B\to C))\vee\bot \qquad \dfrac{\Box(A\to(B\to C))^3 \qquad \Box A \qquad \dfrac{A\to(B\to C)^1 \qquad A^2}{B\to C}\,{\to}\mathscr{E}}{\Box(B\to C)}\,\Box\mathscr{I}{:}1,2 \qquad \dfrac{\bot^4}{\Box(B\to C)}\,\bot_J}{\Box(B\to C)}\,{\vee}\mathscr{E}{:}3,4$$

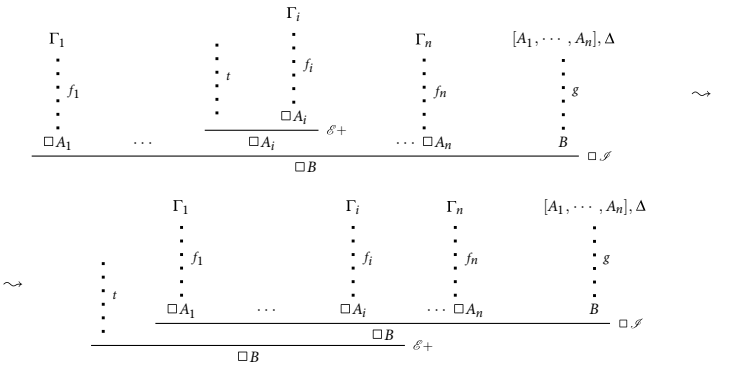
Then we see that the IEL^--deduction

$$\dfrac{\begin{array}{ccc}\vdots & & \dfrac{B\to C^5 \qquad B^6}{C}\,{\to}\mathscr{E}\\ f & & \\ \Box(B\to C) & \Box B & \end{array}}{\Box C}\,\Box\mathscr{I}{:}5,6$$

is normal with respect to $\Xi + \rho_\Box$. In spite of this, the formula $\Box(B \to C)$ is not a subformula of either the conclusion $\Box C$ or of the undischarged hypotheses $\{\Box A, \Box B, \Box(A \to (B \to C)) \vee \bot\}$.

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Thus, we add to the system $\Xi + \rho_\Box$ – previously discussed in Perini Brogi (2021) – the following permutation conversions $\pi_\Box$:



where $\mathscr{E}+$ is $\vee\mathscr{E}$ or $\bot_J$.

We now prove that IEL^--deductions strongly normalise with respect to this extended system of rewritings. Unfortunately, the translation $|-|$ does not preserve the required additional rewritings, but a variant of Dragalin–Friedman translation does the job.

Theorem 13. *Deductions in* IEL^- *strongly normalise with respect to the rewriting system* $\Xi + \rho_\Box + \pi_\Box$.

Proof. We define a new translation $\langle - \rangle$ from our modal λ-calculus to simple type theory with products, sums, unit and empty types as follows:

$$\begin{aligned}
\langle \bot \rangle &:= \bot \\
\langle \top \rangle &:= \top \\
\langle p \rangle &:= p \\
\langle A \to B \rangle &:= \langle A \rangle \to \langle B \rangle \\
\langle A \wedge B \rangle &:= \langle A \rangle \wedge \langle B \rangle \\
\langle A \vee B \rangle &:= \langle A \rangle \vee \langle B \rangle \\
\langle \Box A \rangle &:= \langle A \rangle \vee q
\end{aligned}$$

$\langle x \rangle := x$

$\langle * \rangle := *$

$\langle \bot_J(f) \rangle := \bot_J(\langle f \rangle)$

$\langle \lambda x f \rangle := \lambda x.\langle f \rangle$

$\langle fg \rangle := \langle f \rangle \langle g \rangle$

$\langle (f, g) \rangle := (\langle f \rangle, \langle g \rangle)$

$\langle \pi_i(f) \rangle := \pi_i(\langle f \rangle)$

$\langle \mathsf{C}(f, x.f_1, y.f_2) \rangle := \mathsf{C}(\langle f \rangle, x.\langle f_1 \rangle, y.\langle f_2 \rangle)$

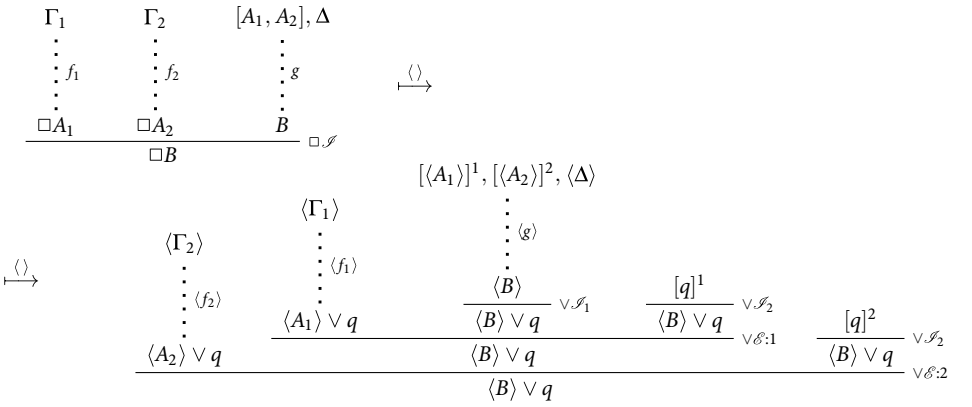
$\langle \mathsf{in}_i(f) \rangle := \mathsf{in}_i(\langle f \rangle)$

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$$\langle \mathsf{box}[x_1, \cdots, x_n].\, g \text{ with } f_1, \cdots, f_n \rangle$$
$$:= \mathsf{C}(\langle f_n \rangle, x_n. \cdots \mathsf{C}(\langle f_2 \rangle, x_2.\mathsf{C}(\langle f_1 \rangle, x_1.\mathsf{in}_1(\langle g \rangle)), y_1.\mathsf{in}_2(y_1)), y_2.\mathsf{in}_2(y_2)) \cdots, y_n.\mathsf{in}_2(y_n))$$

where q is an arbitrary atom.

As in the proof of the previous normalisation lemma, for the sake of readability, it is convenient to rephrase the translation of $\Box\mathscr{I}$ in the natural deduction formalism – this time we can consider w.l.o.g. the case with two minor premises to make the pattern explicit:



It is now straightforward to check that both $\rho_\Box$ *and* $\pi_\Box$ are preserved by this translation and recovered by permutation conversions of Ξ.

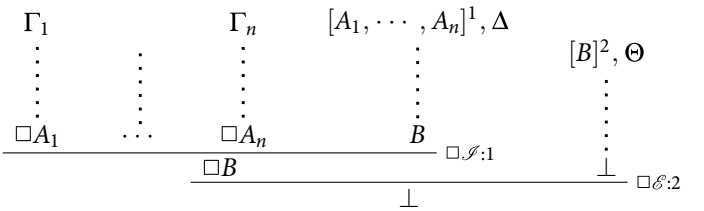
Since deductions in NJp are strongly normalising with respect to Ξ, we have the desired strong normalisation for IEL$^-$-deductions. □

Corollary 14. *Every* IEL$^-$*-deduction uniquely reduces to a normal* IEL$^-$*-deduction with respect to* $\Xi + \rho_\Box + \pi_\Box$.

Proof. We need to prove that every term of the modal λ-calculus for IEL$^-$-deductions has a unique normal form with respect to $\Xi + \rho_\Box + \pi_\Box$. That holds if that rewriting system has the Church–Rosser property. It is not hard to see that it does satisfy the *weak* Church–Rosser property: The desired uniqueness property then follows by Newman's lemma for rewriting systems.[10] □

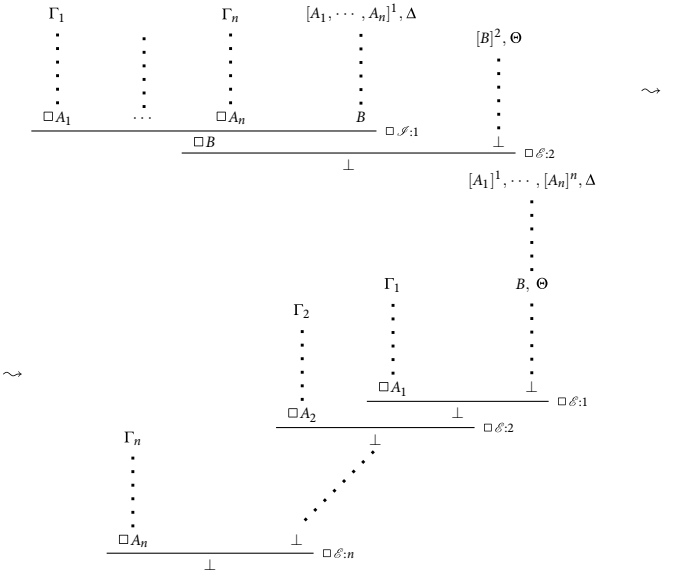
3.2.2 Normal IEL-deductions

The presence of an introduction *and* an elimination rule for the modal operator in our new calculus IEL requires to consider potential detours with the following general pattern

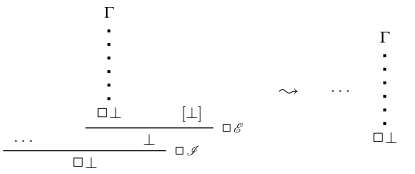


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This situation is easily handled by the following rewriting that we denote by $\delta_\Box$:



Furthermore, we need to consider the following $\gamma_\Box$ rewriting:



We now prove that IEL-deductions strongly normalise with respect to the system of rewritings $\Xi + \rho_\Box + \pi_\Box + \delta_\Box + \gamma_\Box$. All we need to do is to specialise the proof strategy for Theorem 13.

Theorem 15. *Deductions in* IEL *strongly normalise with respect to the rewriting system* $\Xi + \rho_\Box + \pi_\Box + \delta_\Box + \gamma_\Box$.

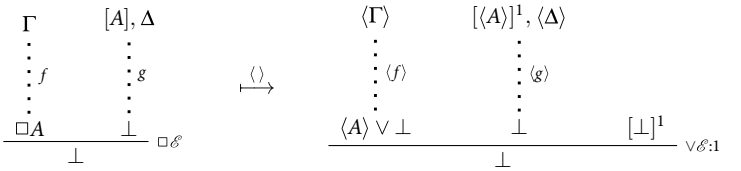
Proof. Tweak the translation $\langle - \rangle$ in the proof of Theorem 13 as follows:

$$\begin{aligned}
\langle \bot \rangle &:= \bot \\
\langle \top \rangle &:= \top \\
\langle p \rangle &:= p \\
\langle A \to B \rangle &:= \langle A \rangle \to \langle B \rangle \\
\langle A \wedge B \rangle &:= \langle A \rangle \wedge \langle B \rangle \\
\langle A \vee B \rangle &:= \langle A \rangle \vee \langle B \rangle \\
\langle \Box A \rangle &:= \langle A \rangle \vee \bot
\end{aligned}$$

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$$\langle x\rangle := x$$
$$\langle *\rangle := *$$
$$\langle \bot_J(f)\rangle := \bot_J(\langle f\rangle)$$
$$\langle \lambda x\, f\rangle := \lambda x.\langle f\rangle$$
$$\langle fg\rangle := \langle f\rangle\langle g\rangle$$
$$\langle (f,g)\rangle := (\langle f\rangle, \langle g\rangle)$$
$$\langle \pi_i(f)\rangle := \pi_i(\langle f\rangle)$$
$$\langle \mathsf{C}(f, x.f_1, y.f_2)\rangle := \mathsf{C}(\langle f\rangle, x.\langle f_1\rangle, y.\langle f_2\rangle)$$
$$\langle \mathsf{in}_i(f)\rangle := \mathsf{in}_i(\langle f\rangle)$$
$$\langle \mathsf{box}[x_1,\cdots,x_n].\, g \text{ with } f_1,\cdots,f_n\rangle$$
$$:= \mathsf{C}(\langle f_n\rangle, x_n.\cdots \mathsf{C}(\langle f_2\rangle, x_2.\mathsf{C}(\langle f_1\rangle, x_1.\mathsf{in}_1(\langle g\rangle)), y_1.\mathsf{in}_2(y_1)), y_2.\mathsf{in}_2(y_2))\cdots, y_n.\mathsf{in}_2(y_n))$$
$$\langle \mathsf{unbox}\, f \text{ with } x.g\rangle := \mathsf{C}(\langle f\rangle, x.\langle g\rangle, y.y).$$

For the sake of readability, it is again convenient to rephrase the translation of $\Box\mathscr{E}$ in the natural deduction formalism:



This translation preserves $\delta_\Box$ and $\gamma_\Box$ too, by means of applications of rewritings in Ξ concerning $\vee$-detours and permutations. Thus, since deductions in NJp are strongly normalising with respect to Ξ (Theorem 5), we have the desired strong normalisation for IEL-deductions. □

3.2.3 Analyticity of normal deductions and further corollaries

In the proofs-as-programs paradigm, a normalisation theorem only states that any execution of a program written in our modal λ-calculi safely terminates. However, from the purely proof-theoretic point of view, our normalisation theorems are much more informative about the structure of the deductions in L, for they let us prove that normal deductions in our calculi satisfy the subformula property. Subformula property is very easy to state:

Theorem 16 (Subformula principle). *Every formula B occurring in a normal* L*-deduction f of A from assumptions Γ is a subformula of A or of some formula in Γ.*

Nevertheless, giving a detailed proof of it is known to be lenghty and quite technical. We thus prefer to focus here on interesting corollaries of normalisation and analyticity for deductions in intuitionistic epistemic logics.[11]

In Artemov and Protopopescu (2016), these properties are established by means of semantic techniques. Here, they are all proven by purely proof-theoretic remarks about the natural deduction systems we introduced for epistemic logics.

Lemma 17. *The logics of intuitionistic belief and intutionistic knowledge are consistent.*

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Proof. If it was not the case, then we would have $\mathsf{L} \vdash \bot$. By normalisation, there would be a normal L-deduction of $\bot$ from no hypothesis, which is impossible because of the subformula principle and the absence of any $\mathscr{I}$-rule for $\bot$. □

Alternative Proof. The consistency of L is a corollary of Lemma 20, without recurring to the subformula principle. □

Lemma 18. *The calculus* IEL^- *is decidable.*

Proof. By Theorem 16, proof search in L is bounded by the degree of the formula one wishes to deduce. □

Let's say – following, for example, Girard et al. (1989) – that a deduction is **neutral** if it consists of an assumption, or its last rule is not an $\mathscr{I}$-rule.

Proposition 19. *In any normal and neutral* L*-deduction of* A *from* Γ, $\Gamma \neq \varnothing$.

Proof. By induction on the height $\mathfrak{h}$ of the given deduction.
The case $\mathfrak{h} = 0$ is trivial.
If $\mathfrak{h} > 0$, we proceed by considering the last rule of the deduction:

a. if that rule is $\rightarrow \mathscr{E}$, we can apply the inductive hypothesis to the major premise, which is neutral by normality of the deduction;
b. if that rule is $\wedge \mathscr{E}$, we can apply the inductive hypothesis to the premise, which is neutral by normality of the deduction;
c. if that rule is $\vee \mathscr{E}$, we can apply the inductive hypothesis to the major premise, which is neutral by normality of the deduction;
d. if that rule is $\bot_J$, we can apply the inductive hypothesis to the premise, which is neutral since there are no $\mathscr{I}$-rules for $\bot$;
e. for IEL only, if the last rule is $\Box \mathscr{E}$, then the inductive hypothesis can be applied to the subdeduction ending in the major premise of that rule.

□

Lemma 20 (Canonicity). *In any normal* L*-deduction of* A*, the last rule applied is the introduction rule for the main connective of* A.

Proof. By Proposition 19, since $\Gamma = \varnothing$ here, and the deduction is normal by assumption, its last rule cannot be an elimination. □

Corollary 21 (Admissibility of the reflection rule). *If* $\mathsf{L} \vdash \Box A$, *then* $\mathsf{L} \vdash A$.

Proof. By Lemma 20, since $\Box A$ is deducible from no assumption, its last rule in the corresponding normal deduction must be $\Box \mathscr{I}$. □

Corollary 22 (Disjunction property). *If* $\mathsf{L} \vdash A \vee B$, *then* $\mathsf{L} \vdash A$ *or* $\mathsf{L} \vdash B$.

Proof. Assume $\mathsf{L} \vdash A \vee B$. Then consider a normal L-deduction of $A \vee B$. By the structure of normal deductions, that deduction must end either by $\bot_J$, $\vee \mathscr{I}_1$ or $\vee \mathscr{I}_2$. But since that very deduction has no assumptions, it cannot end by $\bot_J$. Then we have the result. □

Corollary 23 ($\Box$-primality). *If* $\mathsf{L} \vdash \Box A \vee \Box B$, *then* $\mathsf{L} \vdash A$ *or* $\mathsf{L} \vdash B$.

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Proof. The result follows by the disjunction property and the admissibility of the reflection rule. □

Corollary 24 (Modal disjunction property). *If* $\mathsf{L} \vdash \Box(A \vee B)$, *then* $\mathsf{L} \vdash \Box A$ *or* $\mathsf{L} \vdash \Box B$.

Proof. If we have a deduction of $\Box(A \vee B)$, then by the reflection rule we have a deduction of $A \vee B$. By the disjunction property, we have a deduction of A or a deduction of B. In each case, by applying $\Box\mathscr{I}$ we have the desired result. □

By a pure structural analysis of IEL and IEL^-, we can also show that the the logic of intuitionistic knowledge is a proper extension of the logic of intuitionistic belief, as shown by Artemov and Protopopescu (2016) via a semantic argument.

Lemma 25. $\mathsf{IEL}^- \subsetneq \mathsf{IEL}$.

Proof. Consider the formula $\neg\Box\bot$.

That formula is clearly provable in IEL:

$$\cdot\ \dfrac{\dfrac{[\Box\bot]^2 \qquad [\bot]^1}{\bot}\,\Box\mathscr{E}{:}1}{\neg\Box\bot}\,{\to}\mathscr{I}{:}2$$

Now suppose that $\mathsf{IEL}^- \vdash \neg\Box\bot$ too. Then by the normalisation theorem (Theorem 13), there exists a normal IEL^--deduction of $\neg\Box\bot$. By Lemma 20, the last rule of that deduction must be an $\to\mathscr{I}$ so that $\bot$ is derived from the hypothesis $\Box\bot$ in a *normal* IEL^--deduction. By the structure of normal deduction, and the subformula principle (Theorem 16), that deduction must consist of $\mathscr{I}$-rules, and all formulas occurring in it can be among $\{\Box\bot, \bot\}$ only. But these two requirements cannot be simultaneously satisfied so that $\mathsf{IEL}^- \nvdash \neg\Box\bot$ after all. □

A similar structural and combinatorial reasoning establishes also the following result mentioned by Artemov and Protopopescu (2016):

Lemma 26. *Intuitionistic verifications do not have the disjunction property, that is,* $\mathsf{L} \nvdash \Box(A \vee B) \to \Box A \vee \Box B$.

4. Proofs-as-Arrows for Intuitionistic Epistemic Logics

If λ-calculus gives a computational semantics of proofs in NJp – and, as we showed in the previous section, in our natural deduction systems for intutionistic epistemic logic also – category theory furnishes the tools for an 'algebraic' semantics which is *proof-relevant* – that is, contrary to traditional algebraic semantics based on Heyting algebras and to relational semantics based on Kripke models, it focuses on the very notion of proof, distinguishing between different deductions of the same formula.

We know – after Section 4 and the seminal work by Lambek and Scott (1988) – that cartesian product models conjunction and exponential models implication. Any category having products and exponentials for any of its objects is called **cartesian closed** (CCCat); moreover, if it has also (weak) coproducts – modelling disjunction – it is called **bi-cartesian closed** (bi-CCCat): $\top$ and $\bot$ correspond to empty product – the terminal object 1 – and empty coproduct – the (weak) initial object 0 – respectively.

For our calculi, in order to capture the behaviour of the epistemic modalities, some more structure is required, which is what will be sketched in the next sections, extending the categorical

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semantics of deductions started in our previous work about intuitionistic belief (Perini Brogi 2021) to the new calculus dealing with the knowledge modality.

4.1 Some functors and transformations on monoidal categories

Definition 27. *Given a CCCat $\mathscr{C}$, a monoidal endofunctor consists of a functor $\mathfrak{F} : \mathscr{C} \to \mathscr{C}$ together with*

- *a natural transformation*

$$m_{A,B} : \mathfrak{F}A \times \mathfrak{F}B \to \mathfrak{F}(A \times B);$$

- *a morphism*

$$m_1 : 1 \to \mathfrak{F}1,$$

preserving the monoidal structure of $\mathscr{C}$.[12] *These are called structure morphisms of $\mathfrak{F}$.*

The seminal work by de Paiva and Ritter (2011) proved that a monoidal endofunctor on the category of logical formulas induces a modal operator satisfying K-schema, thus providing a categorical interpretation of the minimal intuitionistic modal logic IK.

Definition 28. *Given any category $\mathscr{C}$, an endofunctor $\mathfrak{F} : \mathscr{C} \to \mathscr{C}$ is pointed iff there exists a natural transformation*

$$\pi : Id_{\mathscr{C}} \Rightarrow \mathfrak{F}$$

$$\pi_A : A \to \mathfrak{F}A$$

$$\begin{array}{ccc} A & \xrightarrow{\pi_A} & \mathfrak{F}A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{F}f} \\ B & \xrightarrow[\pi_B]{} & \mathfrak{F}B \end{array}$$

π is called the point of $\mathfrak{F}$.

In intuitionistic epistemic logics, a pointed endofunctor on the category of logical formulas 'represents' the co-reflection schema.

Since we want to give a semantics of proofs – and not simply of derivability – in L, we need a further notion from category theory.

Definition 29. *Given a monoidal category $\mathscr{C}$, and monoidal endofunctors $\mathfrak{F}, \mathfrak{G} : \mathscr{C} \to \mathscr{C}$, a natural transformation $\kappa : \mathfrak{F} \Rightarrow \mathfrak{G}$ is monoidal when the following commute:*

$$\begin{array}{ccc} \mathfrak{F}A \times \mathfrak{F}B & \xrightarrow{m^{\mathfrak{F}}_{A,B}} & \mathfrak{F}(A \times B) \\ {\scriptstyle \kappa_A \times \kappa_B}\downarrow & & \downarrow{\scriptstyle \kappa_{A\times B}} \\ \mathfrak{G}A \times \mathfrak{G}B & \xrightarrow[m^{\mathfrak{G}}_{A,B}]{} & \mathfrak{G}(A \times B) \end{array}$$

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and

$$\begin{array}{ccc} 1 & \xrightarrow{m_1^{\mathfrak{F}}} & \mathfrak{F}(1) \\ \| & & \downarrow{\scriptstyle \kappa_1} \\ 1 & \xrightarrow[m_1^{\mathfrak{G}}]{} & \mathfrak{G}(1) \end{array}$$

4.2 Categorical interpretation

For our intents, all the previous categorical notions will be packed up in the following structures.[13]

Definition 30. *An* IEL^-*-category is given by a bi-CCCat* $\mathscr{C}$ *together with a monoidal pointed endofunctor* $\mathfrak{K}$ *whose point* κ *is monoidal.*

When $\mathfrak{K}$ *preserves the initial object* 0 *of* $\mathscr{C}$ *up-to-isomorphism, we say that* $\mathfrak{K}$ *is* dense*, and* $\mathscr{C}$ *is an* IEL*-category.*

The reader is invited to check that by taking the poset reflection of an L-model – for $\mathsf{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$ – one obtains a model for *provability* in the logics of intuitionistic belief and knowledge according to the usual algebraic semantics for modal logics, that is, a Heyting algebra with an operator satisfying the axioms and rules of $\mathbb{IEL}^-$ and $\mathbb{IEL}$, respectively.

It is not hard to see that $\rho_\Box$ is what is needed to prove that the construction induced by the modal operator preserves arrow composition. However, that construction cannot be a functor unless one considers the following extensionality principle $\eta_\Box$ for $\Box\mathscr{I}$:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \Box A\end{array} \qquad \begin{array}{c}[A] \\ \end{array}}{\Box A}\,\Box\mathscr{I} \quad \rightsquigarrow \quad \begin{array}{c}\Gamma \\ \vdots \\ \Box A\end{array}$$

IEL^--deductions strongly normalise with respect to the rewriting system $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$: The translation $\langle - \rangle$ is capable of mimicking $\eta_\Box$ in a simple type theory extended by a sum extensionality principle $\eta_\vee$, which in type-theoretic syntax can be defined as

$$\mathsf{C}(f, x.\mathsf{in}_1(x), y.\mathsf{in}_2(y)) \rightsquigarrow f.$$

Normalisation of L-deductions with respect to $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$ is then derived from the fact that $\eta_\vee$-reductions can be postponed in any sequence of $\Xi + \eta_\vee$-rewritings so that simple type theory is normalising with respect to $\Xi + \eta_\vee$ too (Sørensen and Urzyczyn 2006, Ch. 4).

Henceforth, we can now show the adequacy of this categorical interpretation.

Theorem 31 (Soundness). *Let* $\mathscr{C}$ *be an* IEL^-*-category. Then there is a canonical interpretation* $[\![-]\!]$ *of* IEL^- *in* $\mathscr{C}$ *such that*

- *a formula* A *is mapped to a* $\mathscr{C}$*-object* $[\![A]\!]$;
- *a deduction* f *of* $A_1, \cdots, A_n \Rightarrow_{\mathsf{IEL}^-} B$ *is mapped to an arrow*
$$[\![f]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!];$$
- *for any two deductions* f *and* g *which are equal modulo* $\Xi + \rho_\Box + \eta_\Box$*-rewritings, we have* $[\![f]\!] = [\![g]\!]$.

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The analogous soundness result holds for any IEL*-category, by considering the system of rewritings* $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box$ *extended by* $\eta_\Box$.

Proof. By structural induction on $f : \vec{A} \Rightarrow_{\mathsf{IEL}^-} B$.

The intuitionistic cases are interpreted according to the remarks about bi-CCCats at the beginning of this section.

We overload the notation using $\langle \Box, m, \kappa \rangle$ for the monoidal pointed endofunctor of $\mathscr{C}$, its structure morphisms, and its point.

The deduction

$$\frac{f_1 : \Gamma_1 \Rightarrow \Box A_1 \qquad \cdots \qquad f_n : \Gamma_n \Rightarrow \Box A_n \qquad g : [A_1, \cdots, A_n], C_1, \cdots, C_m \Rightarrow B}{\Box B}$$

is mapped to

$$(\Box[\![g]\!]) \circ m_{[\![A_1]\!],\cdots,[\![A_n]\!],[\![C_1]\!],\cdots,[\![C_m]\!]} \circ [\![f_1]\!] \times \cdots \times [\![f_n]\!] \times \kappa_{[\![C_1]\!]} \times \cdots \times \kappa_{[\![C_m]\!]},$$

where $m_{X_1,\cdots,X_n}$ is defined inductively as

$$m_{X_1,\cdots,X_{n-1},X_n} := m_{X_1 \times \cdots \times X_{n-1},X_n} \circ (m_{X_1,\cdots,X_{n-1}}) \times id_{\Box X_n}.$$

It is straightforward to check that identity modulo $\eta_\Box$ holds in the category $\mathscr{C}$ by functoriality of $\Box$.

Identity modulo $\rho_\Box$ is also valid by naturality of m and κ: Lengthy but direct categorical calculations require that κ must be monoidal in order to model correctly the following special case

$$\frac{\dfrac{\Gamma_1 \quad \vdots \quad A_1}{\Box A_1}\,\Box\mathscr{I} \qquad \cdots \qquad \dfrac{\Gamma_n \quad \vdots \quad A_n}{\Box A_n}\,\Box\mathscr{I} \qquad [A_1, \cdots, A_n]^1, C_1, \cdots, C_m \quad \vdots \quad B}{\Box B}\,\Box\mathscr{I}{:}1$$

$$\rightsquigarrow$$

$$\frac{\Gamma_1 \quad \vdots \quad A_1 \qquad \cdots \qquad \Gamma_n \quad \vdots \quad A_n \qquad C_1, \cdots, C_m \quad \vdots \quad B}{\Box B}\,\Box\mathscr{I}$$

For IEL, we furthermore have the density property of the functor $\Box$, which validates the identity of deductions modulo $\delta_\Box$ and $\gamma_\Box$. □

It remains to show that this interpretation is also complete.

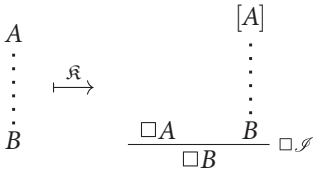
Theorem 32 (Completeness). *If the interpretation of two* L*-deductions is equal in all* L*-categories, then the two deductions are equal modulo* $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$.

Proof. We proceed by constructing a term model for the modal λ-calculus for L-deductions. Consider the following category $\mathscr{M}$:

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- its objects are formulas;
- an arrow $f : A \to B$ is an L-deduction of B from A modulo rewritings in $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$;
- identities are given by assuming a hypothesis;
- composition is given by transitivity of deductions.

Then $\mathscr{M}$ has a bi-cartesian closed structure given by Ξ-rewritings for NJp.
Moreover, the modal operator $\Box$ induces a functor $\mathfrak{K}$ by mapping A to $\Box A$, and



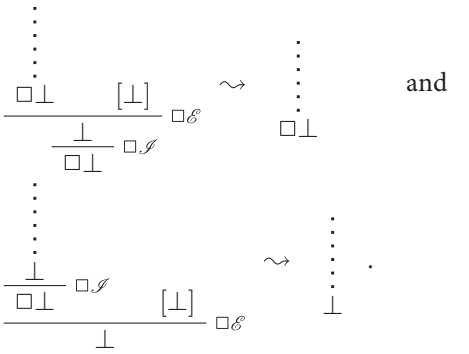
which preserves identities by $\eta_\Box$ and preserves composition as a special case of $\rho_\Box$.

The structure morphism is given by

$$\dfrac{\dfrac{\Box A \wedge \Box B}{\Box A}\wedge\mathscr{E} \quad \dfrac{\Box A \wedge \Box B}{\Box B}\wedge\mathscr{E} \quad \dfrac{[A] \quad [B]}{A \wedge B}\wedge\mathscr{I}}{\Box(A \wedge B)}\Box.\mathscr{I}$$

whose properties follow as a special case of $\rho_\Box$.

The point is given by $\frac{A}{\Box A}$ and its characteristic property is given as a special case of $\rho_\Box$. Finally, such a point is monoidal by $\rho_\Box$ up to Ξ-rewritings. For IEL-categories, we furthermore have that $\Box\bot \simeq \bot$ in virtue of the following rewritings



Thus, if an equation between interpreted L-deductions holds in all L-categories, then it holds also in $\mathscr{M}$ so that those deductions are equal whenever they normalise to the same proof with respect to $\Xi + \rho_\Box + \delta_\Box + \gamma_\Box + \eta_\Box$-rewritings. □

On verificationist and linear modalities. It is instructive to briefly contrast the categorical and proof-theoretic behaviour of our epistemic modality $\Box$ with the exponential modalities of linear logic (Girard 1987; Benton et al. 1993; Blute et al. 1996), particularly the 'why not' modality (?).[14] Because our calculi are governed by the verificationist co-reflection principle ($A \to \Box A$), our modality shares a superficial structural similarity with the linear ?, whose dereliction rule establishes a map $A \to ?A$. This is fundamentally opposite to the comonadic dereliction ($!A \to A$) of the 'of course' modality (!).

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However, the formal parallels end there, owing to three fundamental divergences with proof-theoretic roots. Structurally, the linear ? is formulated as a full *monad*, which necessitates a multiplication map *questionmarkquestionmarkA* $\rightarrow ?A$. In contrast, our epistemic operator is interpreted as a *monoidal pointed endofunctor*. It is equipped with the natural point $\kappa_A : A \rightarrow \Box A$ but deliberately lacks monadic multiplication, accurately reflecting the fact that verificationist epistemic states do not trivially collapse nested modalities. Second, our endofunctor $\mathfrak{R}$ is monoidal with respect to products (to capture the K schema for conjunctions), whereas the linear ? interacts strictly with the multiplicative disjunction ($\parr$). Finally, from a purely computational perspective, the linear exponentials act as substructural gatekeepers to locally restore the structural rules of weakening and contraction within a resource-bounded architecture (Bierman 1995). In IEL^- (and IEL as well), structural rules are globally available across the underlying bi-cartesian closed category of deductions – where discharges can be vacuous and multiple by the calculus design, and $\Box$ acts solely as an epistemic tracker of the verification processes.

5. Conclusion

Intuitionistic logic, through its emphasis on constructive proofs and the identification of truth with provability – as articulated by the BHK interpretation – offers a framework for reasoning that departs fundamentally from its classical counterpart. This difference in the underlying conception of truth motivates the intuitionistic epistemic logics introduced by Artemov and Protopopescu (2016), which bring the constructive standpoint to bear on the formalisation of knowledge and belief.

The present paper contributes to this programme by developing natural deduction calculi for intuitionistic epistemic logics, situating them within the Curry–Howard–Lambek correspondence. The proof-theoretic analysis carried out here – in particular, the normalisation theorems established with respect to several rewriting systems – confirms the structural well-behaviour of these calculi and supports their interpretation as computational systems. Furthermore, the categorical reading of the epistemic modalities as functors over syntactic models forges a precise connection between intuitionistic epistemic reasoning, on the one hand, and categorical proof theory and the abstract semantics of functional programming, on the other.

Several directions remain open for future investigation. These include: a closer study of the categorical and computational content of our systems, in particular the reading of the schemas for intuitionistic belief in terms of applicative functors (McBride and Paterson 2008); a systematic comparison – aimed at inter-translatability results – with the sequent calculi of Krupski and Yatmanov (2016) and van der Giessen (2023); and an exploration of extensions of the calculus IEL^- to proof systems for logics beyond the epistemic domain, as anticipated in Perini Brogi (2022, Ch. 4).

5.1 Related work

The present work extends our earlier investigation of intuitionistic belief (Perini Brogi 2021), from which it inherits the modular methodology, applying it to the design, proof-theoretic analysis and categorical interpretation of the natural deduction calculus for intuitionistic knowledge. This line of research remains relatively novel, and its pursuit is motivated by the deep connections between natural deduction, type theory and categorical semantics that constitute the computational trinitarianism tradition.

Existing proof-theoretic treatments of the logics of Artemov and Protopopescu (2016) – and of various extensions thereof – have been conducted predominantly in the sequent calculus framework; see Krupski and Yatmanov (2016), Su and Sano (2019), Fiorino (2023), van der Giessen (2023) and Hagemeier and Kirst (2022).

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The broader community's growing interest in intuitionistic and constructive modal logic[15] has also stimulated the development of natural deduction calculi – together with their categorical interpretation – for a range of modal systems. Significant contributions in this direction include the foundational results of Alechina et al. (2001) and de Paiva and Ritter (2011) on intuitionistic K and S4, the proof-irrelevant categorical interpretation of IEL given by Rogozin (2021), and, more recently, the treatment of intuitionistic S5 by Guerrini et al. (2023).

Acknowledgements. I would like to thank the anonymous referee for their valuable comments and recommendations.

Special thanks go to Joost Joosten and the participants of the *Seminari Cuc (Seminar on Proof Theory and Foundations of Mathematics)* for highly stimulating discussions in sunny Barcelona.

I am also grateful to the organisers and participants of OVerLAY@ECAI2025 in Bologna for the opportunity to present my research on verificationist epistemic modalities during the conference poster session.

This work was partially supported by the European Union – NextGenerationEU under the PNRR project 'SERICS: Security and Rights in the CyberSpace' (PE0000014, M4C2 I. 1. 3), the project Educating Future Citizens (EFC), the European Mathematical Society, the Istituto Nazionale di Alta Matematica (INdAM), GNSAGA Group, the International Research Network 'Logic and Interaction' and the Proof Society.

Competing interests. The author has no relevant financial or non-financial interests to disclose.

Notes

1 For instance, Dummett (2000) advocates a purely philosophical justification of the whole current of intuitionistic mathematics.

2 Notice, however, that Nozick's account denies also the closure under known logical entailment – that is, the schema K – that is endorsed in Artemov and Protopopescu (2016). This denial is due to Nozick's original aim of criticising the basic assumptions of Gettier (1963) questioning the *JTB* theory on knowledge. That Nozickian position about K is, in any case, challenged by several views that still adopt Nozick's proposal for defining knowledge: see, for example, Lipton and Worrall 2000.

3 As it is common practice, we omit the rules for negation, since we can define $\neg A$ as $A \to \bot$.

4 For other systems, further simplifications are required: see Prawitz (1971) for their definition.

5 The judgements for the other typed terms follow the structure of the corresponding rules in NJp, as the reader might expect.

6 Type annotations are omitted for the sake of readability.

7 Refer to Sørensen and Urzyczyn (2006) for the proofs.

8 Notice that this calculus differs from the system introduced by de Paiva and Ritter (2011) by allowing the set Δ of additional hypotheses in the subdeduction of B.

9 This function is introduced by Kakutani (2016) to prove detour elimination for the implicational fragment of basic intuitionistic modal logic $\mathbb{IK}$ by reducing the problem to normalization of simple type theory. Here, we adopt the mapping to consider also product, sum, unit and empty types, keeping the original strategy due to de Groote (2002) based on Dragalin–Friedman A-translation (Bezhanishvili and Holliday 2016).

10 A proof of Newman's result relating strong normalization and Church–Rosser property is given in Sørensen and Urzyczyn 2006, Prop. 3.6.2).

11 The thesis (Perini Brogi 2022) contains the proofs of that property for both IEL$^-$ and IEL, by making explicit the details of the proof sketch given by Prawitz (1971) for NJp. The reader is referred to Mancosu et al. (2021) for an in-depth presentation of that property for non-modal systems.

12 See (MacLane 2013, Section VII.1) for the corresponding commuting diagrams and the definition of monoidal category.

13 We borrow the term *dense* from Bezhanishvili and Holliday (2016).

14 We thank a reviewer for the insightful recommendation of disambiguating the epistemic modalities discussed here from the exponential modalities of linear logic: this move indeed clarifies the structural and conceptual boundary of our work.

15 This interest is reflected in a substantial and expanding literature; see, for example Fischer Servi (1977), Plotkin and Stirling (1986), Wijesekera (1990), Litak et al. (2017), Stewart et al. (2018), de Paiva and Artemov (2021), Balbiani et al. (2021), Afshari et al. (2024), Pacheco (2024), Balbiani (2025) and Santiago-Fernández et al. (2026).

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Cite this article: Perini Brogi C (2026). Proofs, programs and arrows for epistemic logics. *Mathematical Structures in Computer Science* **36**(e35), 1–31. https://doi.org/10.1017/S0960129526100747

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