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Why Read the Classics (of Mathematical Proofs)?

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Synopsis

Inspired by Italo Calvino's (almost) homonymous writing, this short essay suggests a formal perspective to complement the typical categorisation of classical results in mathematics on aesthetic bases. The traditional proof that $\sqrt{2}$ is not a rational number provides a simple example of proof *mining*. When studied through the tools of logic, that proof has also led to designing new systems for formal proving based on *cyclic proofs*. This example thus supports the central thesis of this essay that reading the classics of mathematical proofs enables us to acquire a taste for mathematical beauty and can open new directions in mathematics as well.

> I classici sono libri che quanto più si crede di conoscerli per sentito dire, tanto più quando si leggono davvero si trovano nuovi, inaspettati, inediti.[0]
>
> *Italo Calvino*

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[0] Translated in [8, pag. 6] as: "Classics are books which, the more we think we know them through hearsay, the more original, unexpected, and innovative we find them when we actually read them."

Introduction

According to Wallace [24], "mathematics may be the last of the acquired tastes." The author suggests a shared aesthetic appreciation among art, literature, music, and mathematics. This aesthetic trait is seen as a key hallmark in elevating a mathematical work to a first-class exemplar in its genre.

Many contributors to a special issue of the *Journal of Humanistic Mathematics* focused on mathematical beauty [13] would likely agree with this characterisation of outstanding mathematical achievements.

Elegance, beauty, and other aesthetic qualities are also characteristic of the proofs collected in the famous Book that, according to Paul Erdős, supposedly contains exceptionally unique specimens of those mathematical entities that every mathematician handles, analyses, and reimagines daily [1].

However, I always remain somewhat sceptical about evaluating ideas and intellectual constructions based mainly on aesthetic grounds, particularly in the rigorous sciences.

I would instead lean towards heuristically assessing them using criteria such as their *functionality*, *novelty*, *fertility*, and *exactitude*, which I find surprisingly akin to the characterisation via *progressive approximation* given by Calvino [8] to the classics of worldwide literature.

Indeed, there are instances where something *qualitatively different* from artistic values captivates the mathematician's palate for proofs. For instance, none of my peers believe Gentzen's cut-elimination theorem possesses an aesthetically pleasing proof. Yet, all agree that the theorem and its proof stand among the most significant outcomes in logic and the foundations of mathematics.[1]

In this brief article, I aim to persuade you that proofs are mathematical objects which can communicate much more than aesthetic stimulation: often, by carefully reading different proofs of the same theorem, we can do more than ranking them according to an innate or, more realistically, acquired

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[1] Mancosu and colleagues [17] give a detailed introduction to this pivotal result of mathematical logic. For further insights into Gerhard Gentzen's life and legacy, you may refer to Menzler-Trott [18] and Rathjen and Sieg [19], respectively.

mathematical taste, cultivated over years of study. This is especially applicable to those theorems we encounter from a tender age, whose proofs, though often concise, we frequently accept as given when they possess, in fact, much to divulge. For the most celebrated *classics* of the art of proving, in short.

1. Hippasus's proof

Take, for instance, the proof that $\sqrt{2}$ is not a rational number, which is traditionally ascribed – e.g. by Kline [15, Volume I, page 32] – to the Pythagorean Hippasus. Should you wish to refresh your memory, the Hamkins family and the musician Hannah Hoffman have prepared an enchanting video on the topic, freely available from [12]. Apart from the melancholy fate of the prover,[2] the ditty elucidates Hippasus' proof admirably.

The proof is straightforward, constructive,[3] ancient, and has exercised a firm hold on mathematicians of yore, to the extent that it is recollected by both Aristotle [2, I-23] and Euclid [9, X].

1.1. Qualitative version

The proof has more *mathematical* information to reveal. Let us first organise it as follows:

Lemma 1. *If p, q are natural numbers, $q \neq 0$, and $\frac{p}{q} = \sqrt{2}$, then p and q are both even.*

Proof. Just follow Hippasus' argument: if $\frac{p}{q} = \sqrt{2}$ with $p, q \in \mathbb{N}$ and $q \neq 0$, then

$$p^2 = 2q^2,$$

hence p is an even number.

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[2] Hippasus is said to have been cast overboard and drowned at sea, for his proof demonstrating the existence of irrational numbers, expounded while aboard a ship, had scandalised his Pythagorean shipmates, who presumed that "all things are made of (natural) numbers." Check out Glaz [10] for a poem about this story: it is aptly titled "$\sqrt{2} = 1.41421\ldots$".) You may refer to Huffman [14] for a comprehensive overview of Pythagoras' philosophy and his followers.

[3] Contrary to the belief of many, the proof *is not* by contradiction: formally, we do not proceed by *reductio ad absurdum* (which is a proof technique not permissible in constructive logic), but by negation-introduction; you may compare [17] for a precise definition of these notions in the context of natural deduction systems.

This means there exists a natural number k such that $p = 2k$. Therefore $p^2 = 4k^2$.

However, this also implies that

$$q^2 = 2k^2,$$

and thus q must be even too, *quod erat demonstrandum.* $\square$

The irrationality of $\sqrt{2}$ then becomes an immediate corollary to Lemma 1:

Corollary 2. $\sqrt{2} \notin \mathbb{Q}$.

Proof. If $\sqrt{2}$ were a rational number, it could be expressed as a fraction $\frac{p}{q}$ where p, q are coprime natural numbers. However, this is impossible due to Lemma 1. Therefore, $\sqrt{2}$ is not a rational number. $\square$

1.2. Quantitative version

We can now achieve something more by thoroughly examining this reshaped classic proof.

Observe, in fact, that $\frac{p}{q} = \sqrt{2}$ is logically equivalent to the statement

$$\forall \delta > 0, \left(|\frac{p}{q} - \sqrt{2}| \leq \delta. \right)$$

for $\delta \in \mathbb{R}$.[4]

Given that Lemma 1 tells us that, for any p, q, $\frac{p}{q} \neq \sqrt{2}$, one would ask: can we explicitly compute the corresponding δ?

To start, it is straightforward to prove the following:

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[4] As one reader pointed out, these claims assume the existence of two distinct mathematical objects: $\sqrt{2}$ and the set of real numbers $\mathbb{R}$. The existence of the former follows from the Pythagorean theorem applied to a right triangle with both legs of unit length, though its formal construction as a real number depends on a development of $\mathbb{R}$. The existence of the latter can be justified by Eudoxus of Cnidus' theory of proportions, which establishes an identity relation between 'magnitudes'', without resorting to the definitions of contemporary mathematics, as discussed by, e.g., Bourbaki [3].

Lemma 3. *For any $x, y, \delta \in \mathbb{R}$, if $x, y > 0$ and $|x^2 - y^2| \geq \delta$, then $|x - y| \geq \frac{\delta}{x+y}$.*

Now, we immediately see that Hippasus' proof allows us to state a *quantitative* version of the theorem proven in music by the Hamkins:

Theorem 4. *Given $p, q \in \mathbb{N}$, with $q \neq 0$. If p and q are not both even, then*

$$|\frac{p}{q} - \sqrt{2}| > \frac{1}{pq + 2q^2}.$$

Proof. Let $p, q \in \mathbb{N}$ with $q \geq 1$, and suppose p, q are not both even.

Then, by Lemma 1, $p^2 \neq 2q^2$; thus $|p^2 - 2q^2| \geq 1$, which implies

$$|\left(\frac{p}{q}\right)^2 - (\sqrt{2})^2| \geq \frac{1}{q^2}.$$

From Lemma 3, we immediately get

$$|\frac{p}{q} - \sqrt{2}| \geq \frac{\frac{1}{q^2}}{\frac{p}{q} + \sqrt{2}} = \frac{1}{q\left(p + q\sqrt{2}\right)} > \frac{1}{pq + 2q^2}. \quad \square$$

Thus, Hippasus' proof not only provides us with the certainty that $\sqrt{2} \notin \mathbb{Q}$, but also allows us to explicitly define a lower bound function for the distance between $\sqrt{2}$ and any other rational number.

2. Proof mining

The previous sections merely showcase an elementary example of proof mining. As `Wikipedia` defines it, "proof mining (or proof unwinding) is a research program that studies or analyzes formalized proofs, especially in analysis, to obtain explicit bounds, ranges or rates of convergence from proofs that, when expressed in natural language, appear to be nonconstructive."[5]

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[5] I learnt proof mining from the lectures given by Paulo Oliva at Swansea University on the occasion of The Proof Society Summer School in 2019. A seminal overview of proof mining is given by Kohlenbach and Oliva [16]. The more recent blog post by Sipoș [22] gently introduces the newcomers to the field.

The discipline's inception dates back to the interests of Georg Kreisel and Dana Scott. Nevertheless, today, the field has been advanced by several mathematicians in various directions, encompassing functional analysis, constructive algebra, and program extraction.

The underlying philosophy of proof mining can be encapsulated as follows: meticulously examine the proof of a theorem to enhance it and extract mathematically pertinent information from the proof, refining the analysis to possibly yield the outlines of a program that returns an actual and concrete computational value out of the theorem.

Proof mining now spans various domains of mathematics, from number theory to optimisation, and shares close affinities with the more computer science facets of logic and the foundations of mathematics in general. It is also, in my personal view, a commendable educational instrument for anyone desiring to assess their analytical acumen, irrespective of their mathematical background and personal preferences.

3. Reasoning in circles

Naturally, the more advanced realms of proof mining demand a significantly more developed technical background than that employed for the example furnished with Hippasus's proof. Nevertheless, this very proof that $\sqrt{2} \notin \mathbb{Q}$ has at least another gem to reveal, as it has played a significant role in the evolution of a novel formal proof technique.

One manner of arriving at the absurd conclusion of Hippasus's argument is to invoke the principle of infinite descent for natural numbers, formalised for the first time by Pierre de Fermat concerning Hippasus's proof [7]:

> If in a proof we manage to construct a strictly decreasing chain of natural numbers, we can dismiss such a proof, since the relation $\leq$ is well-founded on $\mathbb{N}$.

In contemporary structural proof theory, the rationale for infinite descent has facilitated the development of *cyclic proof systems*, wherein informal proofs, familiar to the ordinary mathematician, can be *mechanised* through non-well-founded proof trees, with cycles linking specific nodes according to well-defined validity criteria depending on the mathematical theory under consideration.[6]

These techniques are now employed to design a kernel of automated provers [11] for various formal systems involving principles of induction – e.g., on natural numbers, as in Peano arithmetic [21] – or logics with fixed-point operators as in propositional dynamic logic [23] or modal μ-calculi [4], which allow abstract validations of the behaviour of computer programs and reactive systems.

Conclusion

We have seen how, by carefully reading the classical proof of Hippasus, it is possible to recognise much more than just the aesthetic and cultural value of the demonstrative argument and achieve a more precise and fine-grained result from it (Theorem 4). The same proof, reinterpreted by logicians after Fermat and his reformulation of the proof in terms of infinite descent of natural numbers, has then led to the development of cyclic proof systems, which are now a flourishing area of structural proof theory.[7]

So why read the classics of mathematical proofs? Though beauty remains a highly pertinent criterion for evaluating mathematical achievements, there exist myriad ways to explore proofs, even from a rigorous and purely formal perspective: one must amalgamate aesthetic faculties with the appropriate logical tools (and, perhaps, a modicum of over-analytical spirit), and novel mathematical, philosophical, and genuinely cultural horizons will unfold.

Rephrasing one step in the asymptotic definition of classics in literature given by Calvino [8, page 5], I would say indeed that a classic is a proof that “has never exhausted all it has to say to its readers.”

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