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Addendum: Testing structural balance theories in heterogeneous signed networks



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While we were completing our manuscript, we became aware of a parallel study whose methodology and results overlap with our work¹. The present Addendum is devoted to clarifying the existing relationships between the approach pursued by us in ref. 2 and the one pursued by the authors of ref. 1.

In ref. 2, two, different classes of benchmarks for binary, undirected, signed networks are derived and discussed, i.e. the *free-topology* and the *fixed-topology* ones: the methodological overlap between refs. 1 and 2 concerns the second class of null models, as we explain more detailedly below.

The approach pursued in ref. 2 moves from the Hamiltonian

$$H(\mathbf{A}) = \alpha L^+(\mathbf{A}) + \beta L^-(\mathbf{A}) = \sum_{i=1}^N \sum_{j(>i)} (\alpha a_{ij}^+ + \beta a_{ij}^-) \tag{1}$$

that leads to the partition function

$$Z = \sum_{\substack{\mathbf{A} \in \tilde{\mathcal{A}} \\ (|\mathbf{A}|=|\mathbf{A}^+|)}} e^{-H(\mathbf{A})} = \sum_{\substack{\mathbf{A} \in \tilde{\mathcal{A}} \\ (|\mathbf{A}|=|\mathbf{A}^+|)}} e^{-\sum_{i=1}^N \sum_{j(>i)} (\alpha a_{ij}^+ + \beta a_{ij}^-)} = \prod_{i=1}^N \prod_{j(>i)} \sum_{a_{ij} = \pm 1} e^{-(\alpha a_{ij}^+ + \beta a_{ij}^-)} = (e^{-\alpha} + e^{-\beta})^L; \tag{2}$$

the expression

$$P_{\text{SRGM=FT}}(\mathbf{A}) = \prod_{i=1}^N \prod_{j(>i)} (p^+)^{a_{ij}^+} (p^-)^{a_{ij}^-} = (p^+)^{L^+} (p^-)^{L^-} \tag{3}$$

is, thus, recovered, where

$$p^+ = \frac{e^{-\alpha}}{e^{-\alpha} + e^{-\beta}} \equiv \frac{x}{x+y}, \quad p^- = 1 - p^+ = \frac{e^{-\beta}}{e^{-\alpha} + e^{-\beta}} \equiv \frac{y}{x+y} \tag{4}$$

with $e^{-\alpha} \equiv x$ and $e^{-\beta} \equiv y$.

The approach pursued in ref. 1, instead, moves from re-writing $H(\mathbf{A})$ as

$$H(\mathbf{A}) = \alpha L^+(\mathbf{A}) + \beta L^-(\mathbf{A}) = \alpha L^+(\mathbf{A}) + \beta [L(\mathbf{A}) - L^+(\mathbf{A})] = (\alpha - \beta) L^+(\mathbf{A}) - \beta L(\mathbf{A}); \tag{5}$$

in such a case, the partition function becomes

$$Z = \sum_{\substack{\mathbf{A} \in \tilde{\mathcal{A}} \\ (|\mathbf{A}|=|\mathbf{A}^+|)}} e^{-H(\mathbf{A})} = \sum_{\substack{\mathbf{A} \in \tilde{\mathcal{A}} \\ (|\mathbf{A}|=|\mathbf{A}^+|)}} e^{-\sum_{i=1}^N \sum_{j(>i)} (\alpha - \beta) a_{ij}^+ - \beta a_{ij}} = \prod_{i=1}^N \prod_{j(>i)} \sum_{a_{ij} = \pm 1} e^{-(\alpha - \beta) a_{ij}^+ + \beta a_{ij}} = (1 + e^{-(\alpha - \beta)}) e^\beta \tag{6}$$

and leads to the expressions

$$p^+ = \frac{e^{-(\alpha - \beta)} e^\beta}{(1 + e^{-(\alpha - \beta)}) e^\beta} = \frac{e^{-(\alpha - \beta)}}{1 + e^{-(\alpha - \beta)}} \equiv \frac{x/y}{1 + x/y} \equiv \frac{z}{1 + z}, \quad p^- = 1 - p^+ = \frac{1}{1 + e^{-(\alpha - \beta)}} \equiv \frac{1}{1 + x/y} \equiv \frac{1}{1 + z} \tag{7}$$

evidencing the need for just one multiplier, i.e. $\alpha - \beta$ or equivalently $e^{-(\alpha - \beta)} \equiv z$. Nothing changes from a purely numerical perspective, instead.

A similar reasoning can be repeated for the SCM-FT as well: the two approaches are related by the chain of relationships reading

$$P_{ij}^+ = \frac{x_i x_j}{x_i x_j + y_i y_j} = \frac{x_i x_j / y_i y_j}{x_i x_j / y_i y_j + 1} \equiv \frac{z_i z_j}{z_i z_j + 1}, \forall i < j \quad (8)$$

with $x_i/y_i \equiv z_i, \forall i$. More formally, it comes from considering the expression

$$H(\mathbf{A}) = \sum_{i=1}^N [\alpha_i k_i^+(\mathbf{A}) + \beta_i k_i^-(\mathbf{A})] = \sum_{i=1}^N [\alpha_i k_i^+(\mathbf{A}) + \beta_i (k_i(\mathbf{A}) - k_i^+(\mathbf{A}))] = \sum_{i=1}^N (\alpha_i - \beta_i) k_i^+(\mathbf{A}) + \sum_{i=1}^N \beta_i k_i(\mathbf{A}) \quad (9)$$

and posing $e^{-(\alpha_i - \beta_i)} \equiv z_i, \forall i$. As in the case of the SRGM-FT, the number of unknowns can be reduced upon recognising $\alpha_i - \beta_i, \forall i$ as the only, relevant multipliers.

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References

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