communications physics

Corrections & amendments

https://doi.org/10.1038/s42005-024-01793-5

Addendum: Testing structural balance theories in heterogeneous signed networks

Anna Gallo, Diego Garlaschelli, Renaud Lambiotte, Fabio Saracco & Tiziano Squartini

Addendum to: ***Communications Physics*** **https://doi.org/10.1038/s42005-024-01640-7, published online 13 May 2024**

While we were completing our manuscript, we became aware of a parallel study whose methodology and results overlap with our work[1]. The present Addendum is devoted to clarifying the existing relationships between the approach pursued by us in ref. 2 and the one pursued by the authors of ref. 1.

In ref. 2, two, different classes of benchmarks for binary, undirected, signed networks are derived and discussed, i.e. the *free-topology* and the *fixed-topology* ones: the methodological overlap between refs. 1 and 2 concerns the second class of null models, as we explain more detailedly below.

The approach pursued in ref. 2 moves from the Hamiltonian

$$H(\mathbf{A}) = \alpha L^{+}(\mathbf{A}) + \beta L^{-}(\mathbf{A}) = \sum_{i=1}^{N}\sum_{j(>i)}(\alpha a_{ij}^{+} + \beta a_{ij}^{-}) \tag{1}$$

that leads to the partition function

$$Z = \sum_{\substack{\mathbf{A}\in\mathbb{A}\\(|\mathbf{A}|=|\mathbf{A}^*|)}} e^{-H(\mathbf{A})} = \sum_{\substack{\mathbf{A}\in\mathbb{A}\\(|\mathbf{A}|=|\mathbf{A}^*|)}} e^{-\sum_{i=1}^{N}\sum_{j(>i)}(\alpha a_{ij}^{+}+\beta a_{ij}^{-})} = \prod_{i=1}^{N}\prod_{j(>i)}\sum_{a_{ij}=\pm1} e^{-(\alpha a_{ij}^{+}+\beta a_{ij}^{-})} = (e^{-\alpha}+e^{-\beta})^{L}; \tag{2}$$

the expression

$$P_{\text{SRGM}=\text{FT}}(\mathbf{A}) = \prod_{i=1}^{N}\prod_{j(>i)}(p^{+})^{a_{ij}^{+}}(p^{-})^{a_{ij}^{-}} = (p^{+})^{L+}(p^{-})^{L-} \tag{3}$$

is, thus, recovered, where

$$p^{+} = \frac{e^{-\alpha}}{e^{-\alpha}+e^{-\beta}} \equiv \frac{x}{x+y},\; p^{-} = 1-p^{+} = \frac{e^{-\beta}}{e^{-\alpha}+e^{-\beta}} \equiv \frac{y}{x+y} \tag{4}$$

with $e^{-\alpha} \equiv x$ and $e^{-\beta} \equiv y$.

The approach pursued in ref. 1, instead, moves from re-writing $H(\mathbf{A})$ as

$$H(\mathbf{A}) = \alpha L^{+}(\mathbf{A}) + \beta L^{-}(\mathbf{A}) = \alpha L^{+}(\mathbf{A}) + \beta[L(\mathbf{A}) - L^{+}(\mathbf{A})] = (\alpha-\beta)L^{+}(\mathbf{A}) - \beta L(\mathbf{A}); \tag{5}$$

in such a case, the partition function becomes

$$Z = \sum_{\substack{\mathbf{A}\in\mathbb{A}\\(|\mathbf{A}|=|\mathbf{A}^*|)}} e^{-H(\mathbf{A})} = \sum_{\substack{\mathbf{A}\in\mathbb{A}\\(|\mathbf{A}|=|\mathbf{A}^*|)}} e^{-\sum_{i=1}^{N}\sum_{j(>i)}(\alpha-\beta)a_{ij}^{+}-\beta a_{ij}} = \prod_{i=1}^{N}\prod_{j(>i)}\sum_{a_{ij}=\pm1} e^{-(\alpha-\beta)a_{ij}^{+}+\beta a_{ij}} = (1+e^{-(\alpha-\beta)})e^{\beta} \tag{6}$$

and leads to the expressions

$$p^{+} = \frac{e^{-(\alpha-\beta)}e^{\beta}}{(1+e^{-(\alpha-\beta)})e^{\beta}} = \frac{e^{-(\alpha-\beta)}}{1+e^{-(\alpha-\beta)}} \equiv \frac{x/y}{1+x/y} \equiv \frac{z}{1+z},\; p^{-} = 1-p^{+} = \frac{1}{1+e^{-(\alpha-\beta)}} \equiv \frac{1}{1+x/y} \equiv \frac{1}{1+z} \tag{7}$$

evidencing the need for just one multiplier, i.e. $\alpha-\beta$ or equivalently $e^{-(\alpha-\beta)} \equiv z$. Nothing changes from a purely numerical perspective, instead.

A similar reasoning can be repeated for the SCM-FT as well: the two approaches are related by the chain of relationships reading

$$p_{ij}^{+} = \frac{x_i x_j}{x_i x_j + y_i y_j} = \frac{x_i x_j / y_i y_j}{x_i x_j / y_i y_j + 1} \equiv \frac{z_i z_j}{z_i z_j + 1}, \ \forall i < j \tag{8}$$

with $x_i/y_i \equiv z_i$, $\forall i$. More formally, it comes from considering the expression

$$H(\mathbf{A}) = \sum_{i=1}^{N} [\alpha_i k_i^{+}(\mathbf{A}) + \beta_i k_i^{-}(\mathbf{A})] = \sum_{i=1}^{N} [\alpha_i k_i^{+}(\mathbf{A}) + \beta_i (k_i(\mathbf{A}) - k_i^{+}(\mathbf{A}))] = \sum_{i=1}^{N} (\alpha_i - \beta_i) k_i^{+}(\mathbf{A}) + \sum_{i=1}^{N} \beta_i k_i(\mathbf{A}) \tag{9}$$

and posing $e^{-(\alpha_i - \beta_i)} \equiv z_i$, $\forall i$. As in the case of the SRGM-FT, the number of unknowns can be reduced upon recognising $\alpha_i - \beta_i$, $\forall i$ as the only, relevant multipliers.

Published online: 20 September 2024

References

1. Hao, B. & Kovács, I. A. Proper network randomization is key to assessing social balance. *Sci. Adv.* **10**, eadj0104 (2024).
2. Gallo, A., Garlaschelli, D., Lambiotte, R., Saracco, F. & Squartini, T. Testing structural balance theories in heterogeneous signed networks. *Commun. Phys.* **7**, 154 (2024).