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Abstract

In this paper, we model the economy as a *production network of competitive firms* that interact in a general-equilibrium setup. First, we find that, at the unique Walrasian equilibrium, the profit of each active firm is proportional to (a suitable generalization of) its Bonacich centrality. We also determine consumer welfare at equilibrium and characterize efficient networks. Then we proceed to conduct a broad range of comparative-static analyses. These include the effect on profits and welfare of:
(a) *distortions* (e.g. tax/subsidies) imposed on the whole economy or specific firms;
(b) *structural* changes such as the addition of links and the elimination of nodes;
(c) *productivity* and *preference* changes.

We discover that the induced effects are in general *nonmonotone*, depend on *global* network features, and impinge on each sector depending on the pattern of *incentralities* displayed by its input providers and output users. Furthermore, the inter-sector "linkages" underlying these effects can usually be decomposed – following the heuristic dichotomy proposed by Hirschman (1958) – into a *forward* (push) component and a *backward* (pull) one. Finally, we undertake some preliminary analysis of firm dynamics and illustrate that, when evaluating policies of support and shock mitigation from a dynamic viewpoint, the reliance on strict *market-based criteria* can be quite *misleading* in terms of social welfare.

JEL: C67, D51, D85
Key words: Production, Networks, Distortions, Centrality, Profit

1 Introduction

Any modern economy is a complex network of interfirm buyer-seller relationships that constitute its production structure. There is a growing interest among economists in adopting this network perspective to study a wide variety of economic phenomena: trade, intermediation, innovation, technological diffusion, learning, or the transmission of shocks. This, of course, has prominent precursors in the celebrated work of John von Neumann, Wassily Leontief or Albert O. Hirschman, all of whom stressed the importance of interindustry relationships (or linkages) for a proper understanding of some of the key characteristics of an economic system. In this tradition, the present paper proposes a general-equilibrium model of the economy that, despite being standard in almost every respect, highlights the network of interfirm relationships on its production side. In a nutshell, our objective is to obtain a precise understanding of how the details of the *production network topology* (on which no *a priori* restrictions are imposed) shapes profits, welfare, and the effects (static and dynamic) induced by wide variety of changes and policy interventions.

In order to focus our analysis on the production structure of the economy, the demand side is modeled through a representative consumer while the technology of each firm is assumed of the Cobb-Douglas type. In general, however, each firm uses a diverse range of inputs, whose productivities are individually specific. Under these conditions, our analysis starts by showing that a unique equilibrium exists that displays a very sharp relationship to the network structure of the economy. Specifically, we find that, at equilibrium, the profitability of any given firm is *proportional* to its network centrality, as given by a suitable generalization of the well-known measure of centrality proposed by Bonacich (1987)[1]. On the other hand, we also determine the consumer's welfare at equilibrium, providing closed expressions for how it depends on the different parameters of the model (in particular, the density and symmetry of the production structure).

Next, the paper turns to studying how the equilibrium is affected by a wide variety of different changes in the environment. First, we consider the impact of distortions, formulated in the idealized form of price wedges – either general ones that affect uniformly the whole economy, or individual ones applying to single firms. Then, we turn our attention to structural changes in the production network and consider both the effect of *global* transformations that affect either connectivity or the number of nodes, as well as *local* changes that affect only single nodes or individual links. We then close this part of our analysis with a comparative study of supply-channeled changes (such as technological improvements) with those that operate through variations in demand (e.g. a rise or fall

[1]For example, as compared to the received notion of Bonacich centrality that considers the paths of different lengths that join any given pair of nodes, in our case only those that connect a firm node to a consumption node are considered. The weight attributed to each of these paths reflects not only its length (as in the standard Bonacich centrality, they are discounted by their respective length) but also the relative importance that the consumer's utility function attributes to the end good in question (while in the standard notion all end nodes are given a uniform weight).

in the consumer's relative preference for a particular consumption good).

For all of the cases listed above, we provide formal expressions that capture the total impact of the change in an explicit form. These expressions are conceptually quite simple and should prove helpful in evaluating empirically alternative measures of economic policy. Furthermore, many of them can be understood in a quite parallel fashion, in that they involve a similar integration of constituent effects. Thus, on the one hand, they can be regarded as consisting of a first-order impact on the immediate agent experiencing the change (e.g. the firm subject to an individual tax or subsidy), followed by a spread of such a first-order impact throughout the whole network as captured by a suitable matrix of incentralities. On the other hand, the impact of many of those changes can also be decomposed in terms of a push effect that operates downstream and a pull effect that does so upstream. This dichotomy is reminiscent of the distinction between forward and backward linkages often considered in policy analysis and notably proposed by Hirschman (1958). Here we show how to distinguish precisely between them and also compare their implications: while forward linkages induce resource reallocation downstream but have no effect on profits, revenues, or input demands because of the entailed adjustment of prices, backward linkages alter significantly all those variables – i.e, not only prices but also revenues, profits, and input demands.

The analysis advanced so far is inherently static, i.e. it compares how different changes in the parameters of the model affect *equilibrium magnitudes*. By building on it, however, we can also address some genuinely dynamic issues. In this paper, we simply outline the problem, leaving for future work an exhaustive study of it. We consider, specifically, the problem of how to guide the distribution of firm support when, in the absence of it, some incumbent firms may go bankrupt and disappear. The key dynamic concern here is that, when several firms are at stake and not all can be supported, what particular firms are chosen may lead to subsequent effects that are drastically different. For, as one firm is protected but other fails, the latter may generate a cascade of ensuing failures that has major longer-run effects. The question then is: what is the best criterion to use if the objective is to minimize the overall (intertemporal) impact? In particular, we may ask: are current prices and induced profits the right market signal? In brief, the answer we provide is that, in some cases, profitability may by itself be a very misleading criterion and that other network-based criteria (embodying considerations of intercentrality) will, in general, be much more appropriate.

We end this short introduction with a brief review of related literature. From a methodological viewpoint, our approach is close to recent work by Acemoglu et al. (2012) who, building upon a Cobb-Douglas model proposed by Long and Plosser (1983), study how/whether microeconomic shocks on individual agents or sectors may aggregate into generating significant aggregate effects at the macroscopic level. Previous papers by Horvath (1998), Gabaix (2011) are motivated by a similar concern. As already explained, our objective in this paper is very different and so are some of the key assumptions and questions asked. Thus, just to mention one significant difference, we do not make the assumption of constant returns in production since we are interested in how profit performance is affected by network structure. Besides, our analysis focuses on the effect of distortions, structural changes, or different kinds of demand and support policies rather than the spread of shocks, which are very different kind of phenomena.

Our work is also related to Jones (2011) who, building as well on the framework introduced by Long and Plosser (1983), studies how misallocation at the sector level affects GDP. More generally, our paper relates to the vast literature that has attempted to understand the intersectorial basis of economic development and the sectorial) policies that can mitigate either misallocation or/and coordination problems. The workhorse in this literature has been the input-output methodology originally formulated by Leontief (1936), which has spawned a huge body of work, both theoretical and empirical (see e.g. the monograph Miller and Blair (2009) for a recent account). Early on, building upon this rise of input-output analysis, the influential work of Hirschman (1958) highlighted the importance of some of the notions (e.g. forward and backward linkages) that will help us understand key forces underlying our model. In contrast with the traditional input-output literature, Hirschman's approach emphasizes unbalanced growth rather than equilibrium as the primary tool of *dynamic* analysis. To adopt such a perspective is also our eventual objective, even though in the present paper our analysis is still mostly static.

However, the preliminary dynamic analysis of evolutionary market forces undertaken in the last part of the paper (cf. the summary above) embodies some of the considerations implicitly highlighted by Hirschman's work. It also hints at the role that network-based processes of propagation – e.g. of default through failure cascades – should have in a proper assessment of economic policy in a complex interconnected economy. This, in turn, leads us to the rich and diverse literature on contagion – for example, financial contagion – that has experienced a significant impetus in recent years (see the recent Handbook edited by Bramoulle et al. (2016) and the recent interesting work by Baqaee (2015) on the specific context of production networks). A different, and complementary, aspect in the dynamic study of economic networks concerns the processes of endogenous (payoff-guided) network formation. The network-formation literature is still in too-preliminary a state to provide much help in this endeavor. Our analysis of how certain structural changes on nodes and links affect agents' payoffs is a very preliminary step in this direction. The recent paper by Oberfield (2012) studies a stylized version of the problem where agents endogenously select their production techniques, conceived as links that connect them to a unique supplier.

The rest of the paper is organized as follows. First, in Section 2 we present the benchmark model, followed in Section 3 by the introduction of a collection of basic results. These results include, specifically, an account of how the topological features of the production network shape the corresponding outcomes (i.e. profits of the firms and utility of the consumer). Then, in Section 4 we study how different kinds of distortions (uniform or firm-specific) affect the allocation of resources and relative firm performance. In Section 5 we investigate how does the *global* structure of production network determine the production and the welfare potential of the economy and how *local* perturbations in the network structure impinge on the equilibrium outcomes. Section 6 discusses the influential *push-pull* dichotomy stressed by Hirschman (1958) through the lens of our model. In Section 7 we provide a preliminary exploration of issues pertaining to firm dynamics and point out that market signals can be quite misleading from the point of the long run welfare. We summarize our contribution and conclude in Section 8.

In Appendix A we include the formal proofs of our main results. Then, in the (online) Appendix B, we present the general case with unrestricted heterogeneity, while in Appendix C (also available online) we revisit the comparative-statics analysis on firm distortions when the system is closed and thus the monetary flows involved must be balanced.

2 Benchmark model

We model an economy consisting of a finite set of firms $N = \{1, 2, ..., n\}$ and a single representative consumer. Our focus, therefore, is on the interfirm production relationships – i.e. the production network – through which the economy eventually delivers the net amounts of consumption goods enjoyed by the consumer. In principle, we allow that any specific good may be consumed, used as an intermediate input in the production of some other good, or display both roles simultaneously.

The goods that provide some utility to the consumer are labeled consumption goods, and are included in the set $M \subset N$, with $m = |M|$ and $\boldsymbol{c} = (c_1, c_2, ..., c_m)$ representing a typical consumption bundle. For simplicity, leisure is not in the set M and thus yields no utility. Thus the consumer's endowment of labor (which is normalized to unity) will be inelastically supplied by the consumer and hence only has an instrumental role as a source of income. The consumer's preferences over consumption bundles are represented by a Cobb-Douglas utility function $U(\cdot)$ of the form

$$U(\boldsymbol{c}) = \prod_{i=1}^{m} c_i^{\gamma_i} \tag{1}$$

where each γ_i represents the weight that the consumer's preferences attributes to consumption good i.

Concerning production, we assume that there is a one-to-one correspondence between firms and goods (thus, in particular, we rule out joint production).[2] The production of

[2] Conceptually, we can think of each good as a "sector" consisting of many identical firms. Thus, from

each good k takes place under decreasing returns to scale, and requires both labor and intermediate produced inputs. The set of intermediate inputs that firm k uses in its production is denoted by N_k^+, with $n_k = |N_k^+|$. Let l_k stand for the amount of labor used in the production of good k and let $(z_{jk})_{j\in N_k^+}$ be the associated amount of intermediate goods. Then the amount y_k of good k produced by the homonymous firm is determined by the production function $f_k : \mathbb{R}^{n_k} \times \mathbb{R} \to \mathbb{R}$ which is taken to display the following Cobb-Douglas form:

$$y_k = f_k\left((z_{jk})_{j\in N_k^+}; l_k\right) = A_k l_k^{\beta_k} \left(\prod_{j\in N_k^+} z_{jk}^{g_{jk}}\right)^{\alpha_k} \tag{2}$$

where the vector $(g_{jk})_{k\in N}$ reflects the intensity (assumed positive)[3] with which firm k uses/requires its different inputs, and $\alpha_k > 0$ and $\beta_k > 0$ are the output elasticities of labor and intermediate inputs.[4]

It is convenient to view the intensities g_{jk} of input use as reflecting *relative* magnitudes, so unless mentioned otherwise we shall normalize the corresponding vector to satisfy $\sum_{j\in N} g_{jk} = 1$ – in other words, we assume that the matrix G is column-stochastic. On the other hand, we assume that the production technologies exhibit decreasing returns, hence we posit that $\alpha_k + \beta_k < 1$. It is worth noting that the model allows for full heterogeneity across firms $k \in N$, not only in their pattern of input use $(g_{jk})_{k\in N}$ but also in their production elasticities α_k and β_k. The latter heterogeneity, however, does not raise particularly interesting issues and, therefore, for the sake of formal simplicity, in the main text we shall focus throughout on the case where $\alpha_k = \alpha$ and $\beta_k = \beta$ for some common values α and β. A detailed analysis for the fully heterogeneous case may be found in the online Appendix B.

It is common in economic models to posit that more advanced technologies employ a wider range of intermediate inputs, this being conceived as a reflection of higher "production complexity." Here we choose to formalize this idea in the way suggested by Benassy (1998) (see also Acemoglu et al. (2007)), setting the pre-factor of the production function as

$$A_k = n_k^{\alpha+\nu} \tag{3}$$

with $\alpha + \nu > 0$. With this formulation, it is easy to see that a positive value of the

this perspective, what our model describes is the behavior of a typical firm in its corresponding sector, all firms belonging to a given sector producing perfectly substitute goods. This, of course, is a classical way of rationalizing, and providing foundations for, competitive behavior in equilibrium, as postulated below (see also Definition 2 in Appendix A).

[3] If $g_{jk} = 0$ for some input j used in the production of a certain good k, such an input is neither useful nor required in k's production. Therefore, it might as well be ignored altogether and the corresponding link jk eliminated from the production network.

[4] This formulation implies that labor and all intermediate inputs in N_k^+ are essential in production. When the production technology of $N_k^+ = \emptyset$, we interpret $\prod_{j\in N_k^+} z_{jk}^{g_{jk}} \equiv 1$, thus no production is possible because we then have $A_k = 0$ (see below for details).

parameter ν corresponds to the case where a wider input range enhances productivity, while a negative value reflects the opposite situation.[5]

The inputs $j \in N_k^+$ used in the production of good k can be viewed as the in-neighbors of node k in the production directed network $\Gamma = \{N, L\}$, where the set N of its vertices is identified with the set of firms and there is a directed edge $(i, j) \in L$ whenever j uses good i as an input, i.e. iff $g_{ji} > 0$. This is a discrete (binary) network that only provides a qualitative account of the *production structure* of the economy. A full-fledged description of this structure is provided by the matrix of production intensities $G = (g_{ij})_{i,j \in N}$, which in turn can be regarded as the *adjacency matrix* of the *directed weighted network* that describes completely the production structure. The matrix G, together with the elasticity parameters α and β, and the utility function $U(\cdot)$ of the representative consumer jointly provide a full description of the economy. To study its performance we shall focus on the standard notion of Walrasian (or Competitive) Equilibrium (WE), which consists of a collection of prices and quantities that satisfy the usual optimality and market clearing conditions. More precisely, a WE is an array $[(\boldsymbol{p}^*, w^*), (\boldsymbol{c}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*)]$ such that the following conditions hold:

1. The consumption plan $\boldsymbol{c}^* = (c_1^*, c_2^*, ..., c_m^*)$ maximizes $U(\boldsymbol{c})$ subject to the budget constraint given by the wage and profit income the consumer earns.
2. The production plans given by the outputs $\boldsymbol{y} = (y_1^*, y_2^*, ..., y_n^*)$, the demands for produced inputs $Z^* = (z_{ij}^*)_{i,j \in N}$, and the labor demands $\boldsymbol{l}^* = (l_1^*, l_2^*, ..., l_n^*)$ are all technologically feasible and maximize, for each firm $i \in N$, its respective profit.
3. The labor market and the markets for produced goods clear (i.e. supply equals demand).

A rigorous formalization of WE is provided by Definition 2 in Appendix A. Since the economy satisfies the usual properties contemplated by General Equilibrium Theory, existence of a WE readily follows.

3 Basic results

The form of the production function (2) has the implication that (provided α and β are both positive) labor and at least one intermediate input are essential in the production of any good. No firm, therefore, can be active (i.e. achieve a positive production) unless it

[5] To understand this heuristically, suppose that firm k has a total amount of money M to spend among n_k intermediate inputs used in the production of good k. Then, if each input enters production symmetrically and also bears the same price, firm k should split M equally among the n_k inputs. Fixing the amount of labor used, this implies that production must be proportional to $n_k^{\alpha+\nu} M^\alpha \left(\frac{1}{n_k}\right)^\alpha = n_k^\nu M^\alpha$. Thus, if $\nu > 0$ there are benefits from increasing the range of inputs used in production, and the magnitude of ν quantifies precisely this effect.

relies on some other active firm. This imposes a natural requirement of "systemic balance" on an economic system if all its firms are to be active. And such a condition not only has static implications but, as we shall see in Section 7, it entails dynamic consequences as well. Similar considerations have been found to be important in the evolution of many other systems – biological, ecological, or chemical – where some suitable notion of systemic balance is also key to its static stability and dynamic evolution.[6]

However, an important feature of economic systems that has no clear counterpart in other contexts is that, in a market economy, the "source of value" is not just internal to the production network but, crucially, is also "externally" dependent on consumers' preferences. Preferences provide the standard to measure welfare and also determine the market value that shapes firm performance. To discuss this issue, it is useful to extend the directed production network $\Gamma = \{N, L\}$ with an additional node c that stands for our representative consumer. This node c has an in-link (j, c) originating in every node $j \in M$ that produces a consumption good. Such an extended network is denoted by $\hat{\Gamma}$.

As a preliminary step in our analysis, we want to understand when, given a particular (extended) production structure $\hat{\Gamma}$, a particular firm i can be active at a WE. To this end, we rely on the notion of *Strongly-Connected Component* (SCC). Let the notation $j \twoheadrightarrow k$ indicate that there is a directed path originating in j and ending in k. Then, a subset of nodes $Q \subset N$ is said to be a SCC if, from every pair of nodes $j, k \in Q$, $j \twoheadrightarrow k$. The following result specifies necessary conditions for some particular node to be active at a WE.

Proposition 1
Consider any firm $i \in N$ that is active at some WE. Then we have:
(a) $\exists Q \subset N$ that defines a SCC of the (extended) directed network $\hat{\Gamma}$ s.t. $\forall j \in Q$, $j \twoheadrightarrow i$;
(b) $i \twoheadrightarrow c$.

Proof. See Appendix A.

A simple illustration of the necessary conditions contemplated in Proposition 1 is provided by the production network shown in Figure 1. First we note that in this network the four firms/nodes coded in red – i.e. 1 to 4 – do not satisfy the necessary conditions (a)-(b) specified in Proposition 1 and thus cannot be active at any WE. The violation of these conditions occur because either they do *not* have any intermediate input to rely on (Firm 4) or there is *no* direct or indirect connection to the consumer node (Firms 1 to 3). In the first case, the issue is one of feasibility (production of good 4 is not at all possible), while in the second case the problem is one of incentives (if the goods were produced, they would fetch no market value and hence lead to a zero profit).

In contrast, the remaining blue nodes 5-11 satisfy the aforementioned conditions and,

[6] See, for example, the interesting work of Jain and Krishna (1998), who stress the importance of "autocatalytic balance" (the analogue of what we have called systemic balance above) in processes of growth in biological systems.

in principle, could be active at a WE. For nodes 5-7 this follows from the following two-fold observation: they define a SCC (so they jointly define a feasible production structure), and they also have an (indirect) connection to the consumer node (hence they may contribute to market value). Instead, firms 8 to 11 do not form part of a SCC, and thus have to depend on "external inputs" to undertake production. They can, however, obtain those inputs from the aforementioned SCC and then access market value through firm 11, which is connected to consumer node c (i.e. produces a consumer good).

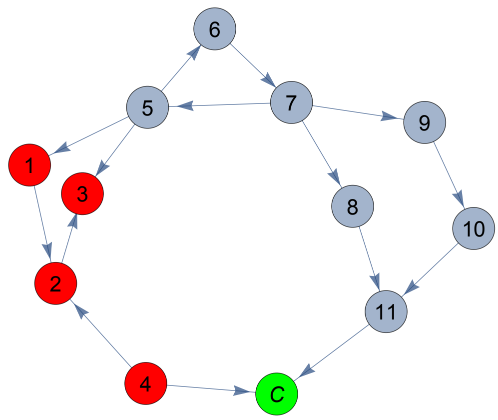

Figure 1: An extended production network $\hat{G}$ that includes the consumer node. The blue nodes are those that satisfy conditions (a)-(b) in Proposition 1, while the red ones do not. The green node represents the consumer demand.

Next, we address the question of whether there are conditions (possibly more stringent than those specified in Proposition 1) that are not only necessary but also sufficient, i.e. *characterize* when a firm is active at a WE. Building upon the ideas underlying Proposition 1 and the Cobb-Douglas specification of the model, we arrive at the following result.

Corollary 1

Consider any given firm $i \in N$. This firm is active at any WE if, and only if, it satisfies (a) and (b) in Proposition 1 and so happens as well for all other firms $j \in N_i^+$ providing inputs to it.[7]

Proof. See Appendix A.

In view of the previous result, throughout this paper we shall assume that all existing firms satisfy (a)-(b). For short, this will be labeled Condition (A). Clearly, this condition can be assumed without loss of generality, for any other firm can be simply ignored in the analysis.

[7]Note that, clearly, if a good i satisfies (b) in Proposition 1, then all its inputs satisfy it as well. Hence the only relevant requirement concerning i's inputs is that they be supported from the production side, i.e. that they satisfy (a).

Of course, the identification of what firms are active at a WE provides only a very partial description of the situation. In general, there will be a wide heterogeneity in performance across firms (specifically, in terms of sales and profits). And if firms are symmetric in every respect except for their network position, it is the overall topology of the production network that should be used to explain such heterogeneity. Can we map the network-performance relationship in a sharp and insightful manner? To answer this question we turn to our first main result of the paper, Proposition 2 below.

Proposition 2
There exists a unique WE for which the vector of equilibrium revenues $\boldsymbol{s}^* = (s_i^*)_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n$ *is given by*

$$\boldsymbol{s}^* = \frac{w^*(1-\alpha)}{\beta}(I - \alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

while the corresponding equilibrium profits $\boldsymbol{\pi}^* = (\pi_i^*)_{i=1}^n = (1 - \alpha - \beta)\boldsymbol{s}^*$.

Corollary 2
Assume $M = N$ (i.e. all goods are consumed). Then, the relative *revenues and profits of the different firms at the WE satisfy:*

$$\left(\frac{s_i^*}{\sum_{j=1}^n s_j^*}\right)_{i=1}^n = \left(\frac{\pi_i^*}{\sum_{j=1}^n \pi_j^*}\right)_{i=1}^n = (1-\alpha)(I - \alpha G)^{-1}\boldsymbol{\gamma}.$$

Proofs. See Appendix A.

Proposition 2 establishes that – under the maintained condition (A) – the unique WE induces an equilibrium revenue (as well as profit) for each firm i that is proportional to a suitable measure of network centrality of this firm given by $(I - \alpha G)^{-1}\boldsymbol{\gamma}$. As we explain next, this centrality notion integrates two main factors:

(i) the utility weight of each of the consumption goods whose production the firm's output contributes to, either directly or/and indirectly;

(ii) a weighted discounted measure of the direct and indirect ways in which the aforementioned contribution takes place.

The measure of network centrality that arises from our analysis is a variation of the widely-used concept of Bonacich centrality. Since this notion of centrality is defined in the literature in slightly different forms, let us start our discussion by introducing the particular version to which we are referring here.

Definition 1
Consider a directed weighted network whose adjacency $(n{\times}n)$-matrix is G. Let $\delta \in (0, 1)$ be a discount factor and $\zeta > 0$ a scale factor. Then, the associated vector of Bonacich centralities *is given by $\boldsymbol{v}(G, \delta, \zeta) = \zeta(I - \delta G)^{-1}\mathbf{1}$, where $\mathbf{1}$ is a suitable column vector whose components are all equal to* 1.

To understand intuitively the notion of Bonacich centrality,[8] suppose first that the network is binary, so that links either have a zero or unit weight, i.e. $g_{ij} \in \{0, 1\}$ for every $i, j \in N$. Then, since the matrix G is assumed column-stochastic, its different columns have to include exactly one unit entry and the other entries must be equal to zero. In this simplified case, it is easy to show that the centrality of a particular node i can be interpreted as (is proportional to) the average of the *discounted number of paths* that start at node i and reach all n nodes (including itself) in r steps ($r = 1, 2, ...$). The discount factor used is $\delta \in (0, 1)$, and the total discount imposed on any given path is tailored to its length, i.e. it is δ^r if its length is r. To see this, rewrite the expression for centrality introduced in Definition 1 as follows:

$$\boldsymbol{v}(G, \delta, \zeta) = (v_i(G, \delta, \zeta))_{i=1}^n = \zeta \left[\sum_{r=0}^{\infty} \delta^r G^r\right] \mathbf{1} \tag{4}$$

so that, for each $i \in N$, its corresponding centrality is given by

$$v_i(G, \delta, \zeta) = \zeta \sum_{j=1}^{n} \left[\sum_{r=0}^{\infty} \delta^r g_{ij}^{[r]}\right]$$

where each $g_{ij}^{[r]}$ represents the ij-th entry of the matrix G^r. The suggested interpretation is then a consequence of the fact that $g_{ij}^{[r]}$ simply counts the number of paths of *exact* length r that start at node i and end at node j. As we have seen (cf. Proposition 2), the discount factor to be used in our case is $\delta = \alpha$ (where α is the output elasticity of intermediate inputs). Therefore, it turns out to be convenient to have a scale factor $\zeta = (1 - \alpha)/n$. Since the matrix G (and therefore every power G^r) is column-stochastic, this scaling amounts to normalizing the centrality vector $\boldsymbol{v}$ to lie in the $n - 1$ simplex, i.e. $\sum_{i=1}^n v_i = 1$.

More generally, when the matrix G is a general column-stochastic matrix with its entries $g_{ij} \in [0, 1]$, an analogous interpretation can be provided but the "number of paths" must then be replaced by the "normalized intensity" that flows between pairs of nodes for every possible path lengths. Applied to our production context, such intensity simply corresponds to the product of the input-demand flows between any given firm i and the firms j that use the former's good as input, directly or indirectly.

The relationship between a firm's performance and its Bonacich centrality is especially stark in the context considered in Corollary 2. There we abstract from any asymmetry associated to the consumption side: all produced goods are assumed to be consumption goods and equivalent for the consumer, which implies that $\boldsymbol{\gamma} = \frac{1}{n}\mathbf{1}$. Then, this result establishes that the *relative* performance of firms is *exclusively* determined by their Bonacich centrality in the production network for a discount factor $\delta = \alpha$ and scale factor $(1 - \alpha)$, where recall that α is the production elasticity of intermediate inputs. In this sense, we

[8] The measure introduced in Bonacich (1987) is defined by the expression $c(G, a, b) = b(I - \alpha G)^{-1} G\mathbf{1}$ while Ballester et al. (2006) use instead $b(G, a) = (I - \alpha G)^{-1}\mathbf{1}$ as the measure they call Bonacich centrality.

may say that, given this elasticity, the profitability of a firm is the reflection of purely "topological" features of the production network.

In the general case where the vector of preferences $\boldsymbol{\gamma}$ is arbitrary and possibly $M \neq N$ (not all goods are necessarily consumption goods), matters must be correspondingly adapted and the notion of centrality considered has to account for those asymmetries. Then, the relevant measure of centrality associates to the paths that arrive to any particular node j the weight γ_j that the consumer attributes to the respective good in her utility function (cf. Proposition 2). Thus, in particular, if good j is not a consumption good, paths of any given length r that connect some node i to such a node j do *not* contribute directly to the centrality (and therefore the profitability) of firm i. They may only do so indirectly to the extent that such paths can be constituent subpaths of longer ones that eventually connect i to some consumption good k (in which case they would be weighted by γ_k and affected by a lower discount factor).

Having characterized the situation on the production side of the economy, now we turn to studying its consumption side. Specifically, our aim is to understand how the features of the network impinge on the consumer's welfare at the WE. Again we find that the vector of centralities plays a prominent role, although in this case additional technological parameters are also important. We start the analysis by providing an explicit expression for the *equilibrium utility* in our next result.

Proposition 3
At the WE the consumer's utility is given by

$$\log U(\boldsymbol{c}) = \sum_{i=1}^{n} \gamma_i \log \gamma_i + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}$$
$$\left((\nu+\alpha) \sum_i v_i \log n_i - (1-\alpha-\beta) \sum_{i\in N} v_i \log v_i + \alpha \sum_{i\in N} \sum_{j\in N_i^+} v_i g_{ji} \log g_{ji} \right) \tag{5}$$

where recall that n_i is the number of inputs used in the production of good i, ν is the parameter modulating the economies of scope through the factor $A_i = n_i^{\alpha+\nu}$, and $\boldsymbol{v}(G, \alpha, 1-\alpha) = (v_i)_{i=1}^n$ is the corresponding vector of Bonacich centralities for $\delta = \alpha$ and $\zeta = 1-\alpha$.
Proof. See Appendix A.

The previous result highlights the three *endogenous* magnitudes that shape welfare in our context:

1. The weighted average (log-)connectivity, $\sum_i v_i \log n_i$, where the degree of each node/firm is weighted by its respective centrality.
2. A measure of heterogeneity/dispersion among firms, as reflected by the entropy of their centralities, $(-\sum_{i\in N} v_i \log v_i)$.
3. The average heterogeneity/entropy in input use (i.e. $-\sum g_{ji} \log g_{ji}$) displayed by the technologies of the different firms $i \in N$, each of these individual magnitudes weighted by the centrality of the respective firm i.

4. The heterogeneity/entropy across goods, $\sum_{i=1}^{n} \gamma_i \log \gamma_i$, displayed by the preferences of the representative consumer.

Specifically, we find that consumer welfare is increasing in the average log-connectivity of firms and in *inter*-firm symmetry, while it is decreasing in the *intra*-firm symmetry displayed by their technologies. These different magnitudes are aggregated through a simple affine function whose coefficients are given by the underlying technological parameters.

Next, to obtain a sharper characterization of the situation, it is useful to reduce the degrees of freedom by postulating the following two symmetry assumptions:

(S1) For any given firm $i \in N$, its different inputs play a symmetric role in its production technology, i.e. $g_{ji} = g_{ki}$ for all j, $k \in N_i^+$.

(S2) All goods are consumption goods and therefore $\gamma_i = 1/n$ for every firm $i \in N$.

Under the previous conditions, the intra-firm heterogeneity of each firm i is dependent on n_i alone, i.e. on the *number* of inputs it uses (which, heuristically, could be understood as the "complexity" of its production technology). Then, the expression in (5) is substantially simplified, as stated by the following corollary.

Corollary 3

Assume (S1) and (S2) above. Then, the utility of the consumer at the WE is given by

$$\log U = -\log n + \log{(1-\alpha)}\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}\left(\nu \sum_{i=1}^{n} v_i \log n_i - (1-\alpha-\beta)\sum_{i=1}^{n} v_i \log v_i\right) \tag{6}$$

Proof. See Appendix A.

Thus, under input symmetry, connectivity and inter-firm entropy are the *sole considerations* that impinge on consumer's welfare. As we shall discuss in Subsection 5.1, this stark conclusion will allow us as well to obtain a similarly sharp assessment of what production structures are welfare optimal in different technological scenarios.

4 Price distortions

In this section, we focus on the study of what could be interpreted as taxes, distortions, or policy/price interventions of different sorts. An obvious, but still important, characteristic of an interconnected economy is that any such distortion or intervention cannot be studied just locally. In general, it is to be expected that its first-order impact could turn out to be quite different from its overall effect on the economy, once the complete chain of indirect effects is taken into account. Such a full-fledged analysis of the situation, however, can be very complex and we need effective tools to carry out a proper analysis of the situation. Here we provide a step in this direction.

Specifically, in this section we consider two different cases. First, in Subsection 4.1, we consider a price distortion that applies directly to *all* firms in the economy in a uniform manner. Then, in Subsection 4.2, we turn to studying distortions that apply directly to just one firm in the economy, so the effects on all others are only indirect. In both cases, uniform or individualized, we focus on situations that, at an abstract level, can be conceived as inducing "price wedges". More concretely, this is an approach that can taken to capture a variety of different cases: government policies that favor a particular sector, *ad-valorem* taxes or subsidies, or the reflection of market power.[9] To stress the intended generality, we shall simply speak of them as *distortions*[10].

A common idea that arises in the analysis of both uniform and individual distortion is that centrality-based measures are key in determining the size and direction of the induced effects. Centrality being an inherently global measure, the point is then that the impact of any change or intervention, no matter how "local" it might appear, must be evaluated globally. In fact, variants of the same general idea will reappear as well in much of the analysis conducted throughout the paper, again an indication of the unavoidable global nature of the issues being studied. In essence, the crucial network magnitudes involved in the analysis will turn out to be what we shall call (bilateral) in- and out-centralities. These are the elements of the matrix $\mathscr{M} = (m_{ij})_{i,j=1}^{n} \equiv (I - \alpha G)^{-1}$ included in the definition of (Bonacich) centrality – cf. Definition 1.

Specifically, for every pair of nodes i and j, the ij-incentrality is identified with m_{ij}. By writing

$$v_i(G, \alpha, \zeta) = \frac{1-\alpha}{n} \sum_{j=1}^{n} \left[\sum_{r=0}^{\infty} \alpha^r g_{ij}^{[r]} \right] = \frac{1-\alpha}{n} \sum_{j=1}^{n} m_{ij} \tag{7}$$

it is indeed apparent that m_{ij} represents (or, more precisely, is proportional to) the contribution of node j to the Bonacich centrality of i. Reciprocally, we refer to the entry m_{ji} as the ij-outcentrality. Note that, within our general economic model (cf. Proposition 2), for each $i \in N$ the entries m_{ij} $(j = 1, 2, ..., n)$ are the weights given to the preference weights γ_j of each good produced in the determination of the equilibrium profits of firm i through the expression

$$\pi_i^* = (1 - \alpha - \beta) \, \frac{w^*(1-\alpha)}{\beta} \sum_{j=1}^{n} m_{ij}\gamma_j. \tag{8}$$

Thus, as explained, the contribution to i's centrality provided by an intermediate product j that is not consumed (whose $\gamma_j = 0$) has no effect on i's equilibrium profits.

[9]See, for example, the paper by Hsieh and Klenow (2009) or Jones (2011) for an elaboration on such a general interpretation of price distortions.

[10]Within our framework one can also analyze input specific distortions (i.e. distortions that lower the marginal product of one input relative to others, see (Hsieh and Klenow, 2009) for details) using the same tools. We do not include analysis of this type of distortions in the paper. The details are available upon request.

4.1 Uniform price distortion

We start our analysis of distortionary effects by considering the implications of a uniform price distortion τ imposed on all firms of the economy. Its effect is to draw a proportional wedge between the price p_i the consumer pays for each good i and the price $(1-\tau)p_i$ received by the firm selling it. In principle, the value of τ might be negative, in which case it could be interpreted, for example, as a subsidy (hence amounting to a proportional increase in the revenue earned from the firm). An issue that arises here is whether the monetary payments (or proceeds) entailed should be distributed back to (or subtracted from) the revenue available to the agents of the economy – that is, whether the system is to be conceived as closed to those monetary flows. In general, the answer must depend on the specific interpretation attributed to those flows (e.g. on whether they are redistributive, or purely distortionary and hence wasteful). Here, in the main text, we shall consider the formally simpler case where they are a pure outflow (or inflow) of resources, while referring the reader to the online Appendix C for a consideration of the alternative closed-system version. None of our results are qualitatively affected by the scenario being considered.

Our first observation (see Lemma 1 in Appendix A for details) is that, under a uniform τ applied to all goods of the economy, the expression that characterizes the vector of equilibrium profits is generalized to

$$\boldsymbol{\pi}^*(\tau) = (1-\tau)(1-\alpha-\beta)\frac{w(1-\alpha)}{\beta}\left(I-\alpha(1-\tau)G\right)^{-1}\boldsymbol{\gamma} \tag{9}$$

This readily implies that, as expected, the effect on equilibrium profits of an increase in τ is unambiguously negative – the reason, of course, is simply that, in our context, any distortion is always detrimental.[11] Indeed, if we consider the marginal effect of increasing τ, we find (see Lemma 2 in Appendix A):

$$\frac{\mathrm{d}\boldsymbol{\pi}^*}{\mathrm{d}\tau}(\tau) = -(1-\alpha-\beta)\boldsymbol{s}^*(\tau) - \alpha(I-\alpha(1-\tau)G)^{-1}G\boldsymbol{\pi}^*(\tau) < 0$$

where $\boldsymbol{\pi}^*(\tau)$ stands for the full vector of profits earned under τ by all firms.

Having settled the question of how a uniform distortion affects firms' profits, next we turn our attention to the much more complex issue of how it can *diversely* impinges on the relative profits of the different firms, as a function of their individual position in the network. To account for this relative performance in a convenient manner, it is useful to focus on the normalized (equilibrium) profits $(\hat{\pi}_j)_{i=1}^n$ of each firm i obtained by setting the nominal wage $w(\tau)$ so that $\sum_{j\in N}\hat{\pi}_j = 1$. A formal characterization of the (marginal) implications of changing τ on the relative profit performance of the different firms is provided by the following result.

Proposition 4
Given any given distortion $\tau \in [0,1]$, let $(\hat{\pi}_j(\tau))_{i\in N}$ be the corresponding profile of nor-

[11] An analogous conclusion obtains for the effect of τ on consumer's welfare, which can be shown to be always negative. Details are available upon request.

malized *equilibrium profits and denote* $(\tilde{m}_{ij}(\tau))_{i,j=1}^n \equiv (I - \alpha(1-\tau)G)^{-1}$. *The marginal effects on profits induced by a change on* τ *are determined by the following system of differential equations:*

$$\frac{d\hat{\pi}_i}{d\tau}(\tau) = \frac{1}{1-\tau}\sum_{j=1}^{n} \tilde{m}_{ij}(\tau)\left(\gamma_j - \hat{\pi}_j(\tau)\right) \qquad (i = 1, 2, ..., n), \tag{10}$$

and hence, at a situation with no distortion ($\tau = 0$)*, the marginal impact of* introducing *it is:*

$$\left.\frac{d\hat{\pi}_i}{d\tau}\right|_{\tau=0} = \sum_{j=1}^{n} m_{ij}\left(\gamma_j - \hat{\pi}_j(0)\right) \qquad (i = 1, 2, ..., n), \tag{11}$$

where recall that $\mathscr{M} = (m_{ij})_{i,j=1}^n$ *is the matrix of incentralities.*

Proof. See Appendix A.

The above result highlights that the firms most *negatively* affected in their *relative* profit performance by the introduction of the distortion are those whose centrality is heavily dependent on firms with high original profits. A particularly stark manifestation of this idea is displayed in (11), which applies to the case where the distortion is just being marginally introduced. This expression can be heuristically understood as follows. The effect on the relative profit standing of any given firm changes as prescribed by an average composition/multiplication of two magnitudes:

(a) the first-order impacts experienced by each one of the firms in the economy, whose sign and size is captured for each firm $j \in N$ by $(\gamma_j - \hat{\pi}_j(0))$ – a comparison of firm j's relative profit and the (relative) weight of its output in consumer's preferences;

(b) the extent to which those impacts are transmitted to the firm i in question, as captured by its respective vector of ij-incentralities $(m_{ij})_{j=1}^n$.

When the base distortion is not zero but is some given $\tau > 0$, this two-fold mechanism exhibits the general form given by (10), which has an analogous interpretation. An interesting feature of this expression is that the sensitivity of relative equilibrium profits to interfirm asymmetries grows steeply as τ approaches unity. Another interesting consideration worth highlighting is that, because the dependence of profits on τ is nonlinear, there is the potential for complex (and, in particular, nonmonotonic) behavior as the distortion changes within its full range $[0, 1]$. The following simple example illustrates that this is indeed a possibility.

Example 1

Consider the simple production network with nine firms depicted in Figure 2, where it is assumed that all firms produce a consumption good and hence the arrows only indicate the flow of input use. Figure 3 traces the relative profits for possible values of $\tau \in [0, 1]$

and a subset of firms (for the sake of readability, not all firms are included). The second diagram shows that, as τ grows (and, naturally, the profit spectrum across firms narrows) there are rank changes in the relative position of some firms. Thus, while Firm 6 remains the highest-profit firm throughout, as τ grows the position of Firm 8 deteriorates, from second to fourth position. Indeed, we find that it is not just the ranking partially changes but even the relative profit of an individual firm may evolve in nonmonotonic way as τ rises. Specifically, as shown more clearly in Figure 4, this happens to Firm 1, whose profit first grows and then decreases.

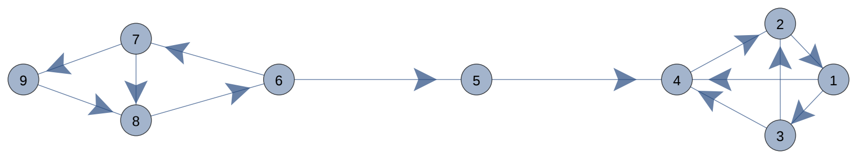

Figure 2: A simple production network illustrating the discussion included in Example 1.

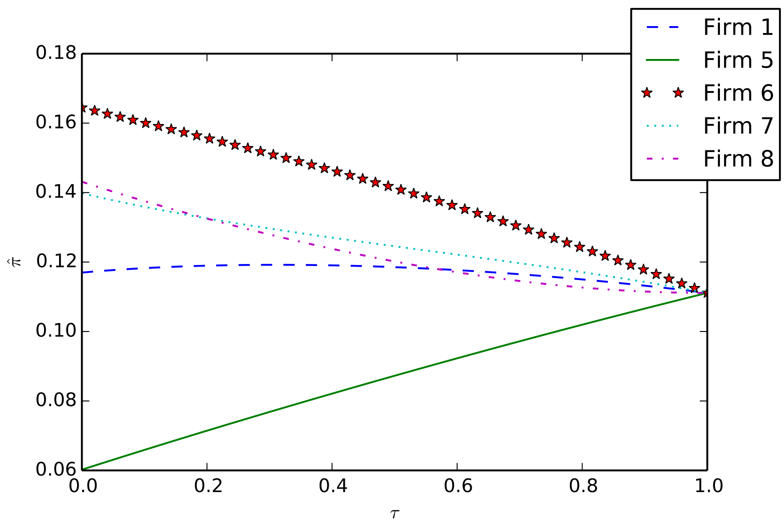

Figure 3: The relative profits of all firms in the production network displayed in Figure 2 for all values of $\tau \in [0, 1]$.

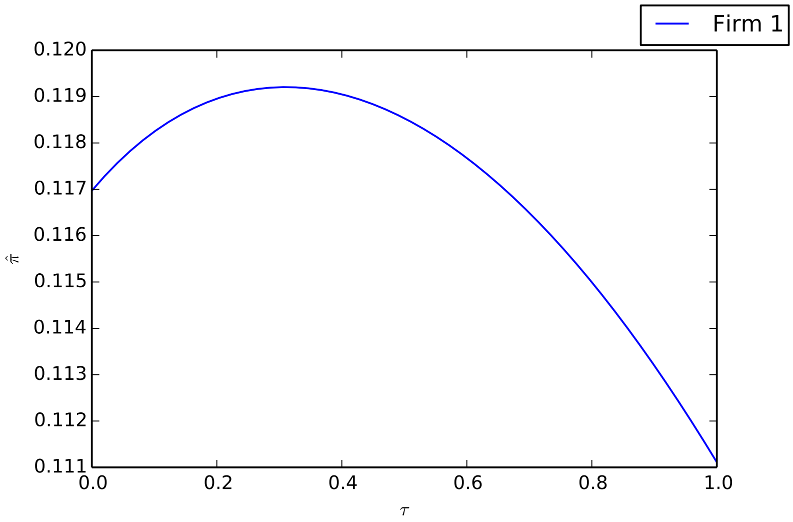

Figure 4: The nonmonotonic behavior of the relative profit of firm 1 in the production network displayed in Figure 2, as τ changes in the range $[0, 1]$.

4.2 Individual price distortion

In this subsection, our focus turns from a common distortion that affects uniformly all firms to one that impinges only on a specific firm k. As shown in Appendix A, such a firm-specific price distortion, denoted by τ_k, introduces just a simple modification in the equilibrium expressions derived for the benchmark model, but one that is of a quite different nature from those obtained for a uniform distortion. For example, in contrast with (9), the induced equilibrium profits $\boldsymbol{\pi}^*(\tau_k^o) = [\pi_i^*(\tau_k)]_{i=1}^n$ are now given by:

$$\boldsymbol{\pi}^*(\tau_k) = (1-\alpha-\beta)\frac{w^*(1-\alpha)}{\beta}\left(I - \alpha\hat{G}(\tau_k)\right)^{-1}\boldsymbol{\gamma}. \tag{12}$$

where $\hat{G}$ represents a modified matrix that replaces the original one, G. These matrices only differ in their respective kth columns, $\hat{\mathbf{g}}_k$ and $\mathbf{g}_k$, which satisfy $\hat{\mathbf{g}}_k = (1-\tau_k)\mathbf{g}_k$.

The difficulty lies in studying the inverse in (12) to obtain its dependence on τ_k. To do this, it is useful to write the modified matrix $\hat{G}(\tau_k)$ as follows:

$$\hat{G}(\tau_k) = G - \tau_k \mathbf{g}_k \mathbf{e}_k', \tag{13}$$

where $\mathbf{e}_k'$ is the n-dimensional row vector that has a 1 in its kth position and 0 elsewhere. This then allows us to rely on the following result in Linear Algebra (cf. Sherman and Morrison (1949) or Hager (1989)):

Sherman-Morrison Formula. Let A be a nonsingular n-dimensional real matrix, and $\boldsymbol{c}$, $\boldsymbol{d}$ two real n-dimensional column vectors such that $1 + \boldsymbol{d}'A^{-1}\boldsymbol{c} \neq 0$. Then,

$$(A + \boldsymbol{c}\boldsymbol{d}')^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{c}\boldsymbol{d}'A^{-1}}{1 + \boldsymbol{d}'A^{-1}\boldsymbol{c}}.$$

Applying the above result to our case – with the particularization $A = I - \alpha G$, $c = \tau_k \boldsymbol{g}_k$, and $d = \boldsymbol{e}_k$ – we arrive at the following full characterization of how the k-specific distortion affects equilibrium profits.

Proposition 5
Consider a distortion τ_k imposed on firm k and let $(\pi_i^*(\tau_k))_{i=1}^n$ stand for the corresponding equilibrium profits. The marginal effects on profits induced by a change on τ_k are determined by the following system of differential equations:

$$\frac{\mathrm{d}\pi_i^*}{\mathrm{d}\tau_k}(\tau_k) = -\pi_k^*(0) \frac{m_{ik}}{(1 - \tau_k(1 - m_{kk}))^2} \tag{14}$$

and hence, at a situation with no distortion ($\tau_k = 0$), the marginal impact of introducing *it is:*

$$\left.\frac{\mathrm{d}\pi_i^*}{\mathrm{d}\tau_k}\right|_{\tau_k=0} = -\pi_k^*(0) m_{ik} \qquad (i = 1, 2, ..., n), \tag{15}$$

Proof. See Appendix A.

The previous result states that the profit decrease experienced by any firm i due to the direct distortion impinging on another firm k depends on

- how profitable k was prior to the distortion;
- the importance of k in determining the centrality of i;
- how much of k's centrality "feeds into" itself.

As intuition would suggest, while the first two considerations increases the profit loss on firm i due to distortion on k, the last one decreases it (with the only exception considered in (15) when there is no distortion to start with). Again we observe that incentralities (as captured by the m_{ik} and m_{kk} mentioned in the last two items) play a prominent role in shaping the overall global effect. It is also interesting to observe from (14) that the function mapping k's distortion into i's profit is convex so that, just as in the uniform-distortion case, the marginal effect of increasing τ_k grows with the level of it. Another feature that was noted for the previous uniform case is that the induced nonlinearities can lead to somewhat paradoxical conclusions. This phenomenon arises as well here, as illustrated below by through two examples.

Example 2
Consider the production network depicted in Figure 5, where the corresponding profile of Bonacich centralities are also shown. Again, for simplicity, all produced goods are assumed to be not only intermediate inputs but consumption goods as well. Firm 1 and then Firm 2 are those with the highest centrality in the production network. Therefore, under no distortion, they are also the firms enjoying the highest profits at the Walrasian equilibrium.

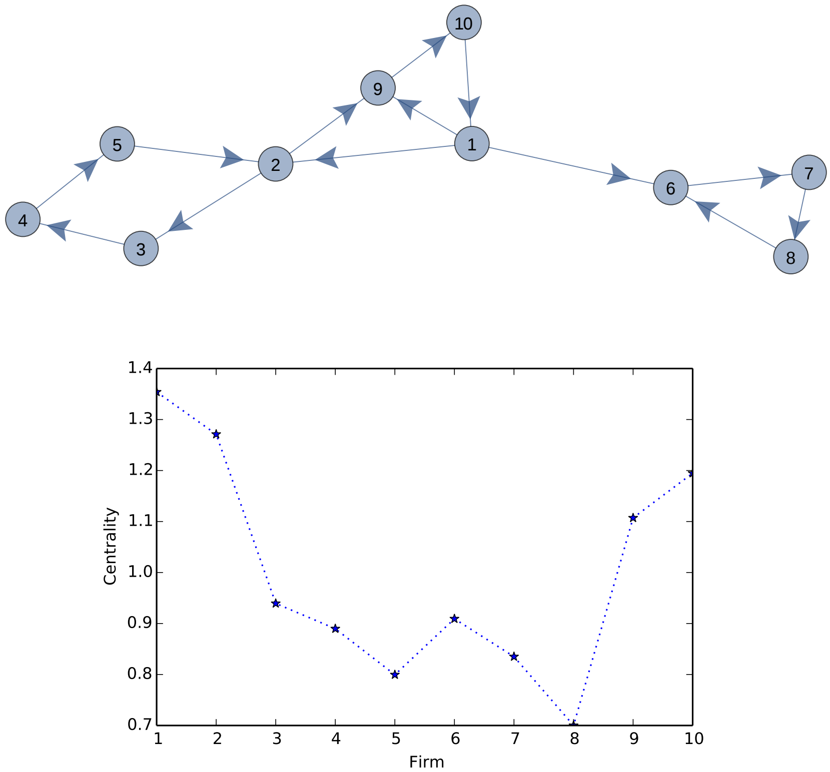

Figure 5: A production network in the upper panel, with the corresponding profile of Bonacich centralities for each of the nodes displayed in the lower panel.

However, as shown in Figure 6, the situation can change in interesting ways if our focus turns to relative profits (normalized to add up to one) and the distortion τ_2 *experienced by Firm* 2 *varies. (Note that here we are allowing* τ_2 *to be negative, varying in the range* $\tau_2 \in [-1, 1]$ *and thus playing possibly the role of a subsidy.) For* $\tau_2 = 0$ *(no distortion) the profit ranking exactly mimics that of centralities, as already explained. In contrast, as* τ_2 *grows and becomes positive, we observe the somewhat paradoxical fact that the relative profit of Firm 2 monotonically grows and, eventually, if* τ_2 *is high enough, even surpasses that of Firm 1 and becomes the highest. The opposite state of affairs is found if Firm 2 is subject to a negative* τ_2*. Then, a higher absolute value for it induces a lower profit for this firm, eventually leading it to fall below that of Firm 10. Why does this happen? The reason is that, as the distortion varies, the effect of this change – which is modulated by the various incentralities – impinges most strongly on the those firms i whose i*2*-incentrality is highest, which then alters the relative position of Firm 2 in the direction opposite to the change of* τ_2*. A further illustration of this role of incentralities, which is particularly simple and stark, is provided by our next example.*

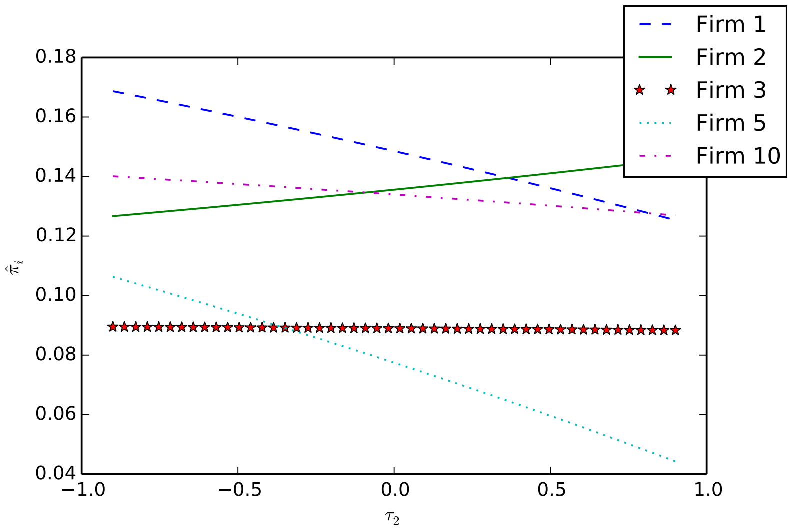

Figure 6: The change in relative profits among the firms placed in the production network depicted in Figure 5 as the distortion experienced by Firm 2 changes in the range $[-1, 1]$.

Example 3

Suppose that ten firms are arranged into a ring production network with each firm $i = 1, 2, ..., 10$ *using the good produced by Firm* $i-1$ *as the sole intermediate input and having its produced good be the sole intermediate input in the production of Firm* $i+1$ *(of course, indices* 0 *and* 11 *are interpreted as* 10 *and* 1*, respectively). As in the previous examples, all produced goods are assumed to be valuable to the consumer. Consider now a distortion experienced by Firm* 1*,* $\tau_1 \in [-1, 1]$*.*

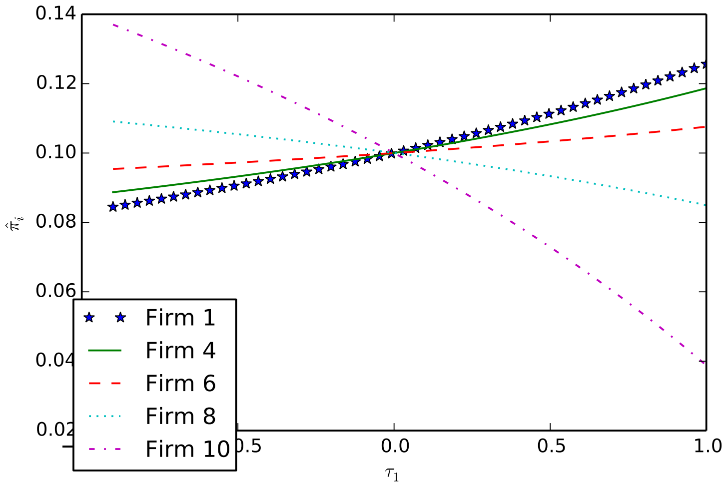

Figure 7: The change in relative profits among the firms placed in a ring production network of ten firms as the distortion experienced by Firm 1 changes in the range $[-1, 1]$. In such a ring network, the input links define the cycle $1 \to 2 \to 3 \to ... \to 10 \to 1$.

Figure 7 depicts how the relative profits of firms change over the whole range of vari-

ation of τ_1. The most affected is Firm 10, either negatively if $\tau_1 > 0$ or positively if $\tau_1 < 0$. The reason is that this firm is the one with the highest $i1$-incentrality. Following it, the firm whose bilateral incentrality relative to 1 is the highest is Firm 9, and thus this firm is the one which is the second most affected by changes in τ_1, and so on along the ring. In the end, we find that, indeed, the firm that is least affected is Firm 1, the firm that is directly subject to the distortion. Of course, this stark conclusion is an artifact of the extreme production network considered but should help clarify the key mechanism underlying global effects in our context.

5 Network structure

In line with our core concern of understanding how economic structure affects performance, here we undertake an analysis of the following issues. First, in Subsection 5.1, we study what features of the production structure (e.g. whether it is more or less connected, or its degree of heterogeneity) impinge on the production and welfare possibilities attainable at equilibrium. Then, in Subsections 5.2 and 5.3, our focus turns to studying the overall effect of single "local perturbations" affecting the production structure. Specifically, we consider two of them: (i) the creation (or elimination) of a link ij; (ii) the elimination of an existing node/firm, or the creation of a new one. As we shall explain, these changes can be understood as reflecting alternative economic phenomena.

5.1 Optimal network structure

The structure of production networks can be studied along many different dimensions. Here we show that, as far as its impact on (consumer) welfare is concerned, the *internode symmetry* displayed by the network and its *connectivity* stand out as two key features. To highlight their effect, it is useful to simplify the analysis by focusing on networks that satisfy the assumptions (S1)-(S2) introduced at the end of Section 3. That is, we postulate that (a) the inputs involved in the production of every good play a symmetric role, and (b) all goods are consumption goods. We also suppose that the number of goods in the economy is given, while postponing to Section 7 an analysis of what are the welfare implications of a change in this feature of the economy. Under these conditions, we can build upon the analysis undertaken in Section 3 to arrive readily at the following conclusion.

Proposition 6
Assume (S.1)-(S.2) in Section 3. Then, the production structure that maximizes consumer utility is a regular network (i.e. all firms have the same number of inputs). Furthermore, if $\nu > 0$ the optimal network is complete (i.e. for all $i \in N$, $n_i = |N_i^+| = n - 1$), while if $\nu < 0$ the optimal network is a minimally connected regular network (for all

$i \in N, n_i = 1)$.

Proof. See Appendix A.

The previous result follows from inspection of the expression determining equilibrium utility in (6), as a function of the the parameters of the economy and the induced profile of firm centralities. On the one hand, the last term of that expression calls for the maximization of the entropy of the simplex-normalized vector of centrality profiles, as given by $-\sum_{i \in N} v_i \log v_i$. This maximization is attained when all nodes have the same centrality, which in turn requires that the network be symmetric across nodes and hence regular. Thus all firms must display the same "input complexity," i.e. they should all use the same *number* of inputs. Then, whether such common complexity should be maximal or minimal sharply depends on the sign of the parameter ν in (3), which reflects the nature of the "economies of input scope" in production. If $\nu > 0$, complexity is beneficial and hence the optimal production structure should be complete, i.e. all other goods should be used in the production of every one of them. Instead, if $\nu < 0$, the exact opposite applies and the optimal production structure should display the minimum complexity consistent with viability (cf. Corollary 2). That is, it should constitute a ring for example.

In principle, of course, it is far-fetched to entertain the possibility that the production structure of the economy might be "designed." Instead, it is more natural to conceive this structure to be a reflection of a wide number of different forces (e.g. technological change or any other modification of the underlying economic environment) whose magnitude and direction can hardly be controlled. Therefore, the discussion in this subsection has to be interpreted mostly as a conceptual exercise. That is, the objective is to gain some understanding of what features of the production structure of an economy have an important impact on welfare. And, from this perspective, what Proposition 6 shows is that (abstracting from some other considerations) two characteristics arise as key: symmetry across all production processes and extreme exploitation/avoidance of all economies/diseconomies of scope.

5.2 Link creation

In the remaining part of this section, we adopt a local perspective to the analysis of changes in the production structure. First, in this subsection, we focus on the impact of a change affecting a single link – for concreteness, we consider the case where the link is created. A preliminary and immediate point to note is that if a link ij is formed, the firm i that consequently sees an expansion in the uses of its output cannot have its *relative* profits decrease. The reason is that its centrality cannot decrease by this change – clearly, the set of paths that access consumer nodes can only grow.

The effect of the new link ij on other firms– in particular, the firm j whose range of inputs has increased – is in general ambiguous (see Example 4 below). It depends, for example, on how the weights $(g_{kj})_{k \in N_j^+}$ are affected by the change. Among the different possibilities one could contemplate in this respect, here we shall assume, for concreteness, that the new matrix $\tilde{G}$ continues to be column stochastic. A possible interpretation of this modeling choice is that the new link entails a new and full-fledged production technology available to firm j that, if adopted by this firm, satisfies the same normalization condition as all other technologies do. In a sense, we view it as embodying a mature new option that can be selected by firm j as a coherent and balanced package.[12]

Proposition 7

Consider an initial production network with adjacency matrix G and suppose that a link ij is added to it. Denote by $\tilde{G}$ the resulting adjacency matrix and let $\boldsymbol{q} = (q_k)_{k=1}^n \in \mathbb{R}^n$ ($\sum_{k \in N} q_k = 0$) be the real vector that reflects the entailed adjustment across the two adjacency matrices, i.e. their respective jth columns, $(g_{kj})_{k=1}^n$ and $(\tilde{g}_{kj})_{k=1}^n$, satisfy $\tilde{g}_{kj} = g_{kj} + q_k$ for all $k = 1, 2, ..., n$ with $\tilde{g}_{ij} = q_i > 0$. The change on equilibrium profits, $(\Delta \pi_k^*)_{k=1}^n$, induced by the new link is given by:

$$\Delta \pi_k^* = \alpha\, \pi_j^* \, \frac{m_{ki}\, q_i + \sum_{\ell \in N_j^+} m_{k\ell}\, q_\ell}{1 + q_i\, m_{ji} + \alpha \sum_{\ell \in N_j^+} m_{j\ell}\, q_\ell}, \tag{16}$$

where π_j^* is the original profit of firm j under adjacency matrix G.

Proof. See Appendix A.

The previous result establishes that if a new link ij is added to the production network, the effect on the equilibrium profit of any particular firm j is given by the composition of two key factors:

- The profit originally obtained by the firm j that provides a new use to good i. Intuitively, this magnitude reflects the importance of firm j in providing the immediate market value generated by the new link ij.
- The impact on the centrality of k of each of the inputs involved in the production of j (including the new input i), as captured by the respective incentralities, m_{ki} and $(m_{k\ell})_{\ell \in N_j^+}$. Each of these incentralities is weighted by the factor q_ℓ that reflects the adjustment made (positive or negative) when transforming the adjacency matrix G into $\tilde{G}$.

Again, therefore, we can understand the changes resulting from the new link as consisting of the composition of two kinds of effects: a *first-order effect* whose magnitude is associ-

[12] Admittedly, this is only one of the possibilities that could be reasonably considered. Alternatively, it could be assumed that the components of the new $(\tilde{g}_{kj})_{k \in N_j^+}$ sum up to some positive magnitude $H \neq 1$. Then, one could still unit-normalize the corresponding j-th column of $\tilde{G}$ by adjusting the overall input elasticity of firm j to the value $\tilde{\alpha}_j = H \alpha_j$. None of these possible modeling variations would affect the essential gist of our analysis.

ated to the equilibrium-induced importance of the directly affected nodes; a *second-order effect* that involves the transmission of the aforementioned first-order effect through the full array of network-based interactions that are captured by the matrix of incentralities.

Example 4

As advanced, in this example we illustrate that the addition of a new link ij can deteriorate the relative position of the firm whose input range is expanded by the new link. Consider the production networks depicted in Figure 8, and suppose that the smaller network gives rise to the larger one that includes the new link $1 \to 4$. Let us assume that all goods are consumer goods and have an equal weight in the utility function of the consumer. Let us also suppose that the new matrix $\tilde{G}$ displayed after the change continues being column stochastic. Then, denoting by $\hat{\pi}^\dagger$ and $\hat{\pi}^\ddagger$ the respective vectors of normalized profits for the original and modified networks, we compute them to be as follows:

$$\hat{\boldsymbol{\pi}}^\dagger = \begin{pmatrix} 0.1000 \\ 0.3265 \\ 0.2960 \\ 0.2775 \end{pmatrix} \qquad \hat{\boldsymbol{\pi}}^\ddagger = \begin{pmatrix} 0.1790 \\ 0.2863 \\ 0.2718 \\ 0.2630 \end{pmatrix} \tag{17}$$

Thus one finds that firm 4 obtains a lower relative profit after the link 14 is added to the original network. Why is this the case? The simple reason is that, as explained, centrality (and hence profitability) is associated to the (discounted) value a product delivers to the consumer, not to the value it allows other firms to attain. That is, profitability is gathered downstream rather that upstream and, given architecture of the network in our example, the discounted downstream "flow" gained by Firm 4 from the new link falls relative to that of the other firms – in particular, that of Firm 1, whose centrality is sure to increase.

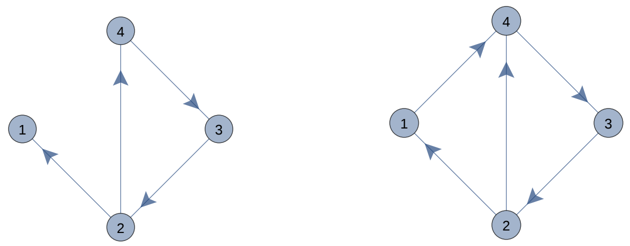

Figure 8: Two different production networks, whose only difference concerns the existence, or not, of the link from node 1 to node 4. All nodes/firms are supposed to produce a consumption good with equal preference weights.

5.3 Node deletion

In this section, we study the alternative (local) change in the production structure given by the addition or removal of a (single) node rather than those of a link. Now, for concreteness, we focus our attention on the case where the change involves the removal of the node. This is, in fact, the kind of change that will be considered in Section 7 when studying the chain implications of a process of node/firm bankruptcy and *endogenous* removal.

Once more we must face the modeling question of how the setup is to be renormalized when the network changes – in this case, the set of nodes. Now it concerns both the utility and the production functions, all of which deliver a value of zero when any of its arguments is equal to zero. For the utility function, the renormalization we choose to tackle the problem is the natural one: we restrict its arguments to the range of consumer goods which can actually be produced given the prevailing set of active firms. In Subsection 7.1, we provide a specific motivation for this choice in a context where the number of firms changes due to a process firm selection.

Concerning the renormalization to be implemented on the production functions when there is a reduction in the inputs available, again for concreteness we choose one specific option but other alternatives could be analogously considered. The alternative considered is different from that adopted in Subsection 5.2, where the change involved an increase in input use. As we now explain, this contrast is motivated by a different view on what is the nature of the phenomenon underlying the situation in either case. Here we conceive the disappearance of a node as a somewhat abrupt/unexpected event that requires some short-run and hence partial adjustment. Thus, unlike what was suggested for the case of link creation, we assume that the columns of the matrix G are not renormalized. Thus the firms j that see one of its inputs vanish somehow adapts (for otherwise the indispensability of all prior inputs would force it to a zero production) but in a partial manner. Along with the point made in Footnote 12, we can interpret this as a downward shift in the productivity in the overall use of the available intermediate inputs.[13] Admittedly, one should not expect real-world situations to be so markedly polar as suggested between the increase and decrease in the use of preexisting inputs. The contrast, however, is plausible and we find it useful to simplify the discussion of either case.

Before stating the main result from this subsection, let us introduce following notation. Let $\tilde{G}$ denote the resulting adjacency matrix after deletion of node i. If the deleted firm produced a consumption good, then after deletion of that node one may expect that weights $\boldsymbol{\gamma}$ that the consumer's preferences attribute to all remaining goods will adjust[14]. Let $\tilde{\boldsymbol{\gamma}}$ denote the vector of resulting preference weights after a node is deleted.

In view of the relationship established between profitability and centrality, it is quite clear that the elimination of a node can only be detrimental to the profits of all firms in the economy. A precise specification of the relative magnitudes of the effects that apply to each of them is given by the following result.

[13] If we proceeded in this way, a full-fledged analysis of the situation should rely on the extended framework considered in the online Appendix B, where full firm heterogeneity is allowed – in particular, on input elasticities.

[14] We discuss this adjustment in more details in 7.1.2

Proposition 8

Consider an initial production network with adjacency matrix G and suppose that node $i \in N = \{1, 2, ..., n\}$ is removed from it. Denote by $\left(\Delta\pi_j^*\right)_{j\neq i}$ the change in equilibrium profits of the remaining firms. Then,

$$\Delta\pi_j^*(G) = \tilde{\pi}_j(G) - \pi_j^*(G) - \tilde{\pi}_i(G)\frac{m_{ji}(G)}{m_{ii}(G)}, \tag{18}$$

where $\tilde{\boldsymbol{\pi}}(G) = (1-\alpha-\beta)\frac{1-\alpha}{\beta}w^*(I-\alpha G)^{-1}\tilde{\boldsymbol{\gamma}}$

Proof. See Appendix A.

The previous result indicates that the effect induced by the removal of a particular firm i on the profits of the remaining firms $j \neq i$ exhibits the usual pattern: a first-order effect (that is quantified by the prominence/profitability of the firm directly affected, i.e. firm i) composed with the effect that the directly affected firm has on the centrality of any other firm (measured by the corresponding incentrality). In the present case, however, the incentrality terms m_{ji} are "normalized" by the effect m_{ii} that firm i has on its own centrality – in this sense, the relevant magnitude scales the outcentralities of i by the extent to which this firm's centrality effect feeds into itself. The additional effect, which may have an opposite sign, is due to potential preference adjustments and is captured by difference $\tilde{\pi}_j(G)-\pi_j^*(G)$. Note that when there is no adjustment in consumer's preferences (when $\tilde{\gamma}_j = \gamma_j \; \forall j \in \{1, 2, .., n\} \setminus \{i\}$) equation (18) becomes

$$\Delta\pi_j^*(G) = -\pi_i^*(G)\frac{m_{ji}(G)}{m_{ii}(G)}.$$

6 Forward and backward linkages

As explained in the Introduction, one of the useful features of the model is that it allows a transparent formalization of the notions of forward and backward linkages (also called push and pull effects) that, as stressed by (Hirschman, 1958), can be important in assessing, for example, public interventions and economic policies. In essence, all of the comparative static results considered so far in Subsections 4.1 through 5.3 can be conceived as involving those two types of linkages, forward and backward. That is, the overall effects considered so far can be seen to include a forward linkage that operates downstream on the production structure, and a backwards one that works upstream.

Rather that revisiting all our previous results in this light, it will prove more useful at this point to discuss these forces/linkages in the context of two particularly clear-cut instances: a (Hicks-neutral) pattern of technological improvements on the production technologies of the different firms; an arbitrary change in the preferences of the consumer across the different goods. As we shall see, while the first case exerts only pure "push" forces on the equilibrium outcome, the second one embodies an array of "pull" forces (some positive and others negative).

6.1 Technological change

Consider a change in the pre-factors $(A_k)_{k\in N}$ of the production functions of each firm k in the economy (cf. (2)-(3)), which become $(\tilde{A}_k)_{k\in N}$. We shall refer to them as the productivities of the respective firms. The key observation to make from Proposition 2 is that the equilibrium conditions imply that the induced sales at equilibrium are *independent* of those productivities. This readily leads to the following two conclusions:

(i) The equilibrium input demands $[(z^*_{jk})_{j\neq k}]_{k\in N}$ by each firm k (which are proportional to sales) remain unaltered by the change in productivities. Thus, in this sense, there are *no pull effects operating upstream* along the production structure.

(ii) The equilibrium prices of the goods enjoying higher productivities decrease proportionally no less than their respective productivities rise. This in turn triggers a downstream push over all sectors that use those goods directly or indirectly, with lower prices all along inducing more output being produced.

Item (i) is quite clear and requires no further elaboration. It is worth mentioning, however, that for the same reason why input demands are unaffected by the change, profits are unaffected as well. Productivity changes, therefore, have no effect on the inter-firm distribution of profits. The reason why this happens is highlighted in Item (ii). In the new equilibrium, price changes adapt to absorb all quantity effects (direct and indirect) resulting from increased productivities. This, to be sure, is crucially dependent on our Cobb-Douglas formulation, which implies that price changes exactly absorb all quantity effects (direct and indirect) resulting from increased productivities. Admittedly, such a consequence of the postulated functional form is quite extreme, but has the advantage of highlighting the contrast between forward and backward linkages in a clear-cut manner. A precise account of how these price-mediated effects propagate upstream is provided by the following result.

Proposition 9
Consider a set of changes in firm productivities such that the new levels, $(\tilde{A}_i)_{i\in N}$, are related to the former ones as follows:

$$\tilde{A}_i = \xi_i\, A_i \qquad (\xi_i > 0,\ i = 1, 2, ..., n). \tag{19}$$

Under a suitable normalization, the corresponding equilibrium prices $(\tilde{p}^*_i)_{i\in N}$ and $(p_i)^*_{i\in N}$, respectively prevailing before and after the change, satisfy:

$$-\left(\log \tilde{p}^*_i - \log p^*_i\right)_{i=1}^n = -(I-\alpha G')^{-1}\left(\log \xi_i\right)_{i=1}^n$$

or

$$\frac{\tilde{p}_i^*}{p_i^*} = \prod_{k=1}^{n} \xi_k^{-m_{ki}} \qquad (i = 1, 2, ..., n), \tag{20}$$

where G'stands for the transpose of the adjacency matrix of the production network and, for each firm i, $(m_{ki})_{k \in N}$ is the vector of its outcentralities.

The key implication of the previous result is that, as explained in (ii), the inter-firm linkages in this case are mediated through prices and quantities alone (i.e. not revenues nor profits) and operate downstream. That is, the first-order effect of the change in the productivity of any given firm k, as embodied by its corresponding ξ_k, propagates through the outcentralities of this firm. Concerning prices, this effects is as described in (9). On the other hand, in view of the constancy of equilibrium revenues, the corresponding effect on quantities is simply given by

$$\frac{\tilde{y}_i^*}{y_i^*} = \prod_{k=1}^{n} \xi_k^{m_{ki}} \qquad (i = 1, 2, ..., n). \tag{21}$$

6.2 Preference changes

Here we illustrate the nature of the backward linkages by focusing on a characteristic example: changes in preferences, as captured by variations in the preference weight vector γ – cf. (1). Recall that, as established by Proposition 2, equilibrium revenues and profits are given by

$$\boldsymbol{s}^* = (s_i^*)_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n = \frac{w(1-\alpha)}{\beta}(I - \alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

and

$$\boldsymbol{\pi}^* = (\pi_i^*)_{i=1}^n = (1 - \alpha - \beta)\boldsymbol{s}^*.$$

Thus both magnitudes, revenues and profits, are determined by linear functions of the preference weights $\boldsymbol{\gamma} = (\gamma_i)_{i \in N}$. This readily leads to the following conclusion, which we state formally for the sake of completeness.

Proposition 10
Suppose that the preference weights change from $\boldsymbol{\gamma}$ to $\tilde{\boldsymbol{\gamma}}$, with $\sum_{j \in N} \Delta\gamma_i \equiv \sum_{j \in N} \tilde{\gamma}_i - \gamma_i = 0$. Then, the induced change in equilibrium profits, $(\Delta\pi_i^*)_{i \in N}$, satisfies:

$$\Delta\pi_i^* \propto \sum_{j \in N} m_{ij}\, \Delta\gamma_j \qquad (i = 1, 2, ..., n). \tag{22}$$

Proof. Omitted □

We find, therefore, that in contrast with the distribution-invariant forward linkages induced by changes in firms' productivities, any variation in preferences not only has consequences on the allocation of resources but also distributional implications on firms' profits. Specifically, we have that the change in the profit of any given firm i is proportional to a weighted average of the changes (positive or negative) experienced by the relative importance that the consumer's utility attributes to each consumption good. Naturally, for each firm i, the weight that a particular consumption good j has in the computation of such an average effect is given by the incentrality m_{ij} that captures the direct and indirect impact of good j on the centrality of firm i. The overall effect on the whole set of firms is *purely redistributive*, in the sense that the total profits are unaffected by the changes in $\boldsymbol{\gamma}$. The following corollary – which is an immediate consequence of the fact that the matrix of incentralities $\mathscr{M} = (m_{ij})_{i,j=1}^{n} \equiv (I - \alpha G)^{-1}$ is column stochastic – states it explicitly.

Corollary 4
Under the conditions specified in Proposition 10, any change in the preference weights $\boldsymbol{\gamma}$ induces a change in equilibrium profits, $(\Delta\pi_i^*)_{i\in N}$, that satisfies $\Sigma_{i\in N}\Delta\pi_i^* = 0$.

7 Firm dynamics

To address in this section the issue of firm dynamics, we start by introducing in Subsection 7.1 a theoretical framework that extends the basic setup considered in Section 2 along two complementary directions. First, we incorporate *fixed labor costs.* This has the main effect of allowing for the possibility that, even at equilibrium, a firm may incur losses. If this occurs, we assume it means that the firm in question becomes bankrupt and thus may disappear. The second extension considers the case where the number of goods in the economy may vary, thus leading to the possibility the *size of the production network* may change.

In Subsection 7.2 we combine the two aforementioned extensions to provide a preliminary analysis of firm dynamics. More specifically, our discussion focuses on comparing the welfare implications of alternative mechanisms governing the disappearance of firms when, say, exogenous shocks may threaten their survival. The main insight we gather through a simple illustrative exercise can be summarized as follows: if one relies on strict market-based indicators (e.g. profits) instead of other network-based criteria, the consequences from the viewpoint of long-run welfare can be quite detrimental.

7.1 A generalized framework: fixed costs and varying network size

7.1.1 Fixed costs

Suppose that, as long as any given firm i remains active, it has to pay some fixed cost $f_i \geq 0$, which is interpreted as some given labor requirements that are independent on the scale of production, This induces two different changes in our basic framework. First, given the wage w, prices $(p_j)_{j=1}^n$ for the intermediate inputs, and the production plan $[y_i, l_i, (z_{ji})_{j=1}^n)]$, the profit of firm i is given by

$$\pi_i = p_i y_i - \sum_{j=1}^{n} p_j z_{ji} - w(l_i + f_i).$$

Second, while the demand functions for labor and intermediate inputs do not change by the introduction of fixed costs, the market-clearing condition for the labor market does change since, in this case, the amount of labor available for production decreases with the number of active firms. Specifically, the new market-clearing condition of the labor market becomes:

$$\sum l_i = 1 - F \tag{23}$$

where $F \equiv \sum_{i=1}^n f_i$ stands for the aggregate fixed cost across all firms.

The former modifications on the theoretical framework in turn induce some changes on the equilibrium analysis of the economy. First, as a counterpart of original Proposition 2, we have the following generalization:

Proposition 11
There exists a unique WE for which the vector of equilibrium revenues $\boldsymbol{s}^* = (s_i^*)_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n$ is given by

$$\boldsymbol{s}^* = \frac{w^*(1-F)(1-\alpha)}{\beta}(I - \alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

while the corresponding equilibrium profits are $\boldsymbol{\pi}^* = (\pi_i^*)_{i=1}^n = (1-\alpha-\beta)\boldsymbol{s}^* - w^*\boldsymbol{f}$, where $\boldsymbol{f} = (f_1, f_2, ..., f_n)'$.

Proof. See Appendix A. □

Arguably, in a market environment such as the one considered here, the profits earned by a firm is one of the benchmarks we would like to contemplate in measuring its performance. Then, quite naturally, the requirement that no negative profits (i.e. no losses) be incurred by a firm arises as well as a natural benchmark to use in assessing its "viability." Under the simplifying assumption that all goods are consumption goods, the implications of such a profit-based criterion are formally spelled out in the following straightforward but useful corollary.

Corollary 5
In the context considered in Proposition 11, the individual firm profits obtained at the WE satisfy:

$$\pi_i^* \geq 0 \quad \Longleftrightarrow \quad v_i \equiv (1-\alpha)\sum_{j=1}^{n} m_{ij}\gamma_j \geq \frac{\beta}{1-\alpha-\beta}\frac{f_i}{1-F} \tag{24}$$

where $(m_{ij})_{j=1}^n$ is the ith row of the matrix of incentralities $\mathscr{M}$ (cf. (7)).

Proof. Omitted □

The above Corollary indicates the that, in order for a firm i to be viable, its network (in-)centrality must be no lower than a certain given threshold, whose magnitude only depends on technological/cost parameters, the size of the economy, and the fixed costs. In Subsection 7.2, we compare this criterion with another one that relies as well on the global architecture of the network but changes focus from how profitable a firm is to how much it impinges on the profits of other firms – that is, from how central a firm is to how it affects the centrality of others. This, in the end, boils down to shifting attention from aggregate incentrality to *normalized outcentrality*, as suggested by Proposition 8. Indeed, this is the approach also pursued in the present context by the following result, which specifies the impact that the disappearance of any given firm has on the profits of all others.

Proposition 12

Consider an initial production network with adjacency matrix G and suppose that node $i \in N = \{1, 2, ..., n\}$ is removed from it. Denote by $\left(\Delta\pi_j^*\right)_{j\neq i}$ the change in equilibrium profits of the remaining firms. Then,

$$\Delta\pi_j^* = \tilde{\pi}_j(G) - \pi_j^*(G) - \tilde{\pi}_i(G)\,\frac{m_{ji}}{m_{ii}} \qquad (j =, 1, 2, ..., n;\ j \neq i), \tag{25}$$

where $\tilde{\boldsymbol{\pi}}(G) = (1-\alpha-\beta)\frac{(1-\alpha)w^*(1-F+f_i)}{\beta}(I-\alpha G)^{-1}\tilde{\boldsymbol{\gamma}}$

Proof. See Appendix A.

An interesting observation that follows from the previous result is that, in contrast with the case with no fixed costs, the removal of a firm i could now conceivably have a positive influence on the profits of other firms $j \neq i$ (and, conceivably, even lead to positive *overall* effects). This occurs because the elimination of a firm relaxes the labor feasibility constraint when there are fixed labor costs. The extent to which this is important for a particular firm j depends on the scales at which firms i and j operated (or, equivalently, their original profits), as well as on the the market pressure that the fixed costs f_i were previously imposing on the labor market.

7.1.2 Network size

Now we focus on the modeling issue of how to measure (consumer) welfare across situations where the network size changes due to variations in the number of firms active in the economy. To this end, we need a formulation of preferences that is of course consistent with our Cobb-Douglas specification (1) but is also able to accommodate a varying number of consumption goods. A useful route to do so is provided by the so-called

Ethier-Dixit-Stiglitz (EDS) preferences, which are represented by the utility function:

$$\tilde{U}(\mathbf{c}\,;\ \rho, N) = \sum_{i \in N} c_i^{\rho} \tag{26}$$

for some $\rho > 0$, where N is the whole universe of *all possible* goods (for simplicity, all assumed to be valued by the consumer). An equivalent representation of the same preferences is embodied by the monotone transformation of $\tilde{U}(\cdot)$ that induces a CES-format counterpart given by:

$$\hat{U}(\mathbf{c}\,;\ \rho, N) = \left[\sum_{i \in N} c_i^{\rho}\right]^{\frac{1}{\rho}}.$$

Clearly, given any ρ, the above formulation can be adapted to any nonempty subset of goods $M \subset N$ by simply changing the set of goods under consideration. Moreover, as it is well-known, the corresponding function $\hat{U}(\cdot\,;\ \rho, M)$ converges to the Cobb-Douglas (CD) utility function (1) with equal weights $\gamma_i = \frac{1}{m}$ in the limit of $\rho \to 0$. It is in this sense that, for *any* given set M of consumption goods, we may view our CD formulation as a representation of EDS preferences with an elasticity of substitution $\frac{1}{1-\rho}$ converging to 1.

The former considerations indicate that one can suitably compare the welfare of the consumer for different sets of goods being consumed, say M and M', by resorting to the functions $\tilde{U}(\cdot\,;\ \rho, M)$ and $\tilde{U}(\cdot\,;\ \rho, M')$, defined respectively as direct generalizations of (26). Then, again taking the limit on ρ for both of them, we arrive at the conclusion that

$$\lim_{\rho \to 0^+} \tilde{U}(\mathbf{c}(\rho)\,;\ \rho, M) = |M| \equiv m; \qquad \lim_{\rho \to 0^+} \tilde{U}(\mathbf{c}'(\rho)\,;\ \rho, M') = |M'| \equiv m' \tag{27}$$

where $\mathbf{c}(\rho) \in \mathbb{R}^m$ and $\mathbf{c}'(\rho) \in \mathbb{R}^{m'}$ stand for the pair of *equilibrium* consumption vectors associated to ρ and, respectively, M and M'. To understand (27), simply note that, under (26), individual optimality requires that the equilibrium consumption of all goods be positively bounded away from zero in ρ. Thus, in the end, we conclude that the welfare comparison of two equilibrium allocations with different sets of available consumption goods, M and M', comes down, in the Cobb-Douglas limit, to a comparison of their respective cardinalities, m and m'. This will be, therefore, the welfare criterion used in our ensuing discussion.

7.2 Firm dynamics: an illustrative example

We now rely on the formalization and results of the previous subsection to undertake a preliminary exploration of firm dynamics. We focus, specifically, on the effects of alternative firm-selection criteria and, to fix ideas, consider the simple example with 6 firms depicted in Figure 9.

To understand the role played by each node i in the network, a natural way to proceed

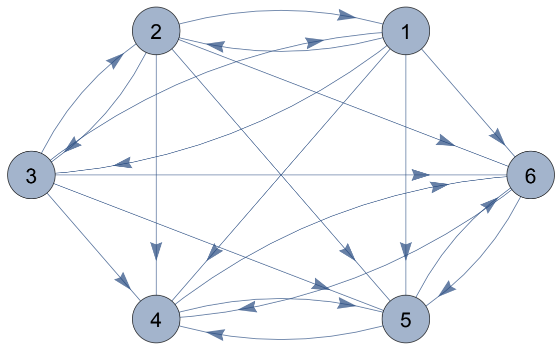

Figure 9: Initial production network. It consists of 6 nodes, with firms $\{1, 2, 3\}$ and $\{4, 5, 6\}$ defining two completely connected cliques.

it is to focus on its bilateral incentralities and outcentralities. On the one hand, we know from Proposition 11 that, if the preference vector $\boldsymbol{\gamma}$ is symmetric, the equilibrium profit of a firm i, is proportional to its aggregate incentrality $m_i^I \equiv \sum_{j\in N} m_{ij}$ (cf. (7)). This magnitude, therefore, can be seen as a market-based measure of firm i's own performance at equilibrium. In contrast, the aggregate outcentrality $m_i^O \equiv \sum_{j\in N} m_{ji}$ is a measure of how important is firm i for the aggregate (in-)centrality of all other firms. In fact, Proposition 12 shows that if the latter magnitude is normalized by m_{ii} (i's contribution to its own centrality), the induced adjusted outcentrality, $\hat{m}_i^O \equiv \frac{1}{m_{ii}} \sum_{j\in N} m_{ji}$, is an important component in the aggregate effect of i's removal on all other firms. This in turn suggests that, in order to anticipate the final impact of the removal of any given firm i, one should compare m_i^I and $\hat{m}_i^O$.

To apply these considerations to our example, in Figure 10 we depict the profiles $\left[m_i^I\right]_{i\in N}$ and $\left[\hat{m}_i^O\right]_{i\in N}$ across all 6 nodes in the production network being considered (for expositional clarity we normalize the wage by setting $w = 1$). The key observation to make is that there is an acute contrast between the incentralities and adjusted outcentralities across nodes, with a polar behavior when one compares the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

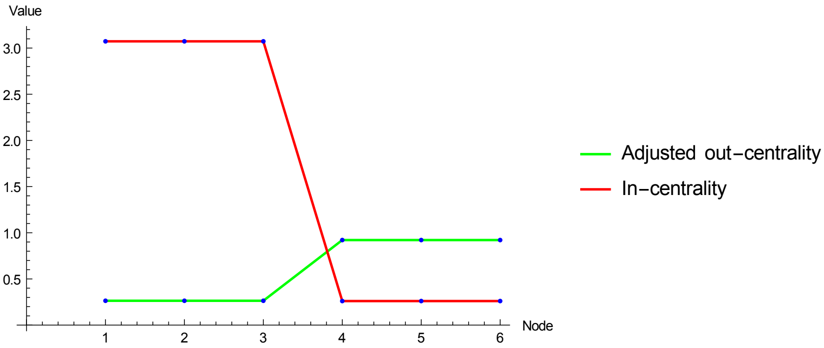

Figure 10: Graphical representation of the incentrality profile and the profile of adjusted outcentrality for the 6 firms of the economy, $\alpha = 0.9$

In view of the aforementioned contrast, the question arises as to what criterion (in- or out-centrality) should be used to evaluate the "systemic importance" of the different firms in our example. This question, as posed, is probably too vague to be really useful. Thus let us frame it in the following more concrete scenario, somewhat artificial but also transparent and clear-cut. Suppose that a common *temporary* shock hits two firms, say Firm 1 and 6, which in the absence of any outside support are sure to go bankrupt and disappear from the economy. Further assume that the funds that may be used to provide such a support are limited in that only one of the two firms can be helped. Which of the two should it be?

To provide all necessary information, suppose that vector of fixed costs $\boldsymbol{f}$ satisfies $f_1 = f_6 = 10^{-8}$ and $f_2 = f_3 = f_4 = f_5 = 0.0235$. Then, it can computed that the equilibrium profit is given by $\boldsymbol{\pi}^* = (0.278, 0.255, 0.255, 9 \times 10^{-5}, 9 \times 10^{-5}, 0.092)'$. Thus, as our theory prescribes, the three first firms with the highest centralities earn the highest profits, while the last three display the lowest profit levels.

Naturally, the answer to the question we have posed must depend on what is the objective function to be optimized. In line with the discussion undertaken in Subsection 7.1.2, let us suppose that the final objective is to maintain the largest possible number of active firms in the economy. Then, one possible criterion to determine what firm should be saved from bankruptcy is given by profitability, i.e. what we have labeled the "market-based measure of performance." If one relies on this criterion, the scarce funds should be used to help Firm 1 and hence it would be this firm that survives while Firm 6 would go under. However, once this has been carried out – interpreted as a once and for all intervention – the market forces continue being at work and can therefore have a further impact on the survival of the remaining firms. In fact, we find that Firms 4 and 5 now incur losses in the Walrasian Equilibrium (WE) of the smaller economy with the set of active firms given by $\{1, 2, 3, 4, 5\}$, and hence those two firms should subsequently exit the market. Thereafter, all the firms in the remaining set $\{1, 2, 3\}$ avoid losses at the corresponding WE and hence the situation may remain stationary in the absence of any further interference or a subsequent entry of new firms. This process is graphically illustrated in Figure 11.

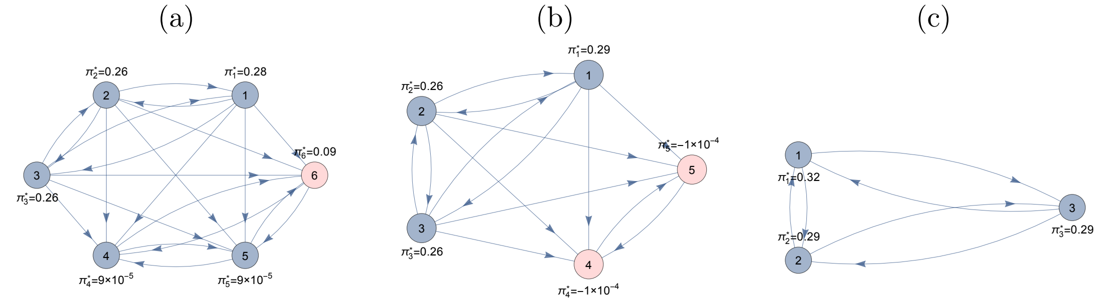

Figure 11: The firm dynamics triggered by the initial elimination of Firm 6 (a), followed by the consecutive step (b) where Firms 4 and 5 incur losses (i.e. obtain negative profits) and are eliminated. In the resulting network (c) all firms make positive net profit

Alternatively, the criterion for selecting the firm to be supported may single out the firm that, if it were to disappear, would have the highest effect on the profit performance of other firms. Then, quantifying such an effect by the adjusted (aggregate) outcentrality, the support to buffer the shock should be enjoyed by Firm 6 rather than Firm 1. Of course, once this choice is implemented, the immediate consequence is that Firm 1 exits the system. The adjustment process ends there. For, in the resulting situation, where the other five firms are still active, all of these enjoy positive profits at the induced Walrasian Equilibrium. This process is graphically illustrated in Figure 12. The system, therefore, reaches a stationary state that dominates (in the sense explained in Subsection 7.1.2) the alternative state that would have been reached by the alternative selection criterion based on firm profitability. This contrast illustrates in a stark manner what has been the leading theme of this paper, namely, that a networked production economy cannot be properly understood, nor suitable policies implemented, unless one adopts a genuine network viewpoint.

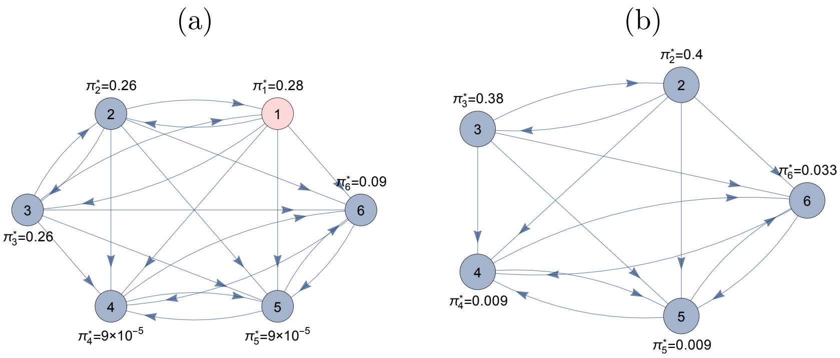

Figure 12: The elimination of Firm 1 (a) does not cause thereafter the exit of any other firm from the network.

8 Summary and conclusions

In this paper, we have studied a general equilibrium model of an economy that focuses primarily on the inter-firm network of input-output relationships that underlie its production structure. Is this modeling approach useful to understand economic performance? A first, and particularly sharp, illustration of its usefulness derives from the connection we have identified between market equilibrium outcomes and an intuitive measure of centrality that combines topological and preference information. Specifically, we have shown that, at equilibrium, the profits of the different firms are directly proportional to their corresponding centralities.

The paper has then turned to extending this approach into the investigation of a number of important comparative-statics questions:
(a) the effects of distortions (local or global);
(b) the implications of alternative network changes (on nodes or links);
(c) the impact of non-network "fundamentals" (preferences or productivities).
While the specifics of each of these cases is quite different, the conclusions obtained exhibit a common format. The overall effect always obtains from a *composition* of a first-order impact on the firms directly affected and a diffusion of this impact throughout the economy as determined by the matrix of cross-firm inter-centralities. A different, but complementary, perspective is gained if we decompose matters into the effects that spread upstream along the network (backward linkages) and those that do it downstream (forward linkages). This alternative perspective has been found useful, for example, when comparing the changes that affect productivities and those that impinge on preferences.

Finally, we have turned to studying the dynamic consequences of incorporating network considerations into the evaluation of simple policy dilemmas – in particular, we have compared the implications of alternative criteria for firm support in the face of exogenous shocks that threaten firm survival. Relying on a simple example, we have illustrated the point that tailoring those supporting decisions only to current market-based information (thus ignoring forward-looking network-based considerations) may be quite misleading and lead to disappointing results. For example, the number of firms *eventually* affected by the shock may be much larger than would have been optimal.

Clearly, the model proposed in this paper is quite stylized and hence leaves ample room for extensions. Interesting possibilities would be to allow for less stringent assumptions on the competitive behavior of agents, or generalize the Cobb-Douglas formulation of preferences and technologies posited here. In fact, a theoretical framework with the aforementioned features has been formulated in the paper by Baqaee (2015) already mentioned in the Introduction. Specifically, that paper considers a context where the set of firms are involved in monopolistic competition, and both preferences and technologies are of the CES type. (On the other hand, it also allows for adjustments in the extensive margin, through the endogenous entry and exit of firms.) Baqaee's concern, however, is similar to that of Acemoglu et al. (2012) and thus focuses his analysis on how microeconomic shocks aggregate at the level of the whole economy. An extension of our analysis to the richer scenario studied in Baqaee (2015) would be very interesting indeed, and is one of our objectives for future research.

More generally, a primary objective for follow-up research should be to apply our network-based approach to inform and shape a variety of measures of economic policy. But to this end, of course, a thorough empirical testing of the model must be conducted first. This, in turn, requires the use of data that are rich and granular enough at the microeconomic level to account for the intricate pattern of inter-firm interactions that characterize a modern economy. Fortunately, these data are now becoming available, as well as the tools (theoretical, algorithmic, and computational) needed to analyze them. As a case in point, we refer to the work in progress that we are currently undertaking with panel data from the Spanish Tax Agency. The data include essentially all bilateral transactions among Spanish firms (and many other economically relevant entities, public or private) during the decade 2004-2013. The analysis conducted so far, still preliminary, has delivered promising results. In particular, we have found a very strong and significant relationship between the profitability of firms and their respective centrality, which is one of the basic predictions of our model (cf. Proposition 2). Subsequent research will test some of its other predictions.

Appendix A: Proofs of the main results

We start this Appendix by providing a formal definition of Walrasian Equilibrium. Then we proceed by providing detailed proofs of the results in the papeper.

Definition 2 (**Walrasian Equilibrium**)
A WE is an array $[(\boldsymbol{p}^*, w^*), (\boldsymbol{c}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*)]$ *that satisfies the following conditions:*

1. *The representative consumer chooses* $\boldsymbol{c}^*$ *by solving:*

$$\begin{aligned} &\underset{\boldsymbol{c}}{max} \quad \prod_{i\ inM} c_i^{\gamma} \\ &s.t. \quad \sum_{i\in M} p_i c_i = w + \sum_{k=1}^{n} \pi_k. \end{aligned}$$

2. *Firms choose* $\boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*$ *by solving:*

$$\begin{aligned} &\underset{(z_{ji})_j, l_i}{max} \quad p_i y_i - \sum_{j\in r_i^+} p_j z_{ji} g_{ji} - w l_i \\ &s.t. \qquad z_i = A_i l_i^{\beta} \left(\prod_{j\in N_i^+} z_{ji}^{g_{ji}} \right)^{\alpha}. \end{aligned}$$

3. *Markets for labour and intermediate goods clear:*

$$y_i = \sum_j z_{ij} + c_i \ \ \forall (i \in N)$$

$$\sum_i l_i = 1.$$

Proof of Proposition 1: The production function (2) implies that in order to be active a firm has to have at least one active in-neighbour (see footnote 4). Consider an upstream walk from i choosing only active firms. As number of firms is finite and each active firm must have at least one active in-neighbor, this means that there must exist an upstream walk from i that visits the same active node more than once. This proves part (a).

To prove part (b), note that, in the end, the demand for produced goods in the model is generated by the consumer. Thus if there is no path from a firm to the consumer then the total demand for that good will be zero, and therefore that firm will choose not to produce.

□

Proof of Corollary 1. Fallows directly from Proposition 1 □

Proof of Proposition 2. To simplify notation, throughout the proofs in the Appendices we shall dispense with the asterisk to identify equilibrium magnitudes. The context should make clear when we refer to equilibrium values.

The first order conditions for the maximization problem of firm i with respect to z_{ji} are given by

$$A_i p_i \alpha g_{ji} l_i^{\beta} z_{ji}^{\alpha g_{ji}-1} \prod_{k,k\neq j} z_{ki}^{\alpha g_{ki}} = p_j \Rightarrow z_{ji} = \frac{p_i \alpha g_{ji}}{p_j} y_i \qquad (j = 1, 2, ..., n), \tag{28}$$

and with respect to labor l_i:

$$A_i p_i \beta l_i^{\beta-1} \prod_k z_{ki}^{\alpha g_{ki}} = w \Rightarrow l_i = \frac{p_i \beta}{w} y_i. \tag{29}$$

Using (28) and (29) we may write the profit of firm i as follows:

$$\pi_i = p_i y_i - \sum_j \left(p_j \frac{p_i \alpha g_{ji}}{p_j} y_i \right) - w \frac{p_i \beta}{w} y_i = (1 - \alpha - \beta) p_i y_i. \tag{30}$$

Next, from the clearing condition for the labor market and (29), we find that $w = \beta \sum_i p_i y_i$, which together with (30) and the FOC for the consumer gives rise to:

$$c_i = \frac{\sum_{j=1}^{n} (1 - \alpha - \beta) p_j y_j + w}{p_i} \gamma_i = \frac{(1 - \alpha) w}{\beta p_i} \gamma_i. \tag{31}$$

Then, if (28) and (31) are inserted into the market clearing condition for good i, this condition can be written in the following manner:

$$s_i = \frac{(1 - \alpha) w}{\beta m} \gamma_i + \alpha \sum_j g_{ij} s_j \tag{32}$$

where $s_i \equiv p_i y_i$ is the revenue of firm i. In vector notation, we may write compactly the system of market clearing conditions for all intermediate goods as follows:

$$\boldsymbol{s} = \frac{(1 - \alpha) w}{\beta} \boldsymbol{\gamma} + \alpha G \boldsymbol{s} \tag{33}$$

which, as the matrix G is column-stochastic and $\alpha < 1$, has the unique solution:

$$\boldsymbol{s} = \frac{(1 - \alpha) w}{\beta} (I - \alpha G)^{-1} \boldsymbol{\gamma}. \tag{34}$$

Finally, also note that (30) directly implies that equilibrium profits satisfy $\boldsymbol{\pi} = (1-\alpha-\beta)\boldsymbol{s}$. This completes the proof. $\square$

Proof of Corollary 2. Suppose that all goods are consumed and define:

$$\boldsymbol{v} \equiv \frac{\beta}{w} \boldsymbol{s} = (1 - \alpha)(I - \alpha G)^{-1} \boldsymbol{\gamma}. \tag{35}$$

Expanding (35) we can write $\boldsymbol{v} = (1 - \alpha) \left(\sum_{k=0}^{\infty} \alpha^k G^k \right) \boldsymbol{\gamma}$. This sum converges since $0 \leq \alpha < 1$ and G is column-stochastic. The latter also implies that $\mathbf{1}' G \boldsymbol{\gamma} = 1$, as $\sum_{i=1}^{n} \gamma_i = 1$. Given that the product of two stochastic matrices is stochastic, the same conclusion applies if instead of G we consider G^k for any $k \in N$, i.e. we also have

$\mathbf{1}'G^k\boldsymbol{\gamma} = 1$. This then implies:

$$\mathbf{1}'\boldsymbol{v} = (1-\alpha)\sum_{k=0}^{\infty}\alpha^k = \frac{1-\alpha}{1-\alpha} = 1. \tag{36}$$

From the previous expression, together with (29) and the labor-market clearing condition, it directly follows that $v_j = \frac{s_j}{\sum_{i=1}^n s_i}$. Thus, recalling that $\pi_i = (1-\alpha-\beta)s_i$, the proof is complete. □

Proof of Proposition 3. To calculate the utility of the consumer we proceed as follows. Inserting (28) and (29) into the production function (2), we have:

$$y_i = A_i\left(\frac{p_i\beta}{w}y_i\right)^{\beta}\prod_j\left(\frac{p_i\alpha g_{ji}}{p_j}y_i\right)^{\alpha g_{ji}} \Rightarrow$$

$$\prod_{j\in N_i^+}\left(p_j^{\alpha g_{ji}}\right)y_i = A_i\left(\frac{p_i\beta}{w}y_i\right)^{\beta}\prod_j\left(p_i\alpha g_{ji}y_i\right)^{\alpha g_{ji}} \Rightarrow$$

$$\frac{\prod_{j\in N_i^+}p_j^{\alpha g_{ji}}}{p_i}s_i = A_i\left(\frac{\beta}{w}s_i\right)^{\beta}\prod_j\left(\alpha g_{ji}s_i\right)^{\alpha g_{ji}}.$$

Then, taking natural logs we get that:

$$\alpha\sum_{j\in N_i}g_{ji}\log p_j - \log p_i = \tag{37}$$

$$\log A_i + \beta\log\beta - (1-\alpha-\beta)\log s_i - \beta\log w + \alpha\log\alpha + \alpha\sum_j g_{ji}\log g_{ji} \quad (i = 1, 2, ..., n)$$

Let us define $u \equiv \alpha\log\alpha + \beta\log\beta + (1-\alpha-\beta)\log\beta$ and write the system (37) in vector notation as follows:

$$\alpha\left(I-\alpha G'\right)\log\boldsymbol{p} = (1-\alpha)\mathbf{1}\log w - (\alpha+\nu)\log\boldsymbol{n} + (1-\alpha-\beta)\log\boldsymbol{v} - \alpha H'\mathbf{1} - u\mathbf{1} \tag{38}$$

where we have used (3) and (35) to expand A_i and s_i respectively. Then, premultiplying (38) with $\boldsymbol{v'} = (1-\alpha)\boldsymbol{\gamma}'(I-\alpha G')^{-1}$ and normalizing $\sum_i \log p_i = 0$ we get (39)

$$(1-\alpha)\log w = (\alpha+\nu)\sum_i v_i\log n_i - (1-\alpha-\beta)\sum_i v_i\log v_i + \alpha\sum_i\sum_j v_i g_{ji}\log g_{ji} + u, \tag{39}$$

Recall now that $c_i = \frac{(1-\alpha)w}{\beta p_i}\gamma_i$. So we can write the utility function of the representative consumer as follows:

$$U(\boldsymbol{c}) = \prod_{i=1}^{n}\left(\frac{(1-\alpha)w}{\beta p_i}\gamma_i\right)^{\gamma_i} = \prod_{i=1}^{n}\gamma_i^{\gamma_i}\frac{1-\alpha}{\beta}w.$$

Substituting w from (39) we can express logarithm of the utility function with the following equation.

$$\log U(\boldsymbol{c}) = \sum_{i=1}^{n} \gamma_i \log \gamma_i + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}$$
$$\left((\nu+\alpha) \sum_i v_i \log n_i - (1-\alpha-\beta) \sum_{i \in N} v_i \log v_i + \alpha \sum_{i \in N} \sum_{j \in N_i^+} v_i g_{ji} \log g_{ji} \right)$$

$\square$

Proof of Corollary 3. In the symmetric case $g_{ji} = g_{ki} = 1/n_i \ \forall (j,k \in N_i^+)$ and $\gamma_i = \frac{1}{n} \ \forall i$. Hence, we have that $\sum_i \sum_j v_i g_{ji} \log g_{ji} = \sum_i v_i \log \frac{1}{n_i} = -\sum_i v_i \log n_i$, and $\sum_{i=1}^{n} \gamma_i \log \gamma_i = -\log n$, which together with Proposition 3 gives:

$$\log U(\boldsymbol{c}) = -logn + \log{(1-\alpha)}\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha} \left(\nu \sum_{i=1}^{n} v_i \log n_i - (1-\alpha-\beta) \sum_{i=1}^{n} v_i \log v_i \right).$$

$\square$

Now, for the sake of completeness, we extend the definition of WE to a context with distortions, as studied in Section 4.

Definition 3 (WE with distortions)
For an exogeneus vector of price distortions $\boldsymbol{\tau} = (\tau_i)_{i=1}^{n}$. A WE is an array $[(\boldsymbol{p}^*, w^*), (\boldsymbol{c}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*)]$ such that:

1. *The representative consumer chooses $\boldsymbol{c}^*$ to solve:*

$$\begin{aligned} &\underset{\boldsymbol{c}}{max} \quad \prod_{i \ inM} c_i^{\gamma} \\ &s.t. \quad \sum_{i \in M} p_i c_i = w + \sum_{k=1}^{n} \pi_k. \end{aligned}$$

2. *Firms choose $\boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*$ to solve:*

$$\begin{aligned} &\underset{(z_{ji})_j, l_i}{max} \quad (1-\tau_i) p_i y_i - \sum_{j \in r_i^+} p_j z_{ji} g_{ji} - w l_i \\ &s.t. \qquad z_i = A_i l_i^{\beta} \left(\prod_{j \in N_i^+} z_{ji}^{g_{ji}} \right)^{\alpha}. \end{aligned}$$

3. *Markets for labor and intermediate goods clear:*

$$y_i^* = \sum_j z_{ij}^* + c_i^* \qquad (i = 1, 2, ..., n)$$
$$\sum_i l_i = 1.$$

Next, we state and prove two separate Lemmas that establish corresponding claims made in Section 4.

Lemma 1
There exists a unique WE with uniform price distortions as long as $(1-\tau)\alpha < 1$. *The vector of equilibrium revenues* $\boldsymbol{s}^*(\tau) = (s_i^*(\tau))_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n$ *is given by*

$$\boldsymbol{s}^*(\tau) = \frac{w^*(1-\alpha)}{\beta}(I - \alpha(1-\tau)G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

while the corresponding equilibrium profits $\boldsymbol{\pi}^*(\tau) = (\pi_i^*)_{i=1}^n = (1-\tau)(1-\alpha-\beta)\boldsymbol{s}^*(\tau)$.

Proof of Lemma 1: The proof is analogous to that of Proposition 2. In the case of a uniform price distortion τ, the former equations (28) and (29) become:

$$z_{ji} = (1-\tau)\frac{p_i \alpha g_{ji}}{p_j} y_i$$

$$l_i = (1-\tau)\frac{p_i \beta}{w} y_i$$

for each $i, j = 1, 2, ..., n$. Therefore, proceeding as before, it readily follows that the vector of equilibrium revenues satisfies

$$\boldsymbol{s}(\tau) = \frac{(1-\alpha)w}{\beta}\left(I - \alpha(1-\tau)G\right)^{-1}\boldsymbol{\gamma}$$

and, correspondingly, the equilibrium profit of each firm i is given by:

$$\pi_i(\tau) = (1-\tau)p_i y_i(\tau) - \sum_{j \in N_i^+} p_j (1-\tau)\frac{p_i \alpha g_{ji}}{p_j} y_i(\tau) - w(1-\tau)\frac{p_i \beta}{w} y_i(\tau) = (1-\tau)(1-\alpha-\beta)s_i(\tau)$$

or in vectorial form:

$$\boldsymbol{\pi}(\tau) = (1-\tau)(1-\alpha-\beta)\boldsymbol{s}(\tau) \tag{40}$$

Finally, the uniqueness of equilibrium follows simply from the fact that G is column-stochastic and $(1-\tau)\alpha < 1$. This completes the proof. $\square$

Lemma 2
In the WE with uniform price distortion τ *the following holds:*

$$\frac{d\boldsymbol{\pi}^*}{d\tau}(\tau) < 0$$

Proof of Lemma 2: Recall that equilibrium revenues satisfy $\boldsymbol{s}(\tau) = \frac{1-\alpha}{\beta}\boldsymbol{\gamma} + \alpha(1-\tau)G\boldsymbol{s}(\tau)$. Thus, differentiating both sides with respect to τ, we get:

$$\frac{d\boldsymbol{s}}{d\tau}(\tau) = -\alpha G\boldsymbol{s}(\tau) + \alpha(1-\tau)G\frac{d\boldsymbol{s}}{d\tau}(\tau) \Rightarrow \frac{d\boldsymbol{s}}{d\tau}(\tau) - \alpha(I - \alpha(1-\tau)G)^{-1}G\boldsymbol{s}(\tau) \tag{41}$$

From (40) and (41), it easily follows that:

$$\begin{aligned}\frac{d\boldsymbol{\pi}}{d\tau}(\tau) &= -(1-\alpha-\beta)\boldsymbol{s}(\tau) - \alpha(1-\tau)(1-\alpha-\beta)(I-\alpha(1-\tau)G)^{-1}G\boldsymbol{s}(\tau) \\ &= -(1-\alpha-\beta)\boldsymbol{s}(\tau) - \alpha(I-\alpha(1-\tau)G)^{-1}G\boldsymbol{\pi}(\tau) < 0.\end{aligned}$$

□

Proof of Proposition 4: Recall, from (40), that

$$\boldsymbol{\pi}(\tau) = (1-\tau)(1-\alpha-\beta)\boldsymbol{s}(\tau) = (1-\tau)(1-\alpha-\beta)\frac{1-\alpha}{\beta}w\left(I-\alpha(1-\tau)G\right)^{-1}\boldsymbol{\gamma} \quad (42)$$

Thus, setting $w = \frac{1-(1-\tau)\alpha}{(1-\tau)(1-\alpha)(1-\alpha-\beta)}\beta$, the induced profits are normalized such that the sum of profits across nodes is equal to 1. We can now write the normalized equilibrium profit vector as (43).

$$\hat{\boldsymbol{\pi}}(\tau) = (1-\alpha(1-\tau))\boldsymbol{\gamma} + \alpha(1-\tau)G\hat{\boldsymbol{\pi}}(\tau) \quad (43)$$

Taking the derivative of (43) with respect to τ we obtain:

$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \alpha\boldsymbol{\gamma} - \alpha G\hat{\boldsymbol{\pi}}(\tau) + \alpha(1-\tau)G\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau),$$

and then an explicit expression for $\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau)$

$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \alpha(I-\alpha(1-\tau)G)^{-1}(\boldsymbol{\gamma} - G\hat{\boldsymbol{\pi}}(\tau)) \quad (44)$$

Equation (44) can be developed as follows:

$$\begin{aligned}
\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) &= \alpha(I-\alpha(1-\tau)G)^{-1}\boldsymbol{\gamma} - \frac{1}{1-\tau}\left((I-\alpha(1-\tau)G)^{-1}\hat{\boldsymbol{\pi}}(\tau) - (1-\alpha(1-\tau)(I-\alpha(1-\tau)G)^{-1}\boldsymbol{\gamma}\right) \\
&= \frac{\alpha}{1-\alpha(1-\tau)}\hat{\boldsymbol{\pi}}(\tau) - \frac{1}{1-\tau}\left((I-\alpha(1-\tau)G)^{-1}\hat{\boldsymbol{\pi}}(\tau) - \hat{\boldsymbol{\pi}}(\tau)\right) \\
&= \frac{1}{1-\tau}\left(\frac{1}{1-\alpha(1-\tau)}\hat{\boldsymbol{\pi}}(\tau) - (I-\alpha(1-\tau)G)^{-1}\hat{\boldsymbol{\pi}}(\tau)\right) \\
&= \frac{1}{1-\tau}(I-\alpha(1-\tau)G)^{-1}\left(\boldsymbol{\gamma} - \hat{\boldsymbol{\pi}}(\tau)\right)
\end{aligned}$$

For a particular firm i we have:

$$\frac{d\hat{\pi}_i}{d\tau}(\tau) = \frac{1}{1-\tau}\sum_{j=1}^{n}\tilde{m}_{ij}(\tau)\left(\gamma_j - \hat{\pi}_j(\tau)\right) \quad (45)$$

If all goods are symmetric consumption goods, (45) becomes:

$$\frac{d\hat{\pi}_i}{d\tau}(\tau) = \frac{1}{1-\tau}\sum_{j=1}^{n}\tilde{m}_{ij}(\tau)\left(\frac{1}{n} - \hat{\pi}_j(\tau)\right)$$

so that, when evaluated at $\tau = 0$, we find:

$$\left.\frac{d\hat{\pi}_i}{d\tau}\right|_{\tau=0} = \sum_{j=1}^{n} m_{ij}\left(\gamma_j - \hat{\pi}_j(0)\right)$$

as desired, thus completing the proof. □

Lemma 3

Let G be a n-dimensional matrix, and $\alpha \in \mathbb{R}$ such that there exist $(I-\alpha G)^{-1}$. Then: $(I-\alpha G)^{-1}G = \frac{1}{\alpha}\left((I-\alpha G)^{-1} - I\right)$

Proof of Lemma 3: $(I-\alpha G)^{-1}G = \left(\sum_{k=0}^{\infty}\alpha^k G^k\right)G = \sum_{k=0}^{\infty}\alpha^k G^{k+1} = \frac{1}{\alpha}\sum_{k=0}^{\infty}\alpha^{k+1}G^{k+1} = \frac{1}{\alpha}\sum_{k=1}^{\infty}\alpha^k G^k = \frac{1}{\alpha}\left(\sum_{k=0}^{\infty}\alpha^k G^k - \alpha^0 G^0\right) = \frac{1}{\alpha}\left((I-\alpha G)^{-1} - I\right)$ $\square$

Proof of Proposition 5: Proceeding in a way analogous to that pursued for a uniform price distortion, we arrive to the following expression for the revenue vector at equilibrium.

$$\boldsymbol{s}(\tau_k) = \frac{1-\alpha}{\beta}w\left(I - (\alpha G - \alpha\tau_k \boldsymbol{g}_k \boldsymbol{e}_k'))\right)^{-1}\boldsymbol{\gamma}$$

Using Sherman-Morrison formula we get (46).

$$\boldsymbol{s}(\tau_k) = \frac{1-\alpha}{\beta}w\left((I-\alpha G)^{-1}\boldsymbol{\gamma} - \frac{(I-\alpha G)^{-1}\alpha\tau_k\boldsymbol{g}_k\boldsymbol{e}_k'(I-\alpha G)^{-1}}{1+\alpha\tau_k\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{g}_k}\boldsymbol{\gamma}\right) \tag{46}$$

Note first that $\frac{1-\alpha}{\beta}w(I-\alpha G)^{-1}\boldsymbol{\gamma} = \boldsymbol{s}(0)$ and $\frac{(1-\alpha)w}{\beta}(I-\alpha G)^{-1}\alpha\tau_k\boldsymbol{g}_k\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{\gamma} = -\alpha\tau_k s_k(0)(I-\alpha G)^{-1}\boldsymbol{g}_k$. Using this to simplify (46) we get:

$$\boldsymbol{s}(\tau_k) = \boldsymbol{s}(0) - s_k(0)\frac{\alpha\tau_k(I-\alpha G)^{-1}\boldsymbol{g}_k}{1+\alpha\tau_k\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{g}_k} = \boldsymbol{s}(0) - s_k(0)\frac{\tau_k((I-\alpha G)^{-1}-I)\boldsymbol{e}_k}{1-\tau_k+\tau_k m_{kk}} \tag{47}$$

where the last equality follows directly from Lemma 3 in Appendix D. For a specific firm $i \neq k$ we have $\pi_i(\tau_k) = (1-\alpha-\beta)\left(s_i(0) - s_k(0)\frac{\tau_k m_{ki}}{1-\tau+\tau m_{kk}}\right)$, and hence

$$\frac{d\pi_i}{d\tau_k}(\tau_k) = -(1-\alpha-\beta)s_k(0)\frac{m_{ki}}{(1-\tau_k(1-m_{kk}))^2}$$

On the other hand, for Firm k, $\pi_k(\tau_k) = (1-\tau_k)(1-\alpha-\beta)\left(s_k(0) - s_k(0)\frac{\tau_k(m_{kk}-1)}{1-\tau+\tau m_{kk}}\right) = (1-\tau_k)(1-\alpha-\beta)s_k(0)\frac{1-\tau_k}{1-\tau_k+\tau_k m_{kk}}$, and therefore

$$\frac{d\pi_k}{d\tau_k}(\tau_k) = -(1-\alpha-\beta)s_k(0)\frac{m_{kk}}{(1-\tau_k(1-m_{kk}))^2}$$

It is then obvious that

$$\left.\frac{\mathrm{d}\pi_i(\tau_k)}{\mathrm{d}\tau_k}\right|_{\tau_k=0} = -\pi_k(0)m_{ki} \;\; (i = 1, 2, ..., n),$$

which completes the proof. $\square$

Proof of Proposition 6: For a fixed $n = |N|$, let us find the network topology that maximizes expression (6). In order to do that it is useful to define $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $\Phi(\boldsymbol{v}, \boldsymbol{n}) = \sum_{i=1}^n v_i \log n_i$. Let us also define function $\Psi : \mathbb{R} \to \mathbb{R}$ as:

$$\begin{aligned} \Psi(\bar{y}) = & \max_{(x_i)_{i=1}^n, (y_i)_{i=1}^n} \sum_{i=1}^n x_i log y_i \\ \text{s.t.} & \sum_{i=1}^n y_i = \bar{y} \\ & \sum_{i=1}^n x_i = 1 \\ & y_i \leq n-1 \wedge y_i \geq 1 \wedge x_i > 0, \;\; i = 1, ..., n \end{aligned}$$

One can easily show that function Ψ is strictly increasing in $\bar{y}$. For any graph with $\hat{n}$ nodes and $\hat{L}$ links ($\sum_i \hat{n}_i = \hat{L}$) it is clear that $\Phi(\hat{v}_i, \hat{n}_i) \le \Psi(\hat{L}) < \Psi(n(n-1))$. Note that for complete network $\forall(i, j \in N)(n_i = n_j) \Rightarrow \Phi(v_i^v, n_i^c) = \Psi(n(n-1))$ (since Ψ is strictly increasing). Thus, the complete network is the unique maximizer of Φ.

The value of expression $-\sum_i^n v_i \log v_i$ which is in fact the entropy measure of simplex vector $\boldsymbol{v}$ will be maximized when $v_i = v_j \ \forall(i, j \in N)$, which will incidentally be true in the case of the complete network and the ring network. It fallows now that utility will be maximized at the complete network when $\nu > 0$ and that the complete network will be the unique network that maximizes the consumer's utility. It is easy to see that when $\nu < 0$ a network in which all nodes have the same centrality v and the same in-degree (equal to 1) will maximize (6), as for such a network the entropy of centrality vector will be maximized and $\eta \sum_{i=1}^n v_i \log n_i = 0$. This will, for instance, be the case when the production network is the ring network. When $\nu = 0$, then any network such that centrality of each node is equal will maximize social welfare. □

Proof of Proposition 7: Using Sherman-Morrison formula we write:

$$\begin{aligned}\tilde{\boldsymbol{s}} &= \frac{1-\alpha}{\beta} w (I - \alpha\tilde{G})^{-1}\boldsymbol{\gamma} = \frac{1-\alpha}{\beta} w (I - \alpha G - \alpha \boldsymbol{q}\boldsymbol{e}_i')^{-1}\boldsymbol{\gamma} \\ &= \frac{1-\alpha}{\beta} w \left((I - \alpha G)^{-1}\boldsymbol{\gamma} + \frac{(I-\alpha G)^{-1}\alpha \boldsymbol{q}\boldsymbol{e}_i'(I-\alpha G)^{-1}}{1 - \alpha \boldsymbol{e}_i'(I-\alpha G)^{-1}\boldsymbol{q}} \gamma \right)\end{aligned} \tag{48}$$

Proceeding analogue to the analysis in Proposition 5, we get that the effect of adding a link (i, j) on the centrality of firm k is:

$$\tilde{s}_k - s_k = \alpha s_j \frac{q_i m_{ki} + \sum_{l \in N_j^+} q_l m_{kl}}{1 + q_i m_{ji} + \alpha \sum_{l \in N_j^+} q_l m_{jl}}.$$

where $\tilde{s}_k$ is the revenue of k after adding link (i, j). Since, as we have shown before, $\boldsymbol{\pi} = (1 - \alpha - \beta)\boldsymbol{s}$ this completes the proof. □

The fallowing important lemma is proven by (Ballester et al., 2006)

Lemma 4 (BZC)
$m_{ji}(G)m_{ik}(G) = m_{ii}(G)(m_{jk}(G) - m_{jk}(G_{-i}))$

Proof of Lemma 4:

$$m_{ii}(G)(m_{jk}(G) - m_{jk}(G_{-i})) = \sum_{p=1}^{\infty} \alpha^p \sum_{\substack{r+s=p \\ r\ge 0, s\ge 1}} g_{ji}^{[r]}(g_{jk}^{[s]} - g_{j(-i)k}^{[s]}) = \sum_{p=1}^{\infty} \alpha^p \sum_{\substack{r+s=p \\ r\ge 0, s\ge 2}} g_{ii}^{[r]} g_{j(i)k}^{[s]}$$

$$= \sum_{p=1}^{\infty} \alpha^p \sum_{\substack{r'+s'=p \\ r'\ge 1, s'\ge 1}} g_{ji}^{[r']} g_{ik}^{[s']} = m_{ji}(G)m_{ik}(G)$$

where G_{-i} is the resulting network after elimination of node i from network G. □

Proof of Proposition 8: We can write the profit of firm j after elimination of node i from the network as

$$\begin{aligned}\pi_j(\tilde{G}) &= (1-\alpha-\beta)\frac{1-\alpha}{\beta}w\sum_{\substack{k=1\\k\neq i}}^{n} m_{jk}(\tilde{G})\gamma_k \\ &= (1-\alpha-\beta)\frac{1-\alpha}{m_{ii}(G)\beta}w\sum_{k=1}^{n}\left(m_{ii}(G)m_{jk}(G)-m_{ji}(G)m_{ik}(G)\right)\tilde{\gamma}_k.\end{aligned}$$

where the second line follows directly from Lemma 4. Furthermore:

$$\begin{aligned}&\pi_j(\tilde{G})-\pi_j(G) = \\ &(1-\alpha-\beta)\frac{1-\alpha}{\beta}w\left(\frac{1}{m_{ii}(G)}\sum_{k=1}^{n}(m_{ii}(G)m_{jk}(G)-m_{ji}(G)m_{ik}(G))\tilde{\gamma}_k-\sum_{k=1}^{n}m_{jk}(G)\gamma_k\right) = \\ &(1-\alpha-\beta)\frac{1-\alpha}{\beta}w\left(\sum_{k=1}^{n}m_{jk}(G)(\tilde{\gamma}_k-\gamma_k)-\frac{m_{ji}(G)}{m_{ii}(G)}\sum_{k=1}^{n}m_{ik}(G)\tilde{\gamma}_k\right) = \\ &\tilde{\pi}_j(G)-\pi_j(G)-\tilde{\pi}_i(G)\frac{m_{ji}(G)}{m_{ii}(G)}\end{aligned}$$

□

Proof of Proposition 9: From (37) by normalizing: $\alpha\log\alpha+\beta\log\beta-\beta\log w=0$ and by writing $b_i\equiv-(1-\alpha-\beta)\log s_i+\alpha\sum_{j=1}^{n}g_{ji}\log g_{ji}$ we get:

$$\log\boldsymbol{p}=(I-\alpha G')^{-1}(\log\boldsymbol{A}+\boldsymbol{b})$$

Consider the technology shock that changes A_i to $\tilde{A}_i\equiv\xi_iA_i$ (we shall use $\tilde{}$ to denote the variables after the technology shock). It is clear that this shock will not affect the relative demand of goods in the equilibrium which implies that (33) will not be affected by the technology shock. So, $s_i=\tilde{s}_i\Rightarrow b_i=\tilde{b}_i$. From here it directly follows:

$$\log\tilde{\boldsymbol{p}}-\log\boldsymbol{p}=(I-\alpha G')^{-1}\left(\log\tilde{\boldsymbol{A}}-\log\boldsymbol{A}\right)=(I-\alpha G')^{-1}\log\boldsymbol{\xi}$$

Where we have used the following notation: $\log\boldsymbol{x}\equiv(\log x_i)_{i=1}^{n}$. □

Proof of Proposition 11: Proceeding analogously to Proposition 2 one can see that the profit of a firm in the equilibrium will be given $\pi_i=(1-\alpha-\beta)s_i-wf_i$. Using this and the market clearing condition for labor (23) we get that the total income of the consumer is: $Y=w+\sum_{i=1}^{n}\pi_i=w+(1-\alpha-\beta)\sum_{i=1}^{n}s_i-\sum_{i=1}^{n}wf_i=(1-\alpha)\frac{1-F}{\beta}w$. Then, the revenue vector in the equilibrium is defined with:

$$\boldsymbol{s}=\frac{(1-\alpha)(1-F)}{\beta}w\boldsymbol{\gamma}+\alpha G\boldsymbol{s}\Rightarrow\boldsymbol{s}=\frac{(1-\alpha)(1-F)}{\beta}w(I-\alpha G)^{-1}\boldsymbol{\gamma}$$

and the corresponding vector of equilibrium profits is $\boldsymbol{\pi}=(1-\alpha-\beta)\boldsymbol{s}-w\boldsymbol{f}$ □

Proof to the Corollary 5: Directly follows from the Proposition 11 and (35). □

Proof of Proposition 12: We can write the profit of firm j after i is eliminated from the network as:

$$\pi_j(\tilde{G}) = (1-\alpha-\beta)\frac{(1-\alpha)(1-F+f_i)}{\beta}\sum_{\substack{k=1\\k\neq i}}^{n} m_{jk}(\tilde{G})\gamma_k$$
$$= (1-\alpha-\beta)\frac{(1-\alpha)(1-F+f_i)}{m_{ii}(G)\beta} w \sum_{k=1}^{n}\left(m_{ii}(G)m_{jk}(G) - m_{ji}(G)m_{ik}(G)\right)\tilde{\gamma}_k.$$

where the second line follows directly from Lemma 4. Furthermore:

$$\pi_j(\tilde{G}) - \pi_j(G) =$$
$$(1-\alpha-\beta)\frac{(1-\alpha)(1-F+f_i)}{\beta}w$$
$$\left(\frac{1}{m_{ii}(G)}\sum_{k=1}^{n}(m_{ii}(G)m_{jk}(G) - m_{ji}(G)m_{ik}(G))\tilde{\gamma}_k - \frac{1-F}{1-F+f_i}\sum_{k=1}^{n} m_{jk}(G)\gamma_k\right) =$$
$$(1-\alpha-\beta)\frac{(1-\alpha)(1-F+f_i)}{\beta}w\left(\sum_{k=1}^{n} m_{jk}(G)\left(\tilde{\gamma}_k - \frac{1-F}{1-F+f_i}\gamma_k\right) - \frac{m_{ji}(G)}{m_{ii}(G)}\sum_{k=1}^{n} m_{ik}(G)\tilde{\gamma}_k\right) =$$
$$\tilde{\pi}_j(G) - \pi_j(G) - \tilde{\pi}_i(G)\frac{m_{ji}(G)}{m_{ii}(G)}$$

$\square$

Appendix B (Online): General framework: model with full inter-firm heterogeneity

Here we present a more general version of our model, and show that relation between centrality and profit will still hold. In this version of the model we still require for production function to be Cobb-Douglas, but we allow general heterogeneity with respect to parameters of the production function (α, β, A). Recall that before we have already allowed for heterogeneity in link weights $(g_{ij})_{i=1,j=1}^{n}$ and in preference weights $(\gamma_i)_{i=1}^{n}$. We write the production function as: $y_i = A_i l_i^{\beta_i}\left(\prod_{j\in N_i} z_{ji}^{g_{ji}}\right)^{\alpha_i}$ with constraint that $\alpha_i + \beta_i \le 1$. The utility function also takes a general form: $U(\boldsymbol{c}) = \prod_{i=1}^{n} c_i^{\gamma_i}$, with $\gamma_i \ge 0\ \forall i \in N$ and $\sum_{i=1}^{n}\gamma_i = 1$. Optimizing, we get that the firm j's demand for intermediate inputs and labor are

$$z_{ij} = \frac{p_j\alpha_j g_{ij}}{p_i}y_j$$
$$l_j = \frac{p_j\beta_j}{w}y_j.$$

The consumer's demand for consumption good i is

$$c_i = \gamma_i\frac{\sum_{j=1}^{n}\pi_j + w}{p_i} = \gamma_i\frac{w + \sum_{j=1}^{n}(1-\alpha_j-\beta_j)p_j y_j}{p_i}.$$

The market clearing condition for good i can be written as (49)

$$s_i = \gamma_i w + \gamma_i \sum_{j=1}^{n} (1 - \alpha_j - \beta_j) s_j + \sum_{j=1}^{n} \alpha_j g_{ij} s_j \tag{49}$$

Before proceeding any further, let us introduce the following notation:

$$\boldsymbol{\xi} := \begin{pmatrix} 1-\alpha_1-\beta_1 \\ 1-\alpha_2-\beta_2 \\ \vdots \\ 1-\alpha_n-\beta_n \end{pmatrix} \text{ and } \aleph \equiv \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix}.$$

Using the above introduced notation we write (49) for every firm i (in matrix notation) as:

$$\boldsymbol{s} = w\boldsymbol{\gamma} + \boldsymbol{\gamma}\boldsymbol{\xi}'\boldsymbol{s} + G\aleph\boldsymbol{s},$$

which implies (50)

$$\boldsymbol{s} = w(I - \boldsymbol{\gamma}\boldsymbol{\xi}' - G\aleph)^{-1}\boldsymbol{\gamma} \tag{50}$$

The equation (50) relates the revenue of the firm i with it's centrality in the network which adjacency matrix is: $\boldsymbol{\gamma}\boldsymbol{\xi}' - G\aleph$. Let us show that inverse matrix in (50) exists. Note that the sum of column i of matrix $\boldsymbol{\gamma}\boldsymbol{\xi}'$ is $(1-\alpha_i-\beta_i)\sum_{j=1}^{n}\gamma_j = (1-\alpha_i-\beta_i) < 1$. The sum of column i of matrix $G\aleph$ is $\sum_k = g_{ki}\alpha_i = \alpha_i < 1$. Then the sum of column i of matrix $\boldsymbol{\gamma}\boldsymbol{\xi}' + G\aleph$ is $0 < 1 - \beta_i < 1$. Furthermore, it is clear that all elements of matrix $\boldsymbol{\gamma}\boldsymbol{\xi}' + G\aleph$ are positive. Thus, matrix $\boldsymbol{\gamma}\boldsymbol{\xi}' + G\aleph$ is a sub-matrix of a column stochastic matrix. From Perron-Frobenius theorem we know that the spectral radius of matrix $\boldsymbol{\gamma}\boldsymbol{\xi}' + G\aleph$ is smaller than 1. This implies that $I - \boldsymbol{\gamma}\boldsymbol{\xi}' - G\aleph$ is invertible (as $(I - \boldsymbol{\gamma}\boldsymbol{\xi}' - G\aleph)^{-1} = \sum_{i=0}^{\infty}\left(\boldsymbol{\gamma}\boldsymbol{\xi}' + G\aleph\right)^i$ converges). As before, it is easy to see that $\pi_i = (1-\alpha_i-\beta_i)s_i$

Appendix C (Online): Distortions with balance conditions

We provide the basic results from Section 4 for the case when the payements of firms are redistributed to final consumers.

Uniform price distortion

In this case the consumer's income is given by (51), as now the consumer receives the collected *tax* from firms.

$$Y = (1-\tau)(1-\alpha-\beta)\sum_{i\in N} p_i y_i + (1-\tau)\sum_{i\in N}\beta p_i y_i + \tau\sum_{i\in N} p_i y_i = \frac{w(1-(1-\tau)\alpha)}{\beta(1-\tau)} \tag{51}$$

From (51) it directly follows that the consumer's demand in the equilibrium must satisfy:

$$c_i = \frac{Y}{p_i}\gamma_i = \frac{w(1-\alpha(1-\tau))}{p_i\beta(1-\tau)}\gamma_i$$

From here it is easy to see that equation (33) in this case becomes:

$$\begin{aligned}\boldsymbol{s}(\tau) &= \frac{w(1-\alpha(1-\tau))}{\beta(1-\tau)}\boldsymbol{\gamma} + \alpha(1-\tau)G\boldsymbol{s} \Rightarrow \\ \boldsymbol{s}(\tau) &= \frac{w(1-\alpha(1-\tau))}{\beta(1-\tau)}\left(I-\alpha(1-\tau)G\right)^{-1}\boldsymbol{\gamma}\end{aligned}$$

The positive effect of the transfer to the consumer is relative to result in Lemma 1 is captured by change of wage scaling factor from $\frac{(1-\alpha)}{\beta}$ to $\frac{(1-\alpha(1-\tau))}{\beta(1-\tau)} > \frac{(1-\alpha)}{\beta}$. However, it is not difficult to see that with this formulation the comparative static results from Subsection 4.1 will (qualitatively) still hold.

Individual price distortion

Suppose firm k is affected by the distortion. The income of the consumer in this case can be written as:

$$\begin{aligned}Y &= \sum_{i\in N}(1-\alpha-\beta)\,p_i y_i - \tau\,(1-\alpha-\beta)\,p_k y_k + \beta\sum_{i\in N} p_i y_i - \tau\beta p_k y_k + \tau p_k y_k \\ &= \frac{1-\alpha}{\beta}w + \tau p_k y_k.\end{aligned}$$

And the centrality equation becomes:

$$\boldsymbol{s}(\tau) = \left(\frac{1-\alpha}{\beta}w + \tau s_k(\tau)\right)\boldsymbol{\gamma} + \alpha(G+Q(\tau))\boldsymbol{s}(\tau) = \frac{1-\alpha}{\beta}w\boldsymbol{\gamma} + \alpha(G+Q(\tau)+V(\tau))\boldsymbol{s}(\tau)$$

where:

$$Q(\tau) \equiv \begin{pmatrix} 0 & \dots & -g_{1k}\tau & \dots & 0 \\ 0 & \dots & -g_{2k}\tau & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & -g_{nk}\tau & \dots & 0 \end{pmatrix} \quad V(\tau) \equiv \begin{pmatrix} 0 & \dots & \gamma_1\frac{\tau}{\alpha} & \dots & 0 \\ 0 & \dots & \gamma_2\frac{\tau}{\alpha} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_n\frac{\tau}{\alpha} & \dots & 0 \end{pmatrix}$$

Let us define $W(\tau) \equiv V(\tau) + Q(\tau)$. The matrix $W(\tau)$ is nxn matrix with non-zero elements only in $k-th$ column, and with (i,k) element equal to $\frac{\tau}{\alpha}(\gamma_i - \alpha g_{ik})$. Let us denote with $\boldsymbol{x} = \boldsymbol{x}(\tau)$ a column vector with elements equal to $\frac{\tau}{\alpha}(\gamma_i - \alpha g_{ik})$. So we can write $W(\tau) = \boldsymbol{x}\boldsymbol{e}_k'$ and $\boldsymbol{s}(\tau) = \frac{(1-\alpha)w}{\beta}(I-(\alpha G+\alpha\boldsymbol{x}\boldsymbol{e}_k'))^{-1}\boldsymbol{\gamma}$. Using the Sherman-Morrison formula we get (52).

$$\boldsymbol{s}(\tau) = \frac{(1-\alpha)w}{\beta}\left((I-\alpha G)^{-1}\boldsymbol{\gamma} + \frac{(I-\alpha G)^{-1}\alpha\boldsymbol{x}\boldsymbol{e}_k'(I-\alpha G)^{-1}}{1-\alpha\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{x}}\gamma\right) \tag{52}$$

Recall now that $\frac{(1-\alpha)w}{\beta}(I-\alpha G)^{-1}\boldsymbol{\gamma} = \boldsymbol{s}(0)$. Furthermore,

$$\frac{(1-\alpha)w}{\beta}(I-\alpha G)^{-1}\alpha\boldsymbol{x}\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{\gamma} = s_k(0)\left(\tau\frac{\beta}{w(1-\alpha)}\boldsymbol{s}(0) - \tau(I-\alpha G)^{-1} - I)\boldsymbol{e}_k\right).$$

On the other hand, the denominator of (52) becomes:

$$
\begin{aligned}
&1-\alpha \boldsymbol{e_k}'(I-\alpha G)^{-1}\boldsymbol{x} = \\
&1-\left(\tau \boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{\gamma} - \tau \boldsymbol{e}_k'((I-\alpha G)^{-1}-I)\boldsymbol{e}_k\right) = \\
&1-\tau\left(\frac{\beta}{(1-\alpha)w}s_k(0)-m_{kk}+1\right).
\end{aligned}
$$

So we can finally write:

$$
\boldsymbol{s}(\tau)=\boldsymbol{s}(0)+\frac{s_k(0)}{1-\tau\left(\frac{\beta}{(1-\alpha)w}s_k(0)-m_{kk}+1\right)}\left(\tau\frac{\beta}{w(1-\alpha)}\boldsymbol{s}(0)-\tau(I-\alpha G)^{-1}-I)\boldsymbol{e}_k\right).
$$

which for a firm i becomes (53)

$$
s_i(\tau)=s_i(0)+\frac{s_k(0)}{1-\tau\left(\frac{\beta}{(1-\alpha)w}s_k(0)-m_{kk}+1\right)}\left(\tau\frac{\beta}{w(1-\alpha)}s_i(0)-\tau m_{ki}\right) \tag{53}
$$

The positive effect of the transfer to the consumer in (53) relative to what we had in Proposition 5 is captured by the expression in the numerator $s_k(0)\left(\tau\frac{\beta}{w(1-\alpha)}s_i(0)\right)$ which reflects the increase in centrality of firm i as demand for good i increases due to the income effect (the consumer receives additional income as the 'tax' is transferred to him). As in (47) this effect is discounted with properly adjusted expression that captures how much distorted firm contributes to it's own centrality. To conclude, the transfer will mild down detrimental consequences of the (negative) distortion, but including it into the model will not bring much more insights in the questions we are studying in the paper.

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