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Computing Ellipsoidal Robust Forward Invariant Tubes for Nonlinear MPC

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Abstract: Min-max differential inequalities (DIs) can be used to characterize robust forward invariant tubes with convex cross-section for a large class of nonlinear control systems. The advantage of using set-propagation over other existing approaches for tube MPC is that they avoid the discretization of control policies. Instead, the conservatism of min-max DIs in tube MPC arises from the discretization of sets in the state-space, while the control law is never discretized and remains defined implicitly via the solution of a min-max optimization problem. The contribution of this paper is the development of a practical implementation of min-max DIs for tube MPC using ellipsoidal-tube enclosures. We illustrate these developments with a spring-mass-damper system.

Keywords: robust control; model predictive control; differential inequalities

1. INTRODUCTION

Robust model predictive control (MPC) is a methodology that accounts for model uncertainty and process noise as part of the control synthesis procedure. Unlike nominal MPC, it provides a certain guarantee that the system will remain feasible under given uncertainty scenarios. Many modern methods for robust MPC are based on set-theoretical approaches (Blanchini, 1999; Blanchini and Miani, 2008), which find their origin in viability theory (Aubin, 1991; Kurzhanski and Filippova, 1993; Kurzhanski and Vályi, 1997). In particular, tube MPC—as formalised by the works by Langson et al. (2004); Rakovic, et al. (2005); Mayne et al. (2009)—have become popular over the last decade. The idea is to replace the predicted trajectory used in standard MPC by a robust forward invariant tube (RFIT), namely a time-varying set-valued function enclosing all possible state trajectories under a given feedback control law, independently of the uncertainty realization (Langson et al., 2004).

Most practical implementations of tube MPC rely on the discretization of the ancillary feedback law associated with the predicted RFIT (Mayne et al., 2009). Affine feedback parameterization are among the most studied directions for discretizing the corresponding computations in robust MPC (Goulart et al., 2006; Goulart and Kerrigan, 2007; Zeilinger et al., 2014). They are also frequently used as the basis for linear matrix inequality (LMI) based robust MPC variants (Kothare et al., 1996). A discretization of the control law comes along with an inevitable loss of accuracy and introduction of conservatism, as discussed in VanParys et al. (2013) in the context of linear systems.

In contrast to the aforementioned tube MPC methods, which are based on a discretization of the feedback control law, Villanueva et al. (2017) introduced an alternative approach based on min-max DIs. There, the main idea is to characterize RFITs with convex cross-sections via a differential inequality for their support functions. This leads to a framework where the conservatism of the RFITs can be controlled by choosing a set parameterization rather than the control discretization. In this paper, we build upon these recent developments by presenting an algorithm and a prototype implementation for the automatic construction of ellipsoidal RFITs for their use in tube MPC. Our main contribution is a derivation of explicit expressions for the extra degrees of freedom introduced in the ellipsoidal tube formulation. The presented methodology enables the formulation of an optimal control problem (OCP) that can be solved numerically using existing software.

The paper is organized as follows. Section 2 reviews min-max DIs for tube MPC and set-discretization methods based on ellipsoids. Section 3 presents the main contribution of the paper, namely a simplified algorithm to construct the tube dynamics based only on the nominal control input. Section 4 presents a simple case study; and Section 5 concludes the paper.

1.1 Notation

The support function of a given compact set Z, denoted by $V[Z] : \mathbb{R}^n \to \mathbb{R}$, is defined as

$$\forall c \in \mathbb{R}^n, \quad V[Z](c) := \max_{z \in Z} \; c^\mathsf{T} z \, .$$

For a compact set-valued and Lipschitz continuous—with respect to the Hausdorff metric—function Z, the notation

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$$\dot{V}[Z(t)](c) = \frac{\mathrm{d}\sigma_{c,Z}(t)}{\mathrm{d}t}$$

is used to denote the weak derivative of the function $\sigma_{c,Z}(t) := V[Z(t)](c)$. The set of $n \times n$ positive semidefinite (resp. definite) matrices is denoted by $\mathbb{S}^n_+$ (resp. $\mathbb{S}^n_{++}$). An n-dimensional ellipsoid with center $q \in \mathbb{R}^n$ and shape matrix $Q \in \mathbb{S}^n_+$ is given by

$$\mathcal{E}(q,Q) := \left\{ q + Q^{\frac{1}{2}} v \,\middle|\, v^\mathsf{T} v \leq 1 \right\} .$$

The sets of interval vectors in $\mathbb{R}^n$ and n-by-m interval matrices are denoted respectively by $\mathbb{IR}^n$, and $\mathbb{IR}^{n\times m}$. The interval extension of a function f is denoted by $f^{\mathbb{I}}$. The upper bound of an interval $y \in \mathbb{IR}$ is denoted by $\max\{y\}$.

By a small abuse of the notation, for a time-varying function, in places where no confusion could arise, we use the function name as a shorthand for the function value.

2. PROBLEM FORMULATION AND BACKGROUND

This section formulates the problem addressed in this paper and reviews the tube MPC approach based on min-max DI as proposed in Villanueva et al. (2017).

2.1 Nonlinear control systems

We consider control systems of the form

$$\dot{x}(t) = g(x(t), u(t), w(t)) := f(x(t), w(t)) + Gu(t) \ , \quad (1)$$

where $x : \mathbb{R} \to X$ and $u : \mathbb{R} \to U$ are Lebesgue integrable state and control trajectories; the closed sets $U \subseteq \mathbb{R}^{n_u}$ and $X \subseteq \mathbb{R}^{n_x}$ denote the control and state constraints; and the exogenous disturbance $w : \mathbb{R} \to W$ is unknown but bounded, with $W \subset \mathbb{R}^{n_w}$ a compact set. Notice that u enters the system affinely, which is the only structural requirement on g, and $G \in \mathbb{R}^{n_x \times n_u}$ is a given constant matrix. The following considerations can be generalized to the case that G depends on $x(t)$. Throughout this paper, f is assumed to be nonlinear and twice-continuously differentiable in both arguments on the domain $X \times W$.

2.2 Robust Forward Invariant Tubes

A compact set-valued function $Y(t) \subset \mathbb{R}^{n_x}$ is called an RFIT, if there exists a feedback control law $\mu : \mathbb{R} \times X \to U$ such that the controlled system

$$\forall t \in \mathbb{R}, \quad \dot{x}(t) = g(x(t), \mu(t, x(t)), w(t)) \quad (2)$$

with $x(t_1) \in Y(t_1)$ satisfies $x(t_2) \in Y(t_2)$ for all $t_1, t_2 \in \mathbb{R}$ with $t_2 \geq t_1$ and all w with $w(t) \in W$. In the following, we denote the set of all RFITs for g by $\mathcal{Y}$; i.e. writing $Y \in \mathcal{Y}$, indicates that Y is an RFIT. Notice that an RFIT is constant if its cross-sections are robust forward invariant sets (Blanchini and Miani, 2008).

2.3 Tube MPC

Tube MPC solves optimization problems of the form

$$\begin{aligned} &\inf_{Y \in \mathcal{Y}} \int_0^T \ell(Y(t))\mathrm{d}t \\ &\text{s.t. } Y(t) \subseteq X \ , \quad \forall t \in [0,T] \\ &\qquad Y(0) = \{\hat{x}\} \ , \end{aligned} \quad (3)$$

where ℓ denotes the worst-case stage cost and $\mathcal{Y}$ the set of all RFITs for (1) on $[0, T]$. Here, $\hat{x}$ denotes the noise-free state measurement. Once the above problem is solved, the optimal control action $u(0) = \mu(0, \hat{x})$ is applied to the process. The current time is always set to 0.

2.4 Min-max Differential Inequalities

Villanueva et al. (2017) have shown that Y is an RFIT if it satisfies the min-max differential inequality

$$\forall c \in \mathbb{R}^{n_x}, \quad \dot{V}[Y(t)](c) \geq \min_{\nu} V[\Gamma_g(\nu, c, Y(t))](c) \ , \quad (4)$$

$$\text{with} \quad \Gamma_g(\nu, c, Z) := \left\{ g(\xi, \nu, \omega) \,\middle|\, \begin{array}{c} c^\mathsf{T}\xi = V[Z](c) \\ \xi \in Z \\ \omega \in W \end{array} \right\} .$$

By invariance of the support function with respect to a rescaling of c, it is enough to enforce the above inequality for all vectors c in the n_x-dimensional unit sphere.

2.5 Ellipsoidal Discretization of Min-Max DIs

Our approach to tackling the min-max DI (4) entails a parameterization of the tube cross-sections as ellipsoids, $Y(t) = \mathcal{E}(q_x(t), Q_x(t))$. The support function is given by

$$V[Y(t)](c) = c^\mathsf{T} q_x(t) + \sqrt{c^\mathsf{T} Q_x(t) c} \ ,$$

and we assume herein that the ellipsoidal cross-sections are non degenerate, $Q_x(t) \in \mathbb{S}^{n_x}_{++}$. The expression for the weak time-derivative of V satisfies

$$\dot{V}[Y(t)](c) = c^\mathsf{T} \dot{q}_x(t) + \frac{c^\mathsf{T} \dot{Q}_x(t) c}{2\sqrt{c^\mathsf{T} Q_x(t) c}} \ ,$$

assuming that q_x and Q_x are weakly differentiable. Conditions for q_x and Q_x such that Y is an RFIT can be derived by substituting these expressions into (4). We also assume that the uncertainty and control sets are ellipsoids,

$$U := \mathcal{E}(q_u, Q_u) \quad \text{and} \quad W := \mathcal{E}(q_w, Q_w) \ ,$$

with $Q_u \in \mathbb{S}^{n_u}_{++}$ and $Q_w \in \mathbb{S}^{n_w}_{++}$. It has been established by Villanueva et al. (2017, Theorem 3) that $Y(t)$ is an RFIT if there exist functions $\lambda, \kappa, \gamma : \mathbb{R} \to \mathbb{R}_{++}$, $u_x : \mathbb{R} \to \mathbb{R}^{n_u}$, and $S : \mathbb{R} \to \mathbb{R}^{n_x \times n_u}$ such that

$$\begin{aligned} &\dot{q}_x(t) = f(q_x(t), q_w) + G u_x(t) \ , \\ &\dot{Q}_x(t) = \Phi_g(q_x(t), Q_x(t), u_x(t), \lambda(t), \kappa(t), \gamma(t), S(t)) \ , \quad (5) \\ &\kappa(t) > 0 \ , \qquad \lambda(t) > 0 \ , \qquad 0 < \gamma \leq 1 \ , \\ &Q_x(t) \succeq 0 \ , \ R_u(u_x(t), \gamma(t)) \succeq 0 \ , \ I - S(t)S(t)^\mathsf{T} \succeq 0 \ . \end{aligned}$$

Here, we have introduced the short-hand notation

$$\begin{aligned} &\Phi_g(q_x, Q_x, u_x, \lambda, \kappa, \gamma, S) := A(t)Q_x + Q_x A(t)^\mathsf{T} \\ &\quad + \lambda^{-1} Q_x + \lambda B(t) Q_w B(t)^\mathsf{T} + \kappa^{-1} Q_x + \kappa \Omega_n(q_x, Q_x) \quad (6) \\ &\quad + Q_x^{\frac{1}{2}} S R_u^{\frac{1}{2}}(u_x, \gamma) G^\mathsf{T} + G R_u^{\frac{1}{2}}(u_x, \gamma) S^\mathsf{T} Q_x^{\frac{1}{2}} \ , \end{aligned}$$

with matrices

$$A(t) := \frac{\partial f}{\partial x}(q_x(t), q_w) \ , \ B(t) := \frac{\partial f}{\partial w}(q_x(t), q_w) \ , \quad (7)$$

$$R_u(u_x, \gamma) := (1-\gamma) Q_u + \left(1 - \frac{1}{\gamma}\right) a(u_x) a(u_x)^\mathsf{T} \ , \quad (8)$$

and $a(u_x) := (u_x - q_u)$. Moreover, $\Omega_n : \mathbb{R}^{n_x} \times \mathbb{S}^{n_x}_+ \to \mathbb{S}^{n_x}_+$ parameterizing the nonlinearity enclosure must satisfy

$$\begin{aligned} &f(\xi, \omega) - f(q_x(t), q_w) - A(t)(\xi - q_x(t)) \\ &\qquad - B(t)(\omega - q_w) \in \mathcal{E}(0, \Omega_n(q_x(t), Q_x(t))) \ , \end{aligned} \quad (9)$$

for all $\xi \in \mathcal{E}(q_x(t), Q_x(t))$ and all $\omega \in W$.

2.6 Tractable Reformulation of Tube MPC

Combining the results from the previous sections, a conservative reformulation of (3) as an OCP, with a finite number of differential and path constraints, is given by

$$\begin{aligned} \inf_{\substack{Q_x,S,\\ q_x,u_x,\gamma,\\ \lambda,\kappa}} \quad & \int_0^T \ell(\mathcal{E}(q_x(t),Q_x(t)))\mathrm{d}\tau \\ \text{s.t.} \quad & \text{Sufficient Conditions (5)} \qquad \forall t\in[0,T] \\ & \mathcal{E}(q_x(t),Q_x(t)) \subseteq X, \qquad \forall t\in[0,T] \\ & u_x(t) \in \mathcal{E}(q_u,Q_u) \quad \forall t\in[0,T] \\ & \mathcal{E}(q_x(0),Q_x(0)) = \{\hat{x}\} \ . \end{aligned} \tag{10}$$

For a tracking objective $l(x) = (x-x_{\mathrm{ref}})^\intercal D(x-x_{\mathrm{ref}})$ with weighting matrix $D\in\mathbb{S}^{n_x}_{++}$, it is possible to find a closed-form expression for the generalized inertia, i.e.

$$\ell(\mathcal{E}(q_x,Q_x)) = (q_x-x_{\mathrm{ref}})^\intercal D(q_x-x_{\mathrm{ref}}) + \frac{\mathrm{Tr}\,(DQ_x)}{n_x+2} \ . \tag{11}$$

Computing a numerical solution of (10) with state-of-the-art optimal control software is challenging, since:

(1) It comprises a large number of extra time-varying degrees of freedom, including S, γ, λ, and κ, which have no clear physical interpretation. The use of local optimization routines requires suitable initial guesses for all the variables, due to the presence of nonconvexities. In practice, finding suitable initial values for these variables can be a difficult task.

(2) The linear matrix inequalities in (5) can be handled by modern LMI solvers. However, to date, there is no generic optimal control software that can deal with such constraints without further reformulation.

3. PRACTICAL CONSTRUCTION OF AN ELLIPSOIDAL RFIT

This section presents a computational approach to constructing a simplified right-hand side function

$$\widehat{\Phi}_g : \mathbb{R}^{n_x}\times\mathbb{S}^{n_x}_{++}\times\mathbb{R}^{n_u}\to\mathbb{S}^{n_x}_{++} \ ,$$

such that $Y(t) = \mathcal{E}(q_x(t),Q_x(t))$ is an RFIT if the ODEs

$$\begin{aligned} \dot{q}_x(t) &= f(q_x(t),q_w) + Gu_x(t) \\ \dot{Q}_x(t) &= \widehat{\Phi}_g(q_x(t),Q_x(t),u_x(t)) \ , \end{aligned} \tag{12}$$

are satisfied for a given control input $u_x\in\mathcal{E}(q_u,Q_u)$. An ad-hoc choice for the functions λ, κ S, and γ involves minimizing

$$\begin{aligned} \min_{\substack{\lambda,\kappa\\ \gamma,S}} \quad & \mathrm{Tr}\,(\Phi_g(q_x,Q_x,u_x,\lambda,\kappa,\gamma,S)) \\ \text{s.t.} \quad & \lambda>0\ , \qquad \kappa>0\ , \qquad \gamma\in(0,1] \\ & Q_x\succeq 0\ , \ \ R_u(u_x,\gamma)\succeq 0\ , \ \ I-SS^\intercal\succeq 0\ , \end{aligned} \tag{13}$$

Possibly after a re-scaling of the states.

In order to proceed systematically, we treat the following four terms of Φ_g independently, which is possible due to the separable structure of the problem. The first term,

$$\Phi_x(t) := A(t)Q_x(t) + Q_x(t)A(t)^\intercal \ , \tag{14}$$

is independent of any auxiliary degree of freedom and thus it does not require any manipulation. The second term,

$$\Phi_w(t) := \lambda(t)^{-1}Q_x(t) + \lambda(t)B(t)Q_wB(t)^\intercal \ , \tag{15}$$

depends on λ and describes the linear effect of the disturbance w. The third term,

$$\Phi_n(t) := \kappa(t)^{-1}Q_x(t) + \kappa(t)\Omega_n(q_x(t),Q_x(t)) \ , \tag{16}$$

depends on κ and collects the nonlinear effects of f, via the nonlinearity bounder Ω_n. The last term,

$$\begin{aligned} \Phi_u(t) := \quad & Q_x^{\frac{1}{2}}(t)S(t)R_u^{\frac{1}{2}}(u_x(t),\gamma(t))G^\intercal \\ & + GR_u^{\frac{1}{2}}(u_x(t),\gamma(t))S(t)^\intercal Q_x^{\frac{1}{2}}(t) \ , \end{aligned} \tag{17}$$

depends on S and γ via R_u and describes describes the effect of the control action on the tube.

Notice that the functions Φ_x, Φ_w, Φ_n, and Φ_u depend on the ellipsoidal states, nominal control and extra degrees of freedom, which are not reported explicitly. Likewise we will often not report these dependencies throughout, e.g. we write $\Omega_n(t) = \Omega_n(q_x(t),Q_x(t))$.

3.1 A closer look at the nonlinearity bounder Ω_n

The construction of the nonlinearity bounder Ω_n may proceed in several ways, including explicit estimates, the use of computer algebras based on polynomial models (Makino and Berz, 2003; Rajyaguru et al. , 2016), and interval analysis tools (Moore and Bierbaum, 1979). Instead, the focus here is on the construction of smooth nonlinearity bounders using the procedure outlined in Villanueva et al. (2017, Lemma 7). This construction requires, for each function f_i, the existence of a bound $F_i \geq \max_{(\xi,\omega)\in\mathcal{D}} \|\nabla^2_{x,w} f_i(\xi,\omega)W_i\|_{\mathrm{F}}$, where $\mathcal{D}\supseteq\mathcal{E}(q_x(t),Q_x(t))\times\mathcal{E}(q_w,Q_w)$ and W_i is some invertible square matrix. The ellipsoidal bound is then given by

$$\Omega_n(t) = \frac{1}{4}\,\mathrm{diag}\left(F_i^2\left\|W_i^{-1}Q(t)\right\|^2_{\mathrm{F}}\right)_{1\leq i\leq n_x} \ , \tag{18}$$

with $Q(t) = \mathrm{diag}\,(Q_x(t),Q_w)$. Lemma 2 provides a computational procedure to construct the nonlinearity estimate. The main difference with Villanueva et al. (2017, Lemma 7) is that the weights W_i are updated at every time instant.

Lemma 1. Let $\widehat{H}_i(t)\in\mathbb{R}^{(n_x+n_w)\times(n_x+n_w)}$ be symmetric and invertible, and let $Q(t)\in\mathbb{S}^{(n_x+n_w)}_{++}$. Then,

$$W_i^*(t) = \left(\widehat{H}_i(t)^{-1}\left(\widehat{H}_i(t)Q^2(t)\widehat{H}_i(t)\right)^{\frac{1}{2}}\widehat{H}_i(t)^{-1}\right)^{\frac{1}{2}}$$

is the minimizer of $\left\|\widehat{H}_i(t)W_i(t)\right\|^2_{\mathrm{F}}\left\|W_i(t)^{-1}Q(t)\right\|^2_{\mathrm{F}}$.

Observe that selecting the weight matrices $W_i^*(t)$ per Lemma 1 minimizes the effect of the nonlinearity in each direction in the state-space, i.e., it minimizes the length of the semi-axes of the ellipsoid $\mathcal{E}(\Omega(t))$.

Lemma 2. Let $H_1,\ldots,H_{n_x}\in\mathbb{IR}^{(n_x+n_w)+(n_x\times n_w)}$ with

$$H_i \supseteq \left\{\nabla^2_{x,w}f_i(\xi,\omega)\,\middle|\, \begin{array}{l}\xi\in\mathcal{E}(q_x(t),Q_x(t))\\ \omega\in\mathcal{E}(q_w(t),Q_w)\end{array}\right\} \ .$$

Choose the invertible Hessian approximation $\widehat{H}_i(t)\approx\nabla^2_{x,w}f_i(q_x(t),q_w)$, and $Q(t) = \mathrm{diag}\,(Q_x(t),Q_w)$. Let $W_1^*(t),\ldots,W_{n_x}^*(t)$ be constructed per Lemma 1. Then, the matrix

$$\Omega_n(t) := \frac{1}{4}\,\mathrm{diag}\left(\bar{F}_i^2(t)\,\mathrm{Tr}\left(\left(\widehat{H}_i(t)Q^2(t)\widehat{H}_i(t)\right)^{\frac{1}{2}}\right)\right)_{1\leq i\leq n_x}$$

with $\bar{F}_i(t) = \sqrt{\max\left\{\mathrm{Tr}^{\mathbb{I}}\left(H_iW_i^{*2}(t)H_i\right)\right\}}$ satisfies (9).

3.2 Effect of disturbances Φ_w and nonlinearity Φ_n

Once the nonlinearity bounder Ω_n has been computed, $\lambda(t)$ and $\kappa(t)$ can be computed as the minimizers of the trace of $\Phi_w(t)$ and $\Phi_n(t)$ respectively. The following lemma provides explicit expressions for these functions.

Lemma 3. The functions $\lambda, \kappa : \mathbb{R} \to \mathbb{R}_{++}$ given by

$$\lambda^*(t) = \frac{\sqrt{\mathrm{Tr}\,(Q_x(t))}}{\sqrt{\mathrm{Tr}\,(B(t)Q_w B(t)^\mathsf{T})}} \quad \text{and} \quad \kappa^*(t) = \frac{\sqrt{\mathrm{Tr}\,(Q_x(t))}}{\sqrt{\mathrm{Tr}\,(\Omega_n(t))}}$$

are the pointwise-in-time minimizers of $\mathrm{Tr}\,(\Phi_w(t))$ s.t. $\lambda(t) > 0$ and $\mathrm{Tr}\,(\Phi_n(t))$ s.t. $\kappa(t) > 0$, respectively.

This result has been established in the context of the computation of (open-loop) reachable tubes with ellipsoidal cross-sections (Villanueva et al., 2015).

3.3 Contributions of the controlled terms Φ_u

Choosing S and γ in order to minimize the trace of $\Phi_u(t)$ given by (17) presents two main difficulties: 1) due to the matrix square root, it is not possible to get explicit expressions for both S and γ simultaneously; and 2) $\Phi_u(t)$ is blind to first order effects of the dynamics. To address these points, we introduce a reparameterization of the problem, restricting the search to stabilizing feedback gains. This is achieved by solving the continuous-time algebraic Riccati equation

$$\widehat{A}(t)^\mathsf{T} P(t) + P(t)\widehat{A}(t) - P(t)GG^\mathsf{T} P(t) + I = 0\,, \qquad (19)$$

with $\widehat{A}(t) = A(t) + \left(\frac{1}{\lambda(t)} + \frac{1}{\kappa(t)}\right) I$, and setting

$$S(t) := -Q_x(t)^{\frac{1}{2}} P(t) G\hat{S}(t) R_u^{-\frac{1}{2}}(u_x(t), \gamma(t)) \qquad (20)$$

with a new parameter $\widehat{S}(t) \in \mathbb{S}_+^{n_u}$ such that

$$\Phi_u(t) = -Q_x(t)P(t)G\widehat{S}(t)G^\mathsf{T} - G\widehat{S}(t)G^\mathsf{T} P(t)Q_x(t)\,. \quad (21)$$

Therefore, we now consider the following problem

$$\begin{aligned} \min_{\widehat{S}(t),\gamma(t)} \quad & -\mathrm{Tr}\left(Q_x(t)P(t)G\widehat{S}(t)G^\mathsf{T}\right) \\ \text{s.t.} \quad & R_u(u_x(t),\gamma(t)) - \widehat{S}(t)\widehat{Q}(t)\widehat{S}(t) \succeq 0\,, \\ & R_u(u_x(t),\gamma(t)) \succeq 0\,, \quad \gamma(t) \in (0,1]\,. \end{aligned} \qquad (22)$$

Problem (22) can now be solved simultaneously for $\gamma(t)$ and $\widehat{S}(t)$, as summarized in the following lemma.

Lemma 4. Let $u_x : \mathbb{R} \to \mathbb{R}^{n_u}$ satisfy $u_x(t) \in \mathcal{E}(q_u, Q_u)$ for all $t \in \mathbb{R}$. The functions $\gamma^* : \mathbb{R} \to (0,1]$ and $\widehat{S}^* : \mathbb{R} \to \mathbb{S}_+^{n_u}$ given by

$$\gamma^*(t) = \frac{\left\| Q_x^{\frac{1}{2}}(t)P(t)G\,(u_x(t) - q_u)\right\|_2}{\sqrt{\mathrm{Tr}\left(Q_x^{\frac{1}{2}}(t)P(t)GQ_uG^\mathsf{T} P(t)Q_x^{\frac{1}{2}}(t)\right)}}$$

$$\widehat{S}^*(t) = \widehat{Q}^{-\frac{1}{2}}(t)\left(\widehat{Q}^{\frac{1}{2}}(t)R_u(u_x(t),\gamma^*(t))\widehat{Q}^{\frac{1}{2}}(t)\right)^{\frac{1}{2}}\widehat{Q}^{-\frac{1}{2}}(t)$$

with $\widehat{Q}(t) := G^\mathsf{T} P(t)Q_x(t)P(t)G$, are the pointwise-in-time minimizers of Problem (22).

3.4 Summary of the Algorithm For Constructing $\widehat{\Phi}_g$

Algorithm 1 summarizes the construction of the right-hand side function $\widehat{\Phi}_g$. The functions λ^* and κ^* constructed per Lemma 3 not only satisfy $\lambda^*(t), \kappa^*(t) > 0$, but are also optimal with respect to the chosen criterion. Moreover, after the reparameterization introduced in the previous section, $\widehat{S}^*$ and γ^* satisfy the required feasibility constraints and minimize the trace of $\Phi_u(t)$.

Theorem 5. Let $u_x : \mathbb{R} \to \mathbb{R}^{n_u}$ be any given reference control with $u_x(t) \in \mathcal{E}(q_u, Q_u)$. If the functions $q_x : \mathbb{R} \to \mathbb{R}^{n_x}$ and $Q_x : \mathbb{R} \to \mathbb{S}_{++}^{n_x}$ satisfy (12) with $\widehat{\Phi}_g$ computed per Algorithm 1, then $Y(t) = \mathcal{E}(q_x(t), Q_x(t))$ describes an RFIT for (1). Moreover, for each $t \in \mathbb{R}$, the shape matrix $Q_x(t)$ has minimal trace.

Algorithm 1. Evaluation of $\widehat{\Phi}_g(q_x(t), Q_x(t), u_x(t))$

Input: Center vector $q_x(t)$ • Shape matrix $Q_x(t)$ • control $u_x(t)$ • Optional: Interval bounders $H_1, \ldots, H_{n_x}$ of $\nabla^2_{x,w} f_1, \ldots, \nabla^2_{x,w} f_{n_x}$ on a compact $\mathcal{D} \subset \mathbb{R}^{n_x \times n_w}$, with $D \supseteq \mathcal{E}(q_x(t), Q_x(t)) \times \mathcal{E}(q_w, Q_w)$, for all t.

1. **Compute** the nonlinearity estimate $\Omega_n(q_x(t), Q_x(t))$:
   - 1a **If** no Hessian bounders are given, compute intervals $\mathcal{D}_x(t) \supseteq \mathcal{E}(q_x(t), Q_x(t))$, $\mathcal{D}_w \supseteq \mathcal{E}(q_w, Q_w)$ and evaluate $H_i = \nabla^2_{x,w} f_i^{\mathbb{I}}(D_x(t), D_w)$ for each $i \in \{1, \ldots, n_x\}$.
   - 1b **Compute** $\Omega_n(q_x(t), Q_x(t))$ according to Lemma 2.
2. **Evaluate** matrices $A(t)$ and $B(t)$ using (7), and **compute**:
$$\Phi_x(t) = A(t)Q_x(t) + Q_x(t)A(t)^\mathsf{T}\,.$$
3. **Evaluate** $\lambda^*(t)$ and $\kappa^*(t)$ per Lemma 3 and **compute**:
$$\begin{aligned}\Phi_{wn}(t) = {} & \frac{1}{\lambda^*(t)}Q_x(t) + \lambda^*(t)B(t)Q_wB(t)^\mathsf{T} \\ & + \frac{1}{\kappa^*(t)}Q_x(t) + \kappa^*(t)\Omega_n(q_x(t), Q_x(t))\,.\end{aligned}$$
4. **Compute** $P(t)$ by solving (19).
5. **Compute** the multiplier $\gamma^*(t)$ per Lemma 4 and **evaluate** $R_u(u_x(t), \gamma^*(t))$ using (8). **Compute** the multiplier $\widehat{S}^*(t)$ per Lemma (4) and **compute**
$$\Phi_u(t) = -Q_x(t)P(t)G\widehat{S}^*(t)G^\mathsf{T} - G\widehat{S}^*(t)G^\mathsf{T} P(t)Q_x(t).$$
6. **Compute** the value of the right-hand side function $\widehat{\Phi}_g$:
$$\widehat{\Phi}_g(t) = \Phi_x(t) + \Phi_{wn}(t) + \Phi_u(t)\,.$$

Return: Value of the right-hand side function $\widehat{\Phi}_g$ at t.

Overall, a robust tube MPC for (1) on $[0, T]$ can be computed by solving the OCP

$$\begin{aligned} \inf_{\substack{q_x, u_x \\ Q_x}} \quad & \int_0^T \ell(\mathcal{E}(q_x(t), Q_x(t)))\mathrm{d}t \\ \text{s.t.} \quad & \dot{q}_x(t) = f(q_x(t), q_w) + Gu_x(t) && \forall t \in [0,T] \\ & \dot{Q}_x(t) = \widehat{\Phi}_g(q_x(t), Q_x(t), u_x(t)) && \forall t \in [0,T] \\ & X \supseteq \mathcal{E}(q_x(t), Q_x(t)) && \forall t \in [0,T] \\ & u_x(t) \in \mathcal{E}(q_u, Q_u) && \forall t \in [0,T] \\ & Q_x(t) \succeq 0 && \forall t \in [0,T] \\ & q_x(0) = \hat{x}\,, \quad Q_x(0) = \epsilon I\,. \end{aligned} \qquad (23)$$

where ϵ is a small positive constant.

Remark 1. The derivations herein are based on the assumption that certain matrices, especially $Q_x(t)$ are positive definite. Thus, a small regularization is needed if $Q_x(t)$ is only positive semi-definite. In our implementation, we have regularized the matrix $\widehat{Q}(t)$ directly, i.e., we use

$$\widehat{Q}(t) = G^\mathsf{T} P(t) Q_x(t) P(t) G + \delta I$$

for a small regularization $\delta > 0$. This ensures that the matrix $\widehat{Q}(t)$ is invertible, i.e., the Ando geometric mean in the Proof of Lemma 4 can be evaluated directly without generalizing this formula to degenerate cases. For a large enough δ, the shape matrix $Q_x(t)$ propagated through Algorithm 1 is guaranteed to remain positive definite, thus the constraint $Q_x(t) \succeq 0$, does not need to be enforced explicitly in (23). One caveat is that the tube cross-sections may only converge to a nondegenerate ellipsoid, whose size depends on δ, even when the controller is able to fully reject the uncertainty.

4. IMPLEMENTATION AND CASE STUDY

We consider a spring-mass-damper system given by

$$\underbrace{\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}}_{\dot{x}(t)} = \underbrace{\begin{pmatrix} x_2(t) + w_1(t) \\ -\frac{k(x)x_1(t)}{M} - \frac{h_d x_2(t)}{M} + \frac{w_2(t)}{M} \end{pmatrix}}_{f(x(t),w(t))} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix}}_{G(x(t))} u(t) \; ,$$

where x_1 and x_2 denote displacement with respect to the equilibrium position [m] and its velocity [m/s], respectively; $M = 1$ kg is the mass; $k(x) := 0.33 \exp(-x_1)$, the stiffness of the spring [N/m]; and $h_d = 1.1$ Ns/m, the damping factor. Bounds for the disturbance and the control sets are given by the ellipsoids $\mathcal{E}(Q_w)$ and $\mathcal{E}(Q_u)$, with $Q_w = \mathrm{diag}(0.1 \text{ m}^2/\text{s}^2, \ 2.5 \text{ N}^2)$ and $Q_u = 36 \text{ N}^2$. The length of the prediction horizon is set to $T = 10$ s, and the initial state of the system is $x_{\text{start}} = (0.7 \text{ m}, 0.7 \text{ m/s})^\mathsf{T}$.

The optimization problem in the tube MPC controller is based on (23) and involves minimizing

$$\int_0^T \left(\|q_x(t)\|_2^2 + \frac{1}{4} \operatorname{Tr}(Q_x(t)) + u_x(t)^2 \right) \mathrm{d}t \; .$$

This cost corresponds to the sum of the generalized rotational inertia and $u_x(t)^2$, a control regularization term. Moreover, we consider the constraint $X := \{x | x_1 \leq 0.85\}$.

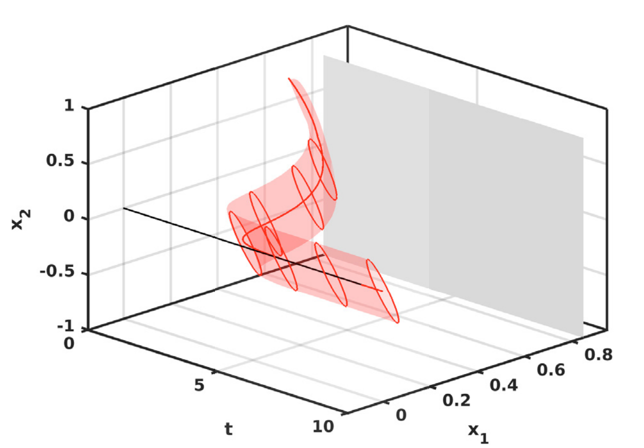

Fig. 1. The optimal ERFIT for $\hat{x} = x_{\text{start}}$. Selected cross sections and the central path q_x are shown in red. The constraint $X = \{x | x_1 \leq 0.85\}$ is shown in grey. The reference trajectory $x_{\text{ref}} = 0$ is shown in black.

Figure 1 presents the result of the tube-base MPC controller (23). The OCP was implemented in ACADO Toolkit (Houska et al., 2011), with a piecewise constant discretization of u_x over 40 equidistant intervals. The numerical parameters used in the implementation were $\delta = 10^{-4}$ and $\epsilon = 10^{-8}$. As expected, the controller is able to steer the central path to the origin at $t = 10$. It is also shown that the controller is not able to completely reject the disturbance. Comparing the results of the proposed approach with those presented in (Villanueva et al., 2017) we can observe that although the controller based on the reduced approach is able to steer the system so as to avoid violating the constraint, eliminating the extra degrees of freedom has an effect on the ability of the tube to rotate. In order to assess the effect of the nonlinearity estimate, we have compared the construction per Lemma 2 to an explicit estimate (See Appendix F). The comparison is not shown as both estimates yielded similar results. A full comparison of the proposed construction to an optimal ERFIT computed using all the extra degrees of freedom is outside the scope of this paper.

5. CONCLUSION

This paper, has presented a practical approach to tube MPC for input-affine nonlinear systems. This approach relies on the construction of RFITs based on min-max differential inequalities, which avoids the discretization and construction of future control policies by parameterizing the RFIT a priori. Similar to Villanueva et al. (2017), the focus is on ellipsoidal tubes, and the main contribution is a simplified computational procedure for such ellipsoidal RFITs based on the solution of an augmented standard optimal control problem using state-of-the-art software.

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Appendix A. ANDO GEOMETRIC MEAN

The derivations below use the following technical lemma.

Lemma 6. Let $A, B \in \mathbb{S}^n_{++}$. The equation $ZAZ = B$ has as solution the Ando geometric mean of A and B,

$$Z = A^{-\frac{1}{2}} \left(A^{\frac{1}{2}} B A^{\frac{1}{2}} \right)^{\frac{1}{2}} A^{-\frac{1}{2}} \in \mathbb{S}^n_{++} \,. \tag{A.1}$$

See Lawson and Lim (2001) for a proof of this statement.

Appendix B. PROOF OF LEMMA 1

Differentiating the objective function, gives

$$\alpha_0^2 Q^2(t) = W_i^2(t) \widehat{H}_i^2(t) W_i^2(t) \,, \quad \alpha_0 = \frac{\|\widehat{H}_i(t) W_i(t)\|_{\mathrm{F}}}{\|W_i^{-1}(t) Q(t)\|_{\mathrm{F}}} \,.$$

Notice that, for any optimal $W_i^*(t)$, $\alpha W_i^*(t)$ is also an optimal weighting matrix for any $\alpha > 0$. Setting $\alpha = \alpha_0^{-\frac{1}{2}}$ and applying Lemma 6, provides the desired result. □

Appendix C. PROOF OF LEMMA 2

Since the interval matrix H_i is a bound of the Hessian $\nabla^2_{xw} f_i$ on $\mathcal{E}(0, Q(t))$, $\bar{F}_i(t)$ is an upper bound for $\max_{(\xi,\omega)\in\mathcal{E}(0,Q(t))} \|\nabla^2_{x,w} f_i(\xi,\omega) W_i(t)\|_{\mathrm{F}}$. Therfore, $\bar{F}_i$ satisfies the assumption of Lemma 7 in (Villanueva et al., 2017). Thus, the result follows from Eq. (18) with

$$\left\| W_i^*(t)^{-1} Q(t) \right\|_{\mathrm{F}}^2 = \mathrm{Tr}\left(\left(\widehat{H}_i(t) Q^2(t) \widehat{H}_i(t) \right)^{\frac{1}{2}} \right) . \qquad \square$$

Appendix D. PROOF OF LEMMA 3

The result for λ^* follows by setting to zero the derivative of $\lambda(t)^{-1}\,\mathrm{Tr}\,(Q_x(t)) + \lambda(t) B(t) Q_w B(t)^{\mathsf{T}}$. Since $Q_x(t)$ is positive definite, its trace is positive, thus $\lambda(t) > 0$ is satisfied and the expression $\lambda^{-1}(t)$ is well defined. A similar argument yields the result for κ^*. □

Appendix E. PROOF OF LEMMA 4

The proof proceeds in two steps, first we minimize the trace of $\Phi_u(t)$ with respect to $\widehat{S}(t)$. Since $\Phi_u(t)$ is linear in $\widehat{S}(t)$, its trace is minimal if $R_u(u_x(t), \gamma(t)) \succeq \widehat{S}(t)\widehat{Q}(t)\widehat{S}(t)$ is active. Applying Lemma 6 to the active constraint, yields a parametric minimizer $\widehat{S}(t, \gamma(t))$.

For the second step we minimize $\mathrm{Tr}\,(\Phi_u(t))$ with respect to $\gamma(t)$ by substituting $\widehat{S}(t, \gamma(t))$ into $\mathrm{Tr}\,(\Phi_u(t))$. As the matrix square-root is monotonic with respect to the Löwner partial order in the positive semidefinite cone, minimizing the trace of $\Phi_u(t)$ is equivalent to

$$\begin{aligned} \min_{\gamma(t)} \quad & -\mathrm{Tr}\left(\widehat{Q}^{\frac{1}{2}}(t) R_u(u_x(t), \gamma(t)) \widehat{Q}^{\frac{1}{2}}(t) \right) \\ \text{s.t} \quad & R_u(u_x(t), \gamma(t)) \succeq 0, \qquad \gamma(t) \in (0, 1] \,. \end{aligned} \tag{E.1}$$

Since the extremum of this problem always occurs in the interior of $(0, 1]$, setting the derivative of the objective function to zero and solving the resulting quadratic equation, yields the desired expression for $\gamma^*(t)$. The result follows from setting $\widehat{S}^*(t) = \widehat{S}(t, \gamma^*(t))$.

Appendix F. EXPLICIT NONLINEARITY ESTIMATE FOR THE SPRING-MASS-DAMPER SYSTEM

The explicit nonlinearity estimate for the spring-mass-damper system used in Section 4 is constructed using the following expressions:

$$\begin{aligned} m_2(q_x(t), Q_x(t)) &:= \frac{1}{2} \left| q_{x_1}(t) + \sqrt{Q_{x_{1,1}}(t)} \right| e^{\sqrt{Q_{x_{1,1}}(t)}} + 1 \\ n_2(q_x(t), Q_x(t)) &:= \frac{1}{3} e^{-q_{x_1}(t)} \left| Q_{x_{1,1}}(t) \right| m_2(q_x(t), Q_x(t)) \\ \Omega_n(q_x(t), Q_x(t)) &= \begin{pmatrix} 0 & 0 \\ 0 & n_2^2(q_x(t), Q_x(t))) \end{pmatrix} . \end{aligned}$$