## Proximal-like algorithms for equilibrium seeking in mixed-integer Nash equilibrium problems

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Abstract—We consider potential games with mixed-integer variables, for which we propose two distributed, proximallike equilibrium seeking algorithms. Specifically, we focus on two scenarios: i) the underlying game is generalized ordinal and the agents update through iterations by choosing an exact optimal strategy; ii) the game admits an exact potential and the agents adopt approximated optimal responses. By exploiting the properties of integer-compatible regularization functions used as penalty terms, we show that both algorithms converge to either an exact or an  $\epsilon$ -approximate equilibrium. We corroborate our findings on a numerical instance of a Cournot oligopoly model.

### I. INTRODUCTION

Tracing back from the seminal work by Monderer and Shapley [1], potential games represent a broad class of noncooperative games characterized by the existence of a real-valued function, the *potential function*, such that any collective strategy profile minimizing the underlying function coincides with a Nash equilibrium of the game. Potential games hence provide a means to naturally model many control-theoretic applications [2] such as routing [3], complex social networks [4] and Cournot competition [5].

We consider generalized ordinal potential games in which part of the decision variables of the agents are constrained to assume integer values. Such mixed-integer (MI) games have been recently proposed as strategic models for the distributed coordination of autonomous vehicles [6], [7], transportation and traffic control [8], and smart grids [9], [10]. In addition, MI restrictions are often encountered in market games [11], [12] and combinatorial congestion games [13], [14] as well.

For this practically relevant, yet intrinsically nonconvex, MI game-theoretic setting the existence of equilibria follows by assuming that a certain master problem admits a solution, and we hence present two distributed, proximal-like equilibrium seeking algorithms. In particular, we consider the following scenarios: i) the underlying game is generalized ordinal and the agents update their control variables iteratively by choosing an exact proximal best-response (BR) strategy; ii) the game admits an exact potential function but we allow agents to choose an inexact proximal BR for their updates.

Similar to Bregman-versions of proximal algorithms [15], we formulate the proximal best-response function using a class of norm-like regularizers, known in the MI optimization community as integer-compatible regularization functions (ICRFs). We choose ICRFs as penalty terms in place of standard quadratic regularizations as we believe they could provide us with a mean to include continuous reformulations of the MI optimization subproblems. We leave this topic for future research. Thus, by exploiting the properties of the ICRFs, acting as penalty terms in the individual agent's BR problems, we prove that both proposed algorithms enjoy convergence guarantees to an equilibrium of the mixedinteger Nash equilibrium problem (MI-NEP). Specifically, in the first scenario considered the computed mixed-integer Nash equilibrium (MI-NE) is exact, while in the second one the algorithm returns an approximate MI-NE.

Since MI-NEPs constitute a rather new class of strategic optimization problems, there are not many solution techniques available. To the best of our knowledge, this work represents a first attempt proposing proximal-like distributed algorithms for a MI game setting. The only alternative applicable algorithm for MI-NEPs is the Gauss-Southwell method designed in [16]. Given the practical relevance of MI-NEPs, this is rather surprising, and completely diametric to continuous Nash equilibrium problems (NEPs), for which a whole arsenal of numerical solution techniques is available [17], [18]. In fact, proximal BR-based algorithms have been extensively studied in both stochastic and deterministic Nash games [19], [20]. All these schemes, however, leverage the variational inequality (VI) reformulation of NEPs [21], and thus require strong (or strict) monotonicity of the VI, assumptions that cannot be structurally satisfied in MI-NEPs. The algorithms we develop, in particular our adaptive update of the penalty parameter regulating the proximal BR, are inspired by the decomposition method proposed in [22]. As main contributions we show i) how a proximal-like BRscheme with ICRFs can be used to compute Nash equilibria satisfying MI restrictions, and ii) we show convergence to an approximate equilibrium even under inexact computations of BR strategies.

The rest of the paper is organized as follows. In §II we introduce the problem addressed, along with some preliminaries, while in §III we present the two methods and related convergence analysis. Finally, in §IV, we test our findings

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on a MI Cournot oligopoly model and we conclude in §V.

### II. PROBLEM FORMULATION AND PRELIMINARIES

Let  $\mathcal{I} := \{1, \ldots, N\}$  be the set indexing the N agents taking part in the noncooperative game  $\Gamma := (\mathcal{I}, (J_i)_{i \in \mathcal{I}}, (\mathcal{X}_i)_{i \in \mathcal{I}})$ . Each agent controls MI variables  $x_i$  belonging to a compact, nonempty set  $\mathcal{X}_i \subseteq \mathbb{R}^{n_i^c} \times \mathbb{Z}^{n_i^d}$ , and aims at minimizing a given cost function  $J_i : \mathcal{X} \to \mathbb{R}$  with  $\mathcal{X} := \prod_{i \in \mathcal{I}} \mathcal{X}_i \subseteq \mathbb{R}^n$  and  $n := \sum_{i \in \mathcal{I}} n_i = \sum_{i \in \mathcal{I}} (n_i^c + n_i^d)$ . The resulting MI-NEP thus reads

$$\forall i \in \mathcal{I} : \min_{x_i \in \mathcal{X}_i} J_i(x_i, \boldsymbol{x}_{-i}), \qquad (1)$$

where  $x_{-i} \coloneqq \operatorname{col}((x_j)_{j \in \mathcal{I} \setminus \{i\}})$ . Given the strategies of the other agents,  $x_{-i}$ , the MI BR of agent  $i \in \mathcal{I}$  is defined as

$$B_i(\boldsymbol{x}_{-i}) \coloneqq \underset{x_i \in \mathcal{X}_i}{\operatorname{argmin}} \ J_i(x_i, \boldsymbol{x}_{-i}) \,. \tag{2}$$

Our goal is to design distributed algorithms, able to drive the set of agents to an MI-NE of the game  $\Gamma$ , according to the definition given next.

Definition 1: (Mixed-integer  $\epsilon$ -Nash equilibrium) Given some  $\epsilon \geq 0$ , a strategy profile  $x^* \in \mathcal{X}$  is an  $\epsilon$ -approximate MI-NE (or  $\epsilon$ -MI-NE) of the game  $\Gamma$  if, for all  $i \in \mathcal{I}$ ,

$$J_i(x_i^*, \boldsymbol{x}_{-i}^*) \le J_i(y_i, \boldsymbol{x}_{-i}^*) + \epsilon \text{ for all } y_i \in \mathcal{X}_i.$$
(3)

 $\Box$ 

If  $\epsilon = 0$ , then we call  $x^*$  an exact MI-NE.

Definition 1 points out that a MI-NE of the game (if it exists) is achieved when all the agents adopt a BR strategy.

### A. Generalized ordinal and exact potential games

Existence theorems for NEPs typically require continuity of the agents' cost functions as well as compactness and convexity of the feasible sets [21]. Since the NEP in (1) is nonconvex, existence of Nash equilibria is, in principle, not guaranteed. We thus focus on the classes of *exact* and *generalized ordinal potential games*, for which existence of solutions can be guaranteed under certain assumptions.

Definition 2: (Potential game) A game  $\Gamma$  is called

exact potential game [1] if there exists a continuous function P : ℝ<sup>n</sup> → ℝ such that, for all i ∈ I,

$$P(x_i, \boldsymbol{x}_{-i}) - P(y_i, \boldsymbol{x}_{-i}) = J_i(x_i, \boldsymbol{x}_{-i}) - J_i(y_i, \boldsymbol{x}_{-i}),$$

for all  $\boldsymbol{x}_{-i}$  and  $x_i, y_i \in \mathcal{X}_i$ ;

• generalized ordinal potential game [22] if, there exists a forcing function  $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$  such that, for all  $i \in \mathcal{I}, \mathbf{x}_{-i}$ , and  $x_i, y_i \in \mathcal{X}_i$ ,

$$\begin{aligned} &J_i(y_i, \boldsymbol{x}_{-i}) - J_i(x_i, \boldsymbol{x}_{-i}) > 0 \text{ implies} \\ &P(y_i, \boldsymbol{x}_{-i}) - P(x_i, \boldsymbol{x}_{-i}) \ge \phi(J_i(y_i, \boldsymbol{x}_{-i}) - J_i(x_i, \boldsymbol{x}_{-i})) \\ &\text{where } \lim_{t \to 0^+} \phi(t) = 0. \end{aligned}$$

Note that any exact potential game is a generalized ordinal potential game. By exploiting the tight relation between firstorder information of the potential function and the local cost functions of the agents, it is well-known that potential functions can be employed in the construction of a suitable master problem facilitating the computation of equilibria. Proposition 1: [16, Th. 2] Let P be a generalized ordinal potential function for the game  $\Gamma$ . Given some  $\epsilon \geq 0$ , any  $\epsilon$ -approximate solution of the optimization problem

$$\min_{\boldsymbol{x}\in\mathcal{X}} P(\boldsymbol{x}) \tag{4}$$

yields an  $\epsilon$ -approximate MI-NE of  $\Gamma$ .

Assumption 1: Problem (4) is solvable, i.e., there exists an  $x^* \in \mathcal{X}$  with  $P(x) > P(x^*)$  for all  $x \in \mathcal{X}$ .

This assumption guarantees that the game  $\Gamma$  admits at least one Nash equilibrium in the nonconvex domain  $\mathcal{X}$ . Clearly, the solutions to the master problem (4) may not contain all possible Nash equilibria of the game  $\Gamma$  [26, Ex. 1].

### B. Integer-compatible regularization functions

Standard regularization techniques in NEPs are based on the proximal BR function obtained from (2) by adding a quadratic penalty term. Motivated by the proximal point interpretation of MI optimization heuristics, we propose a regularization strategy of the individual agents' cost functions via *integer-compatible regularization functions* (ICRFs) [23]. This family of functions has been introduced in the mixed-integer optimization community to control the duality gap [25]. They are also related to penalty methods for mixedinteger optimization [24].

Definition 3: A continuous function  $\rho : \mathbb{R}^n \to \mathbb{R}$  is an *integer-compatible regularization function* (ICRF) if

- i)  $\rho(t) \ge 0$  for all  $t \in \mathbb{R}^n$  and  $\rho(t) = 0 \iff t = 0$ ;
- ii) for  $\gamma \in (0,1)$ , we have  $\rho(\gamma t) < \rho(t)$  for all  $t \neq 0$ ;
- iii) there exists a continuous and strictly increasing function  $s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$  and some  $\overline{K} \in \mathbb{N}$  such that, for all  $K < \overline{K}$ ,  $\rho(t) \leq K \implies ||t||_1 \leq s^{-1}(K)$ , where  $|| \cdot ||_1$  denotes the  $\ell_1$  norm in  $\mathbb{R}^n$ .  $\Box$

Note that any norm defined in  $\mathbb{R}^n$  is an ICRF. A constructive way to design ICRFs is to consider decomposable penalties of the form  $\rho(t) = \sum_{i=1}^n p(|t_i|)$ , where  $p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a concave and strictly increasing function. Prominent examples are  $p(t) = \log(t + \alpha) - \log(\alpha)$ ,  $p(t) = -(t + \alpha)^{-q} + \alpha^{-q}$ ,  $p(t) = 1 - e^{-\alpha t}$ , or  $p(t) = (1 + e^{-\alpha t})^{-1} - \frac{1}{2}$ , for some  $\alpha, q > 0$  (see e.g. [23], [24]). With these choices,  $\rho(\cdot)$  amounts to an ICRF [23, Prop. 3.2]. In particular, for the special case of binary constraints, a sensible formulation of an ICRF is  $\rho(t) = \sum_{i=1}^n \min\{p(|t_i|), p(|1 - t_i|)\}$ .

#### **III. PROXIMAL-LIKE ALGORITHMS FOR MI-NEPS**

We now propose two MI-NE seeking algorithms. Algorithm 1 assumes that agents are able to compute an exact proximal BR at each iteration. Algorithm 2 relaxes this and instead allows for inexact BR computations. To show convergence, in the former case we rely on the fact that the MI-NEP in (1) is generalized ordinal, whereas in the latter case we require the existence of an exact potential function.

Let  $\rho_i$  denote the ICRF employed by agent  $i \in \mathcal{I}$  and let  $\tau > 0$  be a positive regularization parameter. We introduce the *proximal augmented local cost function* as a regularized version of the local cost in (1), which is given by

$$\tilde{J}_{i,\tau}(y_i, x_i; \boldsymbol{x}_{-i}) \coloneqq J_i(y_i, \boldsymbol{x}_{-i}) + \tau \rho_i(y_i - x_i).$$
(5)

**Algorithm 1:** Proximal-like method for MI-NEPs with generalized ordinal potential and exact optimization

In accordance, the proximal BR mapping in (2) turns into

$$\beta_{i,\tau}(\boldsymbol{x}) \coloneqq \underset{y_i \in \mathcal{X}_i}{\operatorname{argmin}} \tilde{J}_{i,\tau}(y_i, x_i; \boldsymbol{x}_{-i}) \,. \tag{6}$$

Setting  $\tau = 0$  allows us to recover the BR mapping as defined in (2), i.e.,  $\beta_{i,0}(\boldsymbol{x}) = B_i(\boldsymbol{x}_{-i})$ . By considering these two new ingredients, in the remainder of this section we design iterative and distributed schemes in which the agents update their own action sequentially, according to  $\mathcal{I}$ . We let  $k \ge 0$ denote the iteration counter of the process and  $\boldsymbol{x}^k$  the iterate at the beginning of round k+1. For an arbitrary agent  $i \in \mathcal{I}$ , we also define the (local) population state as

$$\hat{\boldsymbol{x}}_i(k) \coloneqq \operatorname{col}(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x_i^k, x_{i+1}^k, \dots, x_N^k)$$

This corresponds to the collective vector of strategies at the k-th iteration communicated to agent i when this agent has to perform an update, i.e., it computes the new strategy as a point in the MI proximal BR mapping,  $x_i^{k+1} \in \beta_{i,\tau^k}(\hat{x}_i(k))$ , either exactly (Algorithm 1) or approximately (Algorithm 2). Successively, the next internal state  $\hat{x}_{i+1}(k)$  is updated and passed to the (i + 1)-th agent. Throughout this process, note that, for all  $k \ge 0$ ,  $\hat{x}_1(k) = x^k$  and  $\hat{x}_{N+1}(k) = x^{k+1}$ .

We stress that the proposed algorithms leverage the adaptive update of the regularization parameter  $\tau$  in (7), which produces a monotonically decreasing sequence  $\{\tau^k\}_{k\geq 0}$ , i.e.,  $\tau^{k+1} \leq \tau^k$  for all  $k \geq 0$  [22]. Note that the rate of decrease strongly depends on  $d_{\rho}(\boldsymbol{x}^{k+1}, \boldsymbol{x}^k) \coloneqq \max_{i \in \mathcal{I}} \rho_i(\boldsymbol{x}^{k+1}_i - \boldsymbol{x}^k_i) \geq 0$ , which measures the progress the method is making in the agents' proximal steps at the k-th iteration. As it will be clear from the convergence analysis, this quantity decreases over time, thus inducing a step towards an MI-NE.

# A. Exact BR computation in generalized ordinal potential games

Next, by focusing on MI-NEPs as in (1), we study the convergence of Algorithm 1 under the following assumption.

Assumption 2: The game  $\Gamma = (\mathcal{I}, (J_i)_{i \in \mathcal{I}}, (\mathcal{X}_i)_{i \in \mathcal{I}})$  is a generalized ordinal potential game with potential function  $P(\cdot)$ .

Thus, the Gauss–Seidel sequence of iterations in Algorithm 1 has the following convergence property. We stress that, in our framework, an accumulation point for the sequence  $\{x^k\}_{k\geq 0}$  exists in view of Assumption 1.

Theorem 1: Under Assumption 2, any accumulation point of the sequence of strategy profiles generated by Algorithm 1,  $\{x^k\}_{k\geq 0}$ , is an MI-NE of the game  $\Gamma$  in (1).

*Proof:* We first show that, in case the sequence  $\{x^k\}_{k\geq 0}$  generated by Algorithm 1 admits a limit point  $\bar{x} \in \mathcal{X}$ , then the regularization parameter, adaptively updated via (7), satisfies  $\lim_{k\to\infty} \tau^k = 0$  and there exists an infinite index set  $\mathcal{K}$  such that  $\tau^{k+1} < \tau^k$  for all  $k \in \mathcal{K}$ . Then, we prove that  $\bar{x} \in \mathcal{X}$  is actually an MI-NE of the MI-NEP in (1).

By construction of the regularization parameter sequence defined by (7), we have  $\tau^{k+1} \leq \tau^k$  for all  $k \geq 0$ . Then, for the sake of contradiction, assume that there exists some  $\bar{\tau} > 0$  such that  $\tau^k \geq \bar{\tau}$  for all  $k \geq 0$ . In view of the updating rule at the k-th iteration, for all  $i \in \mathcal{I}$  we have

$$J_{i}(\hat{x}_{i}(k)) - J_{i}(\hat{x}_{i+1}(k)) \geq \tau^{k} \rho_{i}(x_{i}^{k+1} - x_{i}^{k}) \\ \geq \bar{\tau} \rho_{i}(x_{i}^{k+1} - x_{i}^{k}) \geq 0.$$
(8)

By exploiting the definition of a generalized ordinal potential game provided in Definition 2, we hence deduce

$$P(\hat{x}_{i}(k)) - P(\hat{x}_{i+1}(k)) \\ \ge \phi(J_{i}(\hat{x}_{i}(k)) - J_{i}(\hat{x}_{i+1}(k))) \ge 0.$$
(9)

Therefore,  $P(\boldsymbol{x}^{k+1}) = P(\hat{\boldsymbol{x}}_{N+1}(k)) \leq P(\hat{\boldsymbol{x}}_N(k)) \leq \cdots \leq P(\hat{\boldsymbol{x}}_1(k)) = P(\boldsymbol{x}^k)$ , and hence the sequence  $\{P(\boldsymbol{x}^k)\}_{k\geq 0}$  is monotonically non-increasing. By the continuity of P, it follows that the full sequence  $\{P(\boldsymbol{x}^k)\}_{k\geq 0}$  is convergent to a finite value  $\bar{P}$ . Moreover, it follows from (9) that  $\lim_{k\to\infty} \phi(J_i(\hat{\boldsymbol{x}}_i(k)) - J_i(\hat{\boldsymbol{x}}_{i+1}(k))) = 0$ . By definition of the forcing function, we also have

$$\lim_{k \to \infty} (J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) = 0.$$

From (8), we obtain  $\lim_{k\to\infty}\rho_i(x_i^{k+1}-x_i^k)=0$ . Let  $r^k\to 0$  be such that  $\rho_i(x_i^{k+1}-x_i^k)\leq r^k$  for all k sufficiently large. By definition of an ICRF, we deduce  $\|x_i^{k+1}-x_i^k\|_1\leq s_i^{-1}(r^k)$ . We recall that the function  $s_i^{-1}(\cdot)$  is monotonically increasing, and therefore  $\lim_{k\to\infty}\|x_i^{k+1}-x_i^k\|_1=0$ . Consequently, for all k sufficiently large, it holds that  $d_\rho(\boldsymbol{x}^{k+1},\boldsymbol{x}^k)<\bar{\tau}$ . From (7), we have  $\tau^{k+1}=\max\{\omega\tau^k,d_\rho(\boldsymbol{x}^{k+1},\boldsymbol{x}^k)\}$ , and hence we need  $\tau^{k+1}=\omega\tau^k$  for all k. However, this implies that  $\tau^k\to 0$  at a geometric rate, thus contradicting the hypothesis that  $\tau^k\geq\bar{\tau}>0$  for all  $k\geq 0$  and concluding the first part of the proof.

Now, let  $\{x^k\}_{k\geq 0}$  be a convergent subsequence with accumulation point  $\bar{x} \in \mathcal{X}$ . The existence of such a convergent subsequence is guaranteed by the compactness of  $\mathcal{X}$ . By invoking the same arguments as in the first part of the proof, we obtain  $\lim_{k\to\infty} (J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) = 0$  and  $\lim_{k\to\infty} \|x_i^{k+1} - x_i^k\|_1 = 0$  for all  $i \in \mathcal{I}$ . Next, we show by

**Algorithm 2:** Proximal-like method for MI-NEPs with exact potential and inexact optimization

Set  $k \coloneqq 0$ , choose  $\boldsymbol{x}^k \in \boldsymbol{\mathcal{X}}, \, \tau^k > 0, \, \omega \in (0, 1)$  and error tolerance sequence  $\{\delta^k\}_{k \ge 0}$ 

While  $x^k$  is not satisfying a stopping criterion do

For all  $i \in \mathcal{I}$  do Obtain  $\hat{x}_i(k)$ If  $x_i^k \in \hat{\beta}_{i,\tau^k}(\hat{x}_i(k); \delta^k)$  then  $\begin{vmatrix} \text{Set } x_i^{k+1} = x_i^k \\ \text{Else} \\ & \text{Set } x_i^{k+1} \in \hat{\beta}_{i,\tau^k}(\hat{x}_i(k); \delta^k) \\ \text{End} \\ & \text{Update } \hat{x}_{i+1}(k) = (x_i^{k+1}, \hat{x}_{-i}(k)) \\ \text{End} \\ \text{Set } x^{k+1} = (x_1^{k+1}, \dots, x_N^{k+1}) \\ & \text{Update } \tau^{k+1} \text{ as in (7) and set } k \coloneqq k+1 \\ \text{End} \\ \text{End} \\ \end{bmatrix}$ 

contradiction that the accumulation point  $\bar{x} \in \mathcal{X}$  coincides with an MI-NE of (1) with generalized ordinal potential. To this end, let us suppose that there exists an agent  $i \in \mathcal{I}$  that can further minimize its cost function, i.e., for some  $y_i \in \mathcal{X}_i$ ,  $J_i(y_i, \bar{x}_{-i}) < J_i(\bar{x})$  holds. By relying on the update rule in Algorithm 1, we obtain

$$J_i(x_i^{k+1}, \boldsymbol{x}_{-i}^k) + \tau^k \rho_i(x_i^{k+1} - x_i^k) \\ \leq J_i(y_i, \boldsymbol{x}_{-i}^k) + \tau^k \rho_i(y_i - x_i^k).$$

Now, passing to the limit and using the fact that  $\tau^k \to 0$ ,

$$J_i(\bar{\boldsymbol{x}}) \leq J_i(y_i, \bar{\boldsymbol{x}}_{-i}) < J_i(\bar{\boldsymbol{x}}),$$

which denotes a contradiction, thus concluding the proof.

In contrast to the continuous case where a constant penalty parameter can be used (see, e.g., [22, Th. 4.3]), in a MI setting a vanishing regularization parameter is required. In addition, note that Algorithm 1 assumes that agents can compute an exact BR of the proximal augmented local cost function in (5). This requires that, at every single iteration, each agent solves a MI nonlinear optimization problem to optimality. Without additional structural assumptions (e.g., individual convexity), this could render the method inefficient in practice. This reason motivates us to investigate a variant of Algorithm 1 involving inexact computation of a point lying in the perturbed MI BR mapping (6) of each agent.

### B. Inexact BR computation in exact potential games

We now consider the case in which the MI-NEP in (1) admits an exact potential function. In this case, we can prove convergence even if the agents implement only *approximate* BRs at every iteration, according to the following definition:

Definition 4: ( $\delta$ -proximal BR) Given any  $x \in \mathcal{X}$  and tolerance  $\delta \geq 0$ ,  $y_i \in \mathcal{X}_i$  is an  $\delta$ -optimal response to  $x_{-i}$  if

$$\tilde{J}_{i,\tau}(y_i, x_i; \boldsymbol{x}_{-i}) \le \tilde{J}_{i,\tau}(z_i, x_i; \boldsymbol{x}_{-i}) + \delta$$

for all  $z_i \in \mathcal{X}_i$ .

For each agent  $i \in \mathcal{I}$ , we define  $\hat{\beta}_{i,\tau}(\boldsymbol{x}; \delta) \coloneqq \{y_i \in \mathcal{X}_i \mid \tilde{J}_{i,\tau}(y_i, x_i; \boldsymbol{x}_{-i}) \leq \tilde{J}_{i,\tau}(z_i, x_i; \boldsymbol{x}_{-i}) + \delta \text{ for all } z_i \in \mathcal{X}_i\}$  as the set of  $\delta$ -optimal responses, given some collective vector of strategies  $\boldsymbol{x}$ . For the theoretical developments of this subsection, we then make the following assumption.

Assumption 3: The game  $\Gamma = (\mathcal{I}, (J_i)_{i \in \mathcal{I}}, (\mathcal{X}_i)_{i \in \mathcal{I}})$  is an exact potential game with potential function  $P(\cdot)$ .

Algorithm 2 summarizes the main steps of the resulting distributed, Gauss–Seidel type sequence of iterations. For the considered instance, after choosing an initial strategy  $x^0 \in \mathcal{X}$  and a sequence of penalty parameters  $\{\tau^k\}_{k\geq 0}$ , the preliminary step requires to further define a certain error tolerance in computing a  $\delta$ -optimal response. To this end, we let  $\{\delta^k\}_{k\geq 0}$  be a given sequence of positive numbers such that  $\delta^k \to \epsilon$  for some  $\epsilon \geq 0$ . In Algorithm 2 we stick to the sequential update architecture, but instead of requiring that agents pursue an exact BR, we allow the updating agent to choose an inexact BR,  $x_i^{k+1} \in \hat{\beta}_{i,\tau^k}(\hat{x}_i(k); \delta^k)$ , only in case the inexact BR computed at the previous step, i.e., the one obtained by considering  $\hat{x}_i(k-1)$ ,  $\tau^{k-1}$  and  $\delta^{k-1}$ , does not belong to  $\hat{\beta}_{i,\tau^k}(\hat{x}_i(k); \delta^k)$ . The updated strategy is then sent to the agent down the line, and the procedure repeats as long as some stopping criterion is not met.

*Theorem 2:* Let Assumption 3 holds true. Consider an error sequence  $\{\delta^k\}_{k\geq 0}$  satisfying  $\delta^k \to \epsilon$  with  $\epsilon \geq 0$ . Then any accumulation point of the sequence  $\{x^k\}_{k\geq 0}$  generated by Algorithm 2 is an  $\epsilon$ -MI-NE of the game  $\Gamma$  in (1).  $\Box$ 

*Proof:* The proof makes use of similar arguments as the one of Theorem 1. As a starting point, by definition of the update  $x_i^{k+1}$ , we have

$$J_i(\hat{\boldsymbol{x}}_i(k)) - J_i(\hat{\boldsymbol{x}}_{i+1}(k)) \ge \tau^k \rho_i(x_i^{k+1} - x_i^k).$$

Since in this case P is an exact potential function, we have

$$P(\hat{x}_{i}(k)) - P(\hat{x}_{i+1}(k)) \ge \tau^{k} \rho_{i}(x_{i}^{k+1} - x_{i}^{k}) \ge 0,$$

where the last inequality uses the non-negativity of the ICRF. By summing from i = 1 to N, we hence obtain

$$P(\boldsymbol{x}^k) - P(\boldsymbol{x}^{k+1}) \ge 0.$$

In view of [28, Lemma 3.4], it follows that  $\lim_{k\to\infty} (P(\boldsymbol{x}^k) - \min_{\boldsymbol{x}\in\mathcal{X}}P(\boldsymbol{x}))$  exists and is finite. Therefore, since  $\mathcal{X} = \prod_{i\in\mathcal{I}}\mathcal{X}_i$  is compact, the sequence  $\{\boldsymbol{x}^k\}_{k\geq 0}$  admits a convergent subsequence  $\{\boldsymbol{x}^k\}_{k\in\mathcal{K}}$  with indices contained in some countable and infinite set  $\mathcal{K}$ , which has a limit point  $\bar{\boldsymbol{x}} \in \mathcal{X}$ . Therefore, in view of the continuity of the potential function, we have that  $\lim_{k\to\infty} P(\boldsymbol{x}^k) = P(\bar{\boldsymbol{x}})$  and  $\lim_{k\to\infty} (P(\boldsymbol{x}^{k+1}) - P(\boldsymbol{x}^k)) = 0$ . Thus, for all  $i \in \mathcal{I}$ ,

$$\lim_{k \to \infty} \left( P(\hat{\boldsymbol{x}}_{i+1}(k)) - P(\hat{\boldsymbol{x}}_i(k)) \right) = 0$$

By definition of the potential function, it follows that

$$\lim_{k \to \infty} (J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) = 0$$

TABLE I: Simulation parameters

Symbol Un	nit	Description	Value
$\begin{array}{c c} p_i & \in /!\\ m_i & \in /!\\ u_i^d & u_i^c\\ \tau^0 & \\ x_i^0 & \\ \{\delta^k\} & \\ \epsilon & \end{array}$	'good 'good	Selling price Marginal cost Upper bound (discrete) Upper bound (continuous) Regularization parameter Initial goods vector Error tolerance sequence Approximation tolerance	$ \begin{array}{l} \sim \mathcal{U}(10,20) \times 10^{3} \\ \sim \mathcal{U}(7,12) \times 10^{3} \\ \sim \mathcal{U}(200,400) \\ \sim \mathcal{U}(200,400) \\ 5000 \\ 0_{100} \\ \frac{10^{2} + (k^{2} - 1) \times 10^{-6}}{k^{2}} \end{array} $

Again, by exploiting the property of the ICRF in Definition 3.i), we obtain  $\lim_{k\to\infty,k\in\mathcal{K}} \|x_i^{k+1} - x_i^k\|_1 = 0$  for every  $i \in \mathcal{I}$ . It then follows  $\lim_{k\to\infty} x_i^k = \bar{x}_i$  for all  $i \in \mathcal{I}$ . Consequently, we deduce from the first part of the proof of Theorem 1 that also  $\lim_{k\to\infty} \tau^k = 0$ . We now claim that  $\bar{x} \in \mathcal{X}$  is an  $\epsilon$ -approximate Nash equilibrium and argue by contradiction. Suppose there exists an agent  $i \in \mathcal{I}$  such that, for some  $y_i \in \mathcal{X}_i$ ,  $J_i(y_i, \bar{x}_{-i}) + \epsilon < J_i(\bar{x})$ . Resorting to the definition of the update mechanism yields

$$J_i(x_i^{k+1}, \hat{x}_{-i}(k)) + \tau^k \rho_i(x_i^{k+1} - x_i^k) \\\leq J_i(y_i, \hat{x}_{-i}(k)) + \tau^k \rho_i(y_i - x_i^k) + \delta^k.$$

Then, passing to the limit and exploiting the fact that the regularization parameter tends to zero, i.e.,  $\lim_{k\to\infty} \tau^k = 0$ , and that  $\delta^k \to \epsilon$ , we finally obtain

$$J_i(\bar{\boldsymbol{x}}) \leq J_i(y_i, \hat{\boldsymbol{x}}_{-i}) + \epsilon < J_i(\bar{\boldsymbol{x}})$$

This represents a contradiction and concludes the proof.

Note that Algorithm 2 also covers the case in which the error sequence  $\{\delta^k\}_{k\geq 0}$  is not forced by the designer before implementing the procedure, but it naturally arises from, e.g., a learning process. For example, consider a setting in which the agents are endowed with typical learning procedures, such as Gaussian processes or a neural network. As long as  $\delta^k \rightarrow \epsilon$ , Algorithm 2 returns an  $\epsilon$ -approximate MI-NE. If the approximation error  $\delta^k$  vanishes with the iteration index, instead, then Algorithm 2 produces a convergent sequence of strategy profiles to an exact MI-NE. This assumption is not so stringent as it may seem. In the example considered above, i.e., agents endowed with learning procedures, asymptotic consistency bounds on the approximation error can be exploited directly [27].

### **IV. NUMERICAL EXPERIMENTS**

We now test our theoretical findings on a numerical instance of a classic Cournot oligopoly model [5], [11], [26].

Specifically, we consider a market in which N = 20 firms produce  $n_i = 100$  goods each to maximize their profits. Here, the first 50 products,  $x_i^d$ , are indivisible, while the other 50,  $x_i^c$ , are modeled with continuous variables and hence  $x_i = \operatorname{col}(x_i^d, x_i^c)$ . Thus, each firm  $i \in \mathcal{I}$  aims at solving the following MI quadratic program

$$\forall i \in \mathcal{I} : \begin{cases} \min_{x_i} & (m_i - p_i)^\top x_i + (\sum_{j \in \mathcal{I}} C_{i,j} x_j)^\top x_i \\ \text{s.t.} & x_i^d \in \{0, u_i^d\}^{50}, \ x_i^c \in [0, u_i^c]^{50}, \end{cases}$$
(10)

where the main parameters are described in Table I. Also, we define matrices  $C_{i,j} \in \mathbb{R}^{100 \times 100}$ , typically related with the costumers' inverse demand, according to the procedure described in [26, §4] for all  $j \ge i$ , and then we impose  $C_{j,i} = C_{i,j}^{\top}$ . This degree of symmetry ensures the existence of a potential function for the MI-NEP in (10), see, e.g., [9], which has the form

$$P(\boldsymbol{x}) = \sum_{i \in \mathcal{I}} \left( h_i(x_i) + \sum_{j \in \mathcal{I}, j < i} g(x_i, x_j) \right)$$

with  $h_i(x_i) \coloneqq (m_i - p_i)^\top x_i + x_i^\top C_{i,i} x_i$  and  $g(x_i, x_j) \coloneqq x_j^\top C_{j,i} x_i$ . In case the (l, q)-entry of matrix  $C_{i,j}$  is nonnegative, the q-th product of the firm j is a substitute for the l-th product of firm i. On the other hand, if that entry is negative then the q-th product of the firm j is a complement for the l-th product of firm i. This framework resembles the 2-groups partitionable class in [26]. As an ICRF for each agent, we adopted a piecewise affine approximation of the function  $\rho_i(t_i) = \sum_{p=1}^{100} (1 - e^{0.9|t_{i,p}|})$  applied componentwise  $(t_{i,p} = t_i)$  denotes the p-th element of  $t_i \coloneqq y_i - x_i)$ . This is in order to handle each problem in (10) with the solver used.

The numerical results reported in Figure 1 are obtained in Matlab by using Gurobi [29] as a solver on a laptop with a Quad-Core Intel Core i5 2.4 GHz CPU and 8 GB RAM. Specifically, we generate 50 random instances of the considered MI-NEP in (10), and test the behavior of Algorithm 1 and 2, where each agent takes, on average, 0.0788 s to compute a BR strategy. While Fig. 1(a) shows the averaged convergence behavior of the sequence of MI strategy profiles generated by Algorithm 1, which actually converge in less than 30 iterations, Fig. 1(b), instead, illustrates the averaged behavior of the sequence of sub-optimal MI strategy profiles generated by Algorithm 2, which converges to an  $10^{-6}$ approximate MI-NE of the Cournot model in (10). Note that the if-condition in the procedure generates a typical staircase behaviour and the convergence, in general, requires few more iterations. Finally, Fig. 1(c) illustrates the averaged (over the agents) evolution of  $\boldsymbol{\varrho}^k \coloneqq \operatorname{col}((\varrho_i^k)_{i \in \mathcal{I}})$ , which corresponds to the approximation error actually made in computing an inexact BR. Each  $\varrho_i^k$ , indeed, denotes the distance returned by the solver between the inexact and exact BR solutions made by agent i at the k-th iteration. The quantity  $\mathbf{1}^{\top} \boldsymbol{\varrho}^k / N$ is always upper bounded by the error sequence  $\{\delta^k\}_{k\geq 0}$ reported in Table I, as expected.

### V. CONCLUSION

We have presented two proximal-like equilibrium seeking algorithms for NEPs with mixed-integer variables admitting either generalized ordinal or exact potential functions. Exploiting the properties of integer-compatible regularization functions used as penalty terms in the agents' cost functions is key to prove convergence both in case the agents pursue an exact optimal strategy or an approximated one.

Future research directions include, but are not limited to, the extension of the proposed algorithms to generalized MI-NEPs as well as developing their stochastic counterparts. There are also interesting computational questions to investigate. From the proximal-point interpretation of the feasibility



Fig. 1: (a) Distance between the strategy profile  $x^k$  generated by Algorithm 1 and an MI-NE, averaged over the considered 50 randomly generated instances of (10); (b) Distance between the strategy profile  $x^k$  generated by Algorithm 2 and an MI-NE of the Cournot model in (10), averaged over the considered 50 randomly generated instances. The sequence of strategy profiles  $\{x^k\}_{k\geq 0}$  converges to a  $10^{-6}$ -approximate MI-NE; (c) Averaged evolution over k of the approximation error measured by the solver in computing an inexact BR,  $\rho^k$  (red line) – Algorithm 2. This quantity is upper bounded by the error tolerance sequence  $\{\delta^k\}_{k\geq 0}$  in Table I (blue line).

pump, large penalty parameters enforce integer restrictions. One possibility to approach this within the proposed equilibrium seeking algorithms is to design a double loop procedure with large penalties at the beginning of the scheme.

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