SYNCHRONIZATION OF REINFORCED STOCHASTIC PROCESSES WITH A NETWORK-BASED INTERACTION

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Randomly evolving systems composed by elements which interact among each other have always been of great interest in several scientific fields. This work deals with the *synchronization* phenomenon that could be roughly defined as the tendency of different components to adopt a common behavior. We continue the study of a model of *interacting stochastic processes with reinforcement* that recently has been introduced in [Crimaldi et al. (2016)]. Generally speaking, by reinforcement we mean any mechanism for which the probability that a given event occurs has an increasing dependence on the number of times that events of the same type occurred in the past. The particularity of systems of such interacting stochastic processes is that synchronization is induced *along time* by the reinforcement mechanism itself and does not require a large-scale limit. We focus on the relationship between the topology of the network of the interactions and the long-time synchronization phenomenon. After proving the almost sure synchronization, we provide some CLTs in the sense of stable convergence that establish the convergence rates and the asymptotic distributions for both convergence to the common limit and synchronization. The obtained results lead to the construction of asymptotic confidence intervals for the limit random variable and of statistical tests to make *inference on the topology of the network*.

CONTENTS

1. Introduction. The stochastic evolution of systems composed by elements which interact among each other has always been of great interest in several scientific fields: in neuroscience the brain is an active network where billions of neurons interact in various ways in the cellular circuits; many studies in biology focus on the interactions between different subsystems; social sciences and economics deal with individuals that take decisions under the influence of other individuals, and also in engineering and computer science some research questions regard dynamic agents that form a complex pattern of interactions (e.g., [8, 38, 44]). In all these frameworks, an usual phenomenon is the *synchronization* that could be roughly defined as the tendency of different components to adopt a common behavior (we refer to [7] for a detailed and well structured survey on this topic, rich of examples and references). Synchronization has been shown to be of special relevance in neural systems: the study of synchronization in neuronal networks of various level, especially dealing with the role played by the network topology, is crucial for the understanding of the brain functional activities. In social life, preferences and beliefs are partly transmitted by means of various forms of social interaction and opinions are driven by the tendency of individuals to become more similar when they interact. Hence, a collective phenomenon reflects the result of the interactions among different individuals. The underlying idea is that individuals have opinions that change according to the influence of other individuals giving rise to a sort of collective behavior, sometimes grouping together a part of the whole population with similar social attributes. Moreover, in economics, simple rules lead

to interesting collective behaviors and synchronization is one of them, since some of the activities done by individual agents can become correlated in time due to their interaction pattern. For example, the analysis of the International Trade Network (ITN), also known as the World Trade Web (WTW), which is the network related to the trade volume between countries, has revealed a tight relationship between the topology of the ITN and the dynamics of the Gross Domestic Products (GDPs) of the countries. Due to globalization effects, all economies are strongly correlated and they will tend to follow a common trend (what are usually called economic cycle). We can say that economies are synchronized in terms of the GDP. Finally, consensus problems, understood as the ability of a set of interacting dynamic agents to reach a unique and common value in an asymptotic stable state, play a crucial role also in engineering and computer science, particularly in automata theory. Therefore, it is clear that the main goals of different research areas are twofold: (i) to figure out whether and when a (complete or partial) synchronization in a dynamical system of agents can emerge out of initially different statuses and (ii) to understand the interplay between the network topology of the interactions among the agents and the dynamics followed by the agents.

In this paper, we continue the study of synchronization for a model of interacting stochastic processes with reinforcement that recently has been introduced in [21]. Generally speaking, by reinforcement in a stochastic dynamic we mean any mechanism for which the probability that a given event occurs has an increasing dependence on the number of times that events of the same type occurred in the past. The main reason of the attention devoted to reinforced stochastic processes is concerned with their dynamics, which is suitable to describe random phenomena in different scientific areas and can be easily implemented in several fields of application (see, e.g., [41] for a general survey). Our study is motivated by the attempt of understanding the role of the reinforcement mechanism in synchronization phenomena.

More precisely, a *Reinforced Stochastic Process* (RSP) can be defined as a stochastic process in which, along the time-steps, different events occur in such a way that for each event the greater the probability of occurrence at a certain time, the greater the probability of occurrence at the next time. Formally, given a finite set S, for any x in S, we have an S-dimensional stochastic process $X = [X(x) : x \in S]$ such that each component $X(x) = (X_n(x))_{n \geq 1}$ is a stochastic process with values in $\{0, 1\}$ and, for each $n \geq 0$, $\sum_{x \in S} X_n(x) = 1$ and

(1.1) $$P\big(X_{n+1}(x) = 1 | Z_0(x), X_1(x), \ldots, X_n(x)\big) = Z_n(x),$$

where

(1.2) $$Z_n(x) = (1 - r_{n-1})Z_{n-1}(x) + r_{n-1}X_n(x)$$

with $0 \leq r_{n-1} < 1$ and $Z_0(x)$ possibly random. Indeed, the process $X(x)$ describes the sequence of occurrences of the "event" $x \in S$ and, if at time n, the "event" x has

taken place, then the probability of its occurrence at time $(n + 1)$ increases. Therefore, the larger $Z_{n-1}(x)$, the higher the probability of having $Z_n(x)$ greater than $Z_{n-1}(x)$. This *"self-reinforcing property"*, also known as *"preferential attachment rule"*, is a key feature governing the dynamics of many biological, economic and social systems (e.g., [41]). The best known example of reinforced stochastic process is the standard Eggenberger–Pòlya urn [26, 36], which has been widely studied and generalized (some recent variants can be found in [3, 4, 6, 11, 12, 14, 16, 19, 27, 28, 31]). In the standard setting, an urn contains a red and b white balls and, at each discrete time, a ball is drawn out from the urn and then it is put again inside the urn together with one additional ball (or, more generally, with an additional constant number of balls) of the same color. In this case, we have $S = \{0, 1\}$ with 1 representing the color red and 0 the color white and for $Z_n = Z_n(1)$ and $X_n = X_n(1)$, we have

$$Z_n = \frac{a + \sum_{m=1}^{n} X_m}{a + b + n}.$$

It is immediate to verify that

$$Z_0 = \frac{a}{a + b} \quad \text{and} \quad Z_{n+1} = (1 - r_n)Z_n + r_n X_{n+1}$$

with $r_n = (a + b + n + 1)^{-1}$. As shown in [21] and as we will see in this paper, the asymptotic behavior of r_n is essential to determine the results presented in this paper. To this purpose, here we highlight that for the Eggenberger–Pólya urn we have $\lim_n n r_n = 1$. We refer to [21], Example 1.2, for a meaningful case of reinforced stochastic process of the type (1.1)–(1.2) where $\lim_n n^\gamma r_n = c$ with $\gamma < 1$ and $c \in (0, +\infty)$. This example concerns an opinion dynamics in an evolving population, modeled by a graph evolving according to preferential attachment [1, 38, 44].

Whenever S has only two elements, say $S = \{0, 1\}$, there are only two relevant variables, that is, $X_n = X_n(1)$ and $Z_n = Z_n(1)$, since $X_n(0) = 1 - X_n(1)$ and $Z_n(0) = 1 - Z_n(1)$. On the other hand, if S has more than two elements, for each $x \in S$, we can consider an equivalent two-dimensional reinforced stochastic processes $X' = [X'(0), X'(1)]$, with $X'(1) = X(x)$, $Z'(1) = Z(x)$, $X'(0) = \sum_{y \in S, y \neq x} X(y)$ and $Z'(0) = \sum_{y \in S, y \neq x} Z(y)$. Therefore, from now on we assume $S = \{0, 1\}$.

This paper deals with a system of N reinforced stochastic processes that interact according to a given set of relationships among them. More precisely, suppose to have a directed graph $G = (V, E)$ with $V = \{1, \ldots, N\}$ as the set of vertices and $E \subseteq V \times V$ as the set of edges. Each edge $(h, j) \in E$ represents the fact that the vertex h has a direct influence on the vertex j. We also associate a weight $w_{h,j} \geq 0$ to each pair $(h, j) \in V \times V$ in order to quantify how much h can influence j. A weight equal to zero means that the edge is not present. We set $W = [w_{h,j}]_{h,j \in V \times V}$ (*weighted adjacency matrix*) and we assume the weights

to be normalized so that $\sum_{h=1}^{N} w_{h,j} = 1$ for each $j \in V$. Hence, $w_{j,j}$ represents how much the vertex j is influenced by itself and $\sum_{h=1,h\neq j}^{N} w_{h,j} \in [0, 1]$ quantifies how much the vertex j is influenced by the other vertices of the graph. Finally, we suppose to have at each vertex j a reinforced stochastic process described by $X^j = (X_{n,j})_{n\geq 1}$ such that, for each $n \geq 0$, the random variables $\{X_{n+1,j} : j \in V\}$ take values in $\{0, 1\}$ and are conditionally independent given $\mathcal{F}_n$ with

$$(1.3) \qquad P(X_{n+1,j} = 1 | \mathcal{F}_n) = \sum_{h=1}^{N} w_{h,j} Z_{n,h},$$

where, for each $h \in V$,

$$(1.4) \qquad Z_{n,h} = (1 - r_{n-1}) Z_{n-1,h} + r_{n-1} X_{n,h}$$

with $0 \leq r_n < 1$, $Z_{0,h}$ random variables with values in $[0, 1]$ and $\mathcal{F}_n = \sigma(Z_{0,h} : h \in V) \vee \sigma(X_{k,j} : 1 \leq k \leq n, j \in V)$.

As an example, we can imagine that $G = (V, E)$ represents a network of N individuals that at each time-step have to make a choice between two possible alternatives $\{0, 1\}$. We can formalize this setting, assuming to have at each vertex j an urn with red and white balls. The color red represents the choice 1, the proportion $Z_{n,j}$ of red balls at time n in the urn at vertex j represents the inclination of the individual j to adopt the choice 1 at time n and the random variable $X_{n,j}$ represents the choice of j at time n. It is natural to assume a self-reinforcing property for the own inclination of each individual as in (1.4) and, moreover, it is natural to assume that the probability that the individual j will make the choice 1 at time $(n + 1)$ is given by a convex combination of j's own inclination and the inclination of the vertices that have an influence on j according to their weights $w_{k,j}$ as in (1.3). Another example is given by the interacting version of [21], Example 1.2, which could be interpreted as a network of different interacting populations or groups within a given population in the same spirit as in [17].

As already said at the beginning, our study deals with the synchronization phenomenon. More specifically, we prove the almost sure synchronization of the stochastic processes $\{(Z_{n,j})_n : 1 \leq j \leq N\}$ positioned at the vertices. The particularity of systems of interacting reinforced stochastic processes is that *synchronization is induced along time* by the reinforcement mechanism itself (independently of the fixed size N of the network), and so it does not require a large-scale limit (i.e., the limit for $N \to +\infty$), which is usual in statistical mechanics for the study of interacting particle systems. In particular, we focus on the relationship between the topology of the interactions and the long-time synchronization phenomenon: indeed, we show that the eigenvalues and eigenvectors of the weighted adjacency matrix W impact on the synchronization phenomenon. Our theoretical results provide the rates of synchronization and the second-order asymptotic distributions, in which the asymptotic variances have been expressed as functions of the parameters governing the reinforced dynamics and the eigenstructure of the weighted

adjacency matrix. These results lead to the construction of asymptotic confidence intervals for the common limit random variable of the processes $(Z_{n,j})_n$ and to the design of statistical tests to make inference on the topology of the interaction network given the observation of the processes $(Z_{n,j})_n$.

Regarding the literature review, we recall that interacting two-colors urns have been considered in [32, 33]. Their main results are proven when the probability of drawing a ball of a certain color is proportional to ρ^k, where $\rho > 1$ and k is the number of balls of this color. The interaction is of the mean-field type. More precisely, the interacting reinforcement mechanism is the following: at each step and for each urn draw a ball from either all the urns combined with probability p, or from the urn alone with probability $1 - p$, and add a new ball of the same color to the urn. The higher the interacting parameter p, the more memory is shared between the urns. The main results can be informally stated as follows: if $p \geq 1/2$, then all the urns fixate on the same color after a finite time, and if $p < 1/2$, then some urns fixate on a unique color and others keep drawing both colors. In [22, 25, 43], the authors consider interacting urns (precisely, [22] and [25] deal with Pólya urns and [43] regards Friedman's urns) in which the interaction can be defined again as of the mean-field type, but the reinforcement scheme is different from the previous one: indeed, the urns interact among each other through the average composition in the entire system, tuned by the interaction parameter α, and the probability of drawing a ball of a certain color is proportional to the number of balls of that color, rather than to its exponential, leading to quite different results. Synchronization and central limit theorems for the urn proportions have been proven for different values of the tuning parameter α, providing different convergence rates and asymptotic variances. In [21], the same mean-field interaction is adopted, but the analysis has been extended to the general class of reinforced stochastic processes, providing central limit theorems also in functional form. Differently from these works, the model proposed in [2] concerns with a system of generalized Friedman's urns with irreducible mean replacement matrices based on a general interaction structure, which includes the mean-field interaction as a special case. In particular, this interaction acts as follows: the probability to sample a certain color in each urn is a convex combination of the urn proportions of the entire system, and the weights of such combinations are gathered in the interacting matrix. Combining the information provided by the mean replacement matrices and by the interacting matrix, first- and second-order asymptotic results of the urn proportions have been established, from which synchronization phenomenon has not been observed. Moreover, the structure of the interacting matrix allows a decomposition in subsystems of urns evolving with different behaviors.

The present work have some issues in common with [21, 22] and [2], but at the same time some significant differences can be pointed out. In particular, we share with [2] a general interacting framework driven by the interacting matrix (here called weighted adjacency matrix). However, here we mainly consider irreducible interacting matrices, and hence the decomposition of the system in subgroups is

only sketched. Moreover, with respect to [2], we mainly study a class of stochastic processes for which we obtain synchronization toward a random variable. This class does not include the generalized Friedman's urns studied in [2]; while it includes urn models with not-irreducible mean replacement matrices, as Pólya urns. With [21], we share the main class of reinforced stochastic processes considered, which contains Pólya's urns also studied in [22]. However, with respect to [21, 22], we generalize the form of interaction since here we deal with a general weighted adjacency matrix instead of just the mean-field interaction. Indeed, the intent of this work is different from the one of the above papers: after proving synchronization and central limit theorems for some interesting cases, we focus on analyzing the interplay between the topology of the interaction network and the reinforced dynamics of the stochastic processes positioned at the vertices of the network, providing some statistical tools. On the other hand, we do not provide central limit theorems in functional form as in [21] (although it is possible to do it combining the results given here and the methods illustrated in [21]) and we do not cover some cases considered in [21, 22]. Also these cases are interesting for synchronization phenomena, but we decided to not include them in this paper since, as we will explain more deeply in the sequel, they lead to quite different asymptotic results and so we think that it is more appropriate to possibly deal with them separately.

Finally, we mention that in literature we can find other works concerning models of interacting urns that consider interacting mechanisms different from ours and are generally not focused on synchronization. For instance, the model studied in [37] describes a system of interacting units, modeled by Pólya urns, subject to perturbations and which occasionally break down. The authors consider a system of interacting Pólya urns arranged on a d-dimensional lattice. Each urn contains initially b black balls and 1 white ball. At each time step, an urn is selected and a ball is drawn from it: if the ball is white, a new white ball is added to the urn; if it is black a "fatal accident" occurs and the urn becomes unstable and it "topples" coming back to the initial configuration. The toppling mechanism involves also the nearby urns. In [40], a class of discrete time stochastic processes generated by interacting systems of reinforced urns is introduced and its asymptotic properties analyzed. Given a countable set of urns, at each time a ball is independently sampled from every urn in the system and in each urn a random number of balls of the same color of the extracted ball is added. The interaction arises since the number of added balls depends also on the colors generated by the other urns as well as on a common random factor. In [15], the authors consider a network of interacting urns displaced over a lattice. Every urn is Pólya-like and its reinforcement matrix is not only a function of time (time contagion) but also of the behavior of the neighboring urns (spatial contagion), and of a random component, which can represent either simple fate or the impact of exogenous factors. In this way, a non-trivial dependence structure among the urns is built, and the given construction is used to model different phenomena characterized by cascading failures such as power grids and financial networks. In [9, 13, 34], a graph-based model, with urns

at each vertex and pair-wise interactions, is considered. Given a finite connected graph, place a bin at each vertex. Two bins are called a pair if they share an edge. At discrete times, a ball is added to each pair of bins. In a pair of bins, one of the bins gets the ball with probability proportional to its current number of balls raised by some fixed power $\alpha > 0$. The authors characterize the limiting behavior of the proportion of balls in the bins for different values of the parameter α.

The rest of the paper is organized as follows. In Section 2, we introduce the notation, we describe the model and the leading assumptions. Section 3 is concerned with the main results established in the paper, while the relative proofs are gathered in Section 4. Some meaningful examples of reinforced stochastic processes with a network-based interaction are described in Section 5, in order to apply the theoretical results provided in the paper to some practical cases and to establish the corresponding asymptotic behaviors. In Section 6, we illustrate some statistical tools coming from the obtained theoretical results. In particular, we propose an inferential procedure to test the structure of the network which the interaction between the reinforced stochastic processes is based on. Finally, Section 7 is concerned with some possible variants of the model here presented. For the reader's convenience, the paper is also enriched by an exhaustive Appendix containing necessary definitions and technical results.

2. The model.   Throughout the paper, we will adopt the following notation:

(a) Given a complex number z, $\mathcal{R}e(z)$ and $\mathcal{I}m(z)$ denote its real and imaginary parts, respectively, $\overline{z}$ denotes its conjugate and $|z|$ its modulus.

(b) If A is a matrix with complex entries, then $\overline{A}$ denotes its conjugate, that is, the matrix whose entries are the conjugates of the entries of A, and $A^\top$ indicates its transpose. Moreover, we denote by $|A|$ the sum of the modulus of its entries so that, if A is equal to the row-column product of two matrices B, C, we have $|A| \leq |B||C|$. Finally, $\mathrm{Sp}(A)$ indicates its spectrum, that is, the set of all its eigenvalues repeated with multiplicity, and $\lambda_{\max}(A)$ indicates the subset of $\mathrm{Sp}(A)$ containing the eigenvalues with maximum real part, that is, $\lambda^* \in \lambda_{\max}(A)$ whenever $\mathcal{R}e(\lambda^*) = \max\{\mathcal{R}e(\lambda) : \lambda \in \mathrm{Sp}(A)\}$. Moreover, we will denote by I the identity matrix, whose dimension depends on the context.

(c) A vector $\mathbf{v}$ is considered as a matrix with a single column, and hence all the notation stated in (b) apply to $\mathbf{v}$. Moreover, $\|\mathbf{v}\|$ indicates the norm of the vector $\mathbf{v}$, that is, $\|\mathbf{v}\|^2 = \overline{\mathbf{v}}^\top \mathbf{v}$. Finally, we will denote by $\mathbf{1}$ and by $\mathbf{0}$ the vectors whose entries are all ones and all zeros, respectively.

We now present the model. Suppose to have a directed graph $G = (V, E)$ with $V = \{1, \ldots, N\}$ as the set of vertices and $E \subseteq V \times V$ as the set of edges. Each edge $(h, j) \in E$ represents the fact that the vertex h has a direct influence on the vertex j. We assume also to associate a weight $w_{h,j} \geq 0$ to each pair $(h, j) \in V \times V$ in order to quantify how much h can influence j. A weight equal to zero means

that the edge is not present. We set $W = [w_{h,j}]_{h,j \in V \times V}$ (*weighted adjacency matrix*) and we assume the weights to be normalized so that $\sum_{h=1}^{N} w_{h,j} = 1$ for each $j \in V$. Finally, we suppose to have at each vertex j a reinforced stochastic process described by $X^j = (X_{n,j})_{n \geq 1}$ such that, for each $n \geq 0$, the random variables $\{X_{n+1,j} : j \in V\}$ take values in $\{0, 1\}$ and are conditionally independent given $\mathcal{F}_n$ with

$$P(X_{n+1,j} = 1 | \mathcal{F}_n) = \sum_{h=1}^{N} w_{h,j} Z_{n,h},$$

where, for each $h \in V$,

$$Z_{n,h} = (1 - r_{n-1})Z_{n-1,h} + r_{n-1}X_{n,h}$$

with $0 \leq r_{n-1} < 1$ constants, $Z_{0,h}$ random variables with values in $[0, 1]$ and $\mathcal{F}_n = \sigma(Z_{0,h} : h \in V) \vee \sigma(X_{k,j} : 1 \leq k \leq n, j \in V)$.

To express the above dynamics in a compact form, let us define the vectors $\mathbf{X}_n = (X_{n,1}, \ldots, X_{n,N})^\top$ and $\mathbf{Z}_n = (Z_{n,1}, \ldots, Z_{n,N})^\top$. Hence, the dynamics can be expressed as follows:

$$E[\mathbf{X}_{n+1} | \mathcal{F}_n] = W^\top \mathbf{Z}_n,$$

where

(2.1) $$\mathbf{Z}_n = (1 - r_{n-1})\mathbf{Z}_{n-1} + r_{n-1}\mathbf{X}_n.$$

Moreover, the assumption about the normalization of the matrix W can be written as $W^\top \mathbf{1} = \mathbf{1}$.

Throughout all the paper, we assume that the following condition holds.

ASSUMPTION 2.1.   There exist two constants $c > 0$ and $1/2 < \gamma \leq 1$ such that

(2.2) $$\lim_{n \to \infty} n^\gamma r_n = c.$$

When $\gamma = 1$, for a particular case covered by our analysis, we will require a slightly stricter condition than (2.2), that is,

(2.3) $$nr_n - c = O(n^{-1}).$$

We will explicitly mention this assumption in the statement of the theorems when it is required.

This paper is concerned with the case $1/2 < \gamma \leq 1$, while the case $\gamma \leq 1/2$ is not considered. Indeed, in [21] it was established that, under soft assumptions on the initial distribution, if the mean-field interaction is present and $\sum_n r_n^2 = +\infty$, then all the stochastic processes $\{(Z_{n,j})_n : 1 \leq j \leq N\}$ converge almost surely to

the same random variable $Z_\infty \in \{0, 1\}$ a.s. Hence, although this case is interesting for synchronization, we decided to focus here on the case $1/2 < \gamma \le 1$, for which soft assumptions on the initial distribution lead to a limit random variable not concentrated only on $\{0, 1\}$.

In addition, thoughout the paper, we assume that the following condition holds.

ASSUMPTION 2.2. *The weighted adjacency matrix W is irreducible and diagonalizable.*

The irreducibility of W reflects a situation in which all the vertices are connected among each others, and hence there are no subsystems with independent dynamics (see [2] and Section 7.2 for further details). The diagonalizability of W guarantees the existence of a nonsingular matrix $\widetilde{U}$ such that $\widetilde{U}^\top W (\widetilde{U}^\top)^{-1}$ is diagonal with elements $\lambda_j \in \text{Sp}(W)$. Notice that each column $\mathbf{u}_j$ of $\widetilde{U}$ is a left eigenvector of W associated to λ_j. Without loss of generality, we set $\|\mathbf{u}_j\| = 1$. Moreover, when the multiplicity of some λ_j is bigger than one, we set the corresponding eigenvectors to be orthogonal. Then, if we define $\widetilde{V} = (\widetilde{U}^\top)^{-1}$, we have that each column $\mathbf{v}_j$ of $\widetilde{V}$ is a right eigenvector of W associated to λ_j such that

$$(2.4) \qquad \mathbf{u}_j^\top \mathbf{v}_j = 1, \quad \text{and} \quad \mathbf{u}_h^\top \mathbf{v}_j = 0, \qquad \forall h \ne j.$$

These constraints combined with the above assumptions on W (precisely, $w_{h,j} \ge 0$, $W^\top \mathbf{1} = \mathbf{1}$ and the irreducibility) imply, by the Frobenius–Perron theorem, the following proposition.

PROPOSITION 2.1. *The eigenvalue $\lambda_1 := 1$ of W has multiplicity one,* $\lambda_{\max}(W) = \{1\}$ *and*

$$(2.5) \qquad \begin{aligned} \mathbf{u}_1 &= N^{-1/2}\mathbf{1}, \qquad N^{-1/2}\mathbf{1}^\top \mathbf{v}_1 = 1 \quad \text{and} \\ [\mathbf{v}_1]_j &:= v_{1,j} \in (0, +\infty) \qquad \forall j = 1, \dots, N. \end{aligned}$$

Finally, throughout all the paper, we will use U and V to indicate the submatrices of $\widetilde{U}$ and $\widetilde{V}$, respectively, whose columns are the left and the right eigenvectors of W associated to $\text{Sp}(W) \setminus \{1\}$, that is, $\{\mathbf{u}_2, \dots, \mathbf{u}_N\}$ and $\{\mathbf{v}_2, \dots, \mathbf{v}_N\}$, respectively, and we will denote by λ^* an eigenvalue belonging to $\text{Sp}(W) \setminus \{1\}$ such that

$$\mathcal{R}e(\lambda^*) = \max\{\mathcal{R}e(\lambda_j) : \lambda_j \in \text{Sp}(W) \setminus \{1\}\}.$$

We will see throughout the paper that the vector $\mathbf{v}_1$, defined and characterized above, will play a key role in order to establish the synchronization result.

3. Main results. In this section we present our main results, which regard the asymptotic behavior of the process $\mathbf{Z}_n$. We refer to the Appendix for a brief review of the notion of stable convergence.

Let us recall the assumptions stated in Section 2. We start by providing a first-order asymptotic result concerning the almost sure convergence of $\mathbf{Z}_n$.

THEOREM 3.1 (Synchronization). *There exists a random variable Z_∞ with values in $[0, 1]$ such that*

$$\mathbf{Z}_n \xrightarrow{a.s.} Z_\infty \mathbf{1}.$$

This result states that the stochastic processes $\{(Z_{n,j})_n : 1 \leq j \leq N\}$ located at the different vertices synchronize, that is, all of them converge almost surely toward the same random variable Z_∞. It is interesting to note that this result holds true without any assumption on the initial configuration $\mathbf{Z}_0$ and for any choice of the weighted adjacency matrix W with the required assumptions.

We now focus on the second-order asymptotic results concerning the process $(\mathbf{Z}_n)_n$. First, we present a central limit theorem in the sense of stable convergence that establishes the rate of convergence to the limit $Z_\infty \mathbf{1}$ determined in Theorem 3.1 and the relative asymptotic random variance.

THEOREM 3.2 (CLT for convergence). *The following hold*:

(a) *For $1/2 < \gamma < 1$, then*

$$n^{\gamma - \frac{1}{2}}(\mathbf{Z}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N}\big(\mathbf{0}, Z_\infty(1 - Z_\infty)\widetilde{\Sigma}_\gamma\big) \qquad \textit{stably},$$

where

$$(3.1) \qquad \widetilde{\Sigma}_\gamma := \widetilde{\sigma}_\gamma^2 \mathbf{1}\mathbf{1}^\top \quad \textit{and} \quad \widetilde{\sigma}_\gamma^2 := \frac{c^2 \|\mathbf{v}_1\|^2}{N(2\gamma - 1)} > 0.$$

(b) *For $\gamma = 1$, if $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$, then*

$$\sqrt{n}(\mathbf{Z}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N}\big(\mathbf{0}, Z_\infty(1 - Z_\infty)(\widetilde{\Sigma}_1 + \widehat{\Sigma}_1)\big) \qquad \textit{stably},$$

where $\widetilde{\Sigma}_1$ is defined as in (3.1) with $\gamma = 1$,

$$(3.2) \qquad \begin{aligned} &\widehat{\Sigma}_1 := U\widehat{S}_1 U^\top \quad \textit{and} \\ &[\widehat{S}_1]_{h,j} := \frac{c^2}{2c - c(\lambda_h + \lambda_j) - 1}(\mathbf{v}_h^\top \mathbf{v}_j) \qquad \textit{with } 2 \leq h, j \leq N. \end{aligned}$$

(c) *For $\gamma = 1$, if $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$ and (2.3) holds, then*

$$\frac{\sqrt{n}}{\sqrt{\ln(n)}}(\mathbf{Z}_n - Z_\infty \mathbf{1}) \longrightarrow \mathcal{N}\big(\mathbf{0}, Z_\infty(1 - Z_\infty)\widehat{\Sigma}_1^*\big) \qquad \textit{stably},$$

where

$$\widehat{\Sigma}_1^* := U \widehat{S}_1^* U^\top \quad and$$

(3.3)

$$[\widehat{S}_1^*]_{h,j} := \begin{cases} c^2(\mathbf{v}_h^\top \mathbf{v}_j) & if \ \lambda_h + \lambda_j = 2 - c^{-1}, \\ 0 & if \ \lambda_h + \lambda_j \neq 2 - c^{-1}, \end{cases} \qquad with \ 2 \leq h, j \leq N.$$

Notice that the matrix $\widehat{S}_1^*$ defined in (3.3) can never be null, as stated more ahead in Theorem 3.4.

REMARK 3.1. Notice that $\widetilde{\sigma}_\gamma^2$ is decreasing with the size N of the network and so, for cases (a) and (b), the larger the size of the network, the lower the asymptotic variance. Moreover, fixed N and γ, since by (2.4) and (2.5) we have $\|\mathbf{v}_1\|^2 = \|\mathbf{u}_1 + (\mathbf{v}_1 - \mathbf{u}_1)\|^2 = 1 + \|\mathbf{v}_1 - \mathbf{u}_1\|^2 \geq 1$ and $\|\mathbf{v}_1\|^2 \leq N$, we can obtain the following lower and upper bounds for $\widetilde{\sigma}_\gamma^2$ (not depending on W):

$$\frac{c^2}{N(2\gamma - 1)} \leq \widetilde{\sigma}_\gamma^2 \leq \frac{c^2}{(2\gamma - 1)},$$

where the lower bound is achieved when $\mathbf{v}_1 = \mathbf{u}_1$, that is, when W is doubly stochastic.

Given the long-run synchronization stated in Theorem 3.1, it is interesting to establish the rate of synchronization, that is, the convergence rate of the difference $(Z_{n,h} - Z_{n,j})_n$ to zero for $h \neq j$ and to characterize the relative asymptotic distribution. The following result achieves this goal.

THEOREM 3.3 (CLT for synchronization). *For any* $h, j \in \{1, \ldots, N\}$, $h \neq j$, *we have*:

(a) *For* $1/2 < \gamma < 1$, *then*

$$n^{\frac{\gamma}{2}}(Z_{n,h} - Z_{n,j}) \longrightarrow \mathcal{N}(0, Z_\infty(1 - Z_\infty)\Sigma_{\gamma,h,j}) \qquad stably,$$

where $\Sigma_{\gamma,h,j} := [\widehat{\Sigma}_\gamma]_{h,h} + [\widehat{\Sigma}_\gamma]_{j,j} - 2[\widehat{\Sigma}_\gamma]_{h,j}$,

(3.4)
$$\widehat{\Sigma}_\gamma := U \widehat{S}_\gamma U^\top \quad and$$

$$[\widehat{S}_\gamma]_{h,j} := \frac{c}{2 - (\lambda_h + \lambda_j)}(\mathbf{v}_h^\top \mathbf{v}_j) \qquad with \ 2 \leq h, j \leq N.$$

(b) *For* $\gamma = 1$, *if* $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$, *then*

$$\sqrt{n}(Z_{n,h} - Z_{n,j}) \longrightarrow \mathcal{N}(0, Z_\infty(1 - Z_\infty)\Sigma_{1,h,j}) \qquad stably,$$

where $\Sigma_{1,h,j} := [\widehat{\Sigma}_1]_{h,h} + [\widehat{\Sigma}_1]_{j,j} - 2[\widehat{\Sigma}_1]_{h,j}$ *and* $\widehat{\Sigma}_1$ *is defined in* (3.2).

(c) *For $\gamma = 1$, if $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$ and (2.3) holds, then*

$$\frac{\sqrt{n}}{\sqrt{\ln(n)}}(Z_{n,h} - Z_{n,j}) \longrightarrow \mathcal{N}(0, Z_\infty(1 - Z_\infty)\Sigma^*_{1,h,j}) \qquad \text{stably},$$

where $\Sigma^*_{1,h,j} := [\widehat{\Sigma}^*_1]_{h,h} + [\widehat{\Sigma}^*_1]_{j,j} - 2[\widehat{\Sigma}^*_1]_{h,j}$ and $\widehat{\Sigma}^*_1$ is defined in (3.3).

REMARK 3.2.   In the particular case when W is symmetric, the eigenvectors of W are real, $U = V$ and $V^\top V = I$. As a consequence, the matrices $\widehat{S}_\gamma$, $\widehat{S}_1$ and $\widehat{S}^*_1$ are diagonal, with elements $c[2(1 - \lambda_j)]^{-1}$, $c[2(1 - \lambda_j) - c^{-1}]^{-1}$ and $c^2 \mathbb{1}_{\{\lambda_j = 1 - (2c)^{-1}\}}$, respectively, where $\lambda_j \in \text{Sp}(W) \setminus \{1\}$. Moreover, in this case, we have $\mathbf{u}_1 = \mathbf{v}_1 = N^{-1/2}\mathbf{1}$ and $UU^\top = UV^\top = (I - N^{-1}\mathbf{1}\mathbf{1}^\top)$ (see Section 4.1 for details). Notice that, for instance, this is the case of undirected graphs.

In order to ensure that Theorem 3.2 and 3.3 provide the right convergence rates of $(Z_{n,j})_n$ to Z_∞ and of $(Z_{n,h} - Z_{n,j})_n$ to zero, respectively, we prove that $[\widehat{\Sigma}_1]_{j,j} \geq 0$, $[\widehat{\Sigma}^*_1]_{j,j} > 0$, $\Sigma_{\gamma,h,j} > 0$, $\Sigma_{1,h,j} > 0$, $\Sigma^*_{1,h,j} > 0$ and

$$(3.5) \qquad P(Z_\infty = 0) + P(Z_\infty = 1) < 1.$$

The result below deals with the first set of conditions.

THEOREM 3.4.   *We have*:

(a) *For $1/2 < \gamma < 1$, $\widehat{\Sigma}_\gamma$ is a positive semidefinite real matrix of rank $(N - 1)$ and $\mathbf{v}_1^\top \widehat{\Sigma}_\gamma \mathbf{v}_1 = 0$; in addition, $\Sigma_{\gamma,h,j} > 0$ for any $1 \leq h \neq j \leq N$.*
(b) *For $\gamma = 1$, if $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$, then $\widehat{\Sigma}_1$ is a positive semidefinite real matrix of rank $(N - 1)$ and $\mathbf{v}_1^\top \widehat{\Sigma}_1 \mathbf{v}_1 = 0$; in addition, $\Sigma_{1,h,j} > 0$ for any $1 \leq h \neq j \leq N$.*
(c) *For $\gamma = 1$, if $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$ and (2.3) holds, define*

$$(3.6) \qquad A^* := \{\lambda_j \in \text{Sp}(W), \mathcal{R}e(\lambda_j) = 1 - (2c)^{-1}\}$$

and let m^* be the cardinality of A^*; then $\widehat{\Sigma}^*_1$ is a positive semidefinite real matrix of rank m^* and $\mathbf{v}_j^\top \widehat{\Sigma}^*_1 \mathbf{v}_j = 0$ for any j such that $\lambda_j \notin A^*$; moreover, $[\widehat{\Sigma}^*_1]_{jj} > 0$ when $u_{h,j} \neq 0$ for some h such that $\lambda_h \in A^*$ and $\Sigma^*_{1,h,j} > 0$ when $u_{k,h} \neq u_{k,j}$ for some k such that $\lambda_k \in A^*$.

Finally, we give two results concerning the distribution of Z_∞, of which the last one deals with condition (3.5).

THEOREM 3.5.   *We have $P(Z_\infty = z) = 0$ for any $z \in (0, 1)$.*

THEOREM 3.6.   *If we have*

$$(3.7) \qquad P\left(\bigcap_{j=1}^{N}\{Z_{0,j}=0\}\right) + P\left(\bigcap_{j=1}^{N}\{Z_{0,j}=1\}\right) < 1,$$

then condition (3.5) *is verified.*

REMARK 3.3.   In case (a), that is, $1/2 < \gamma < 1$, since $\gamma/2 > \gamma - 1/2$ we have that the rate at which two stochastic processes $(Z_{n,h})_n$, $(Z_{n,j})_n$ positioned in any pair of different vertices (h, j) of the network synchronize is greater than the rate at which they converge to Z_∞, that is, synchronization of the stochastic processes at the vertices is faster then their convergence to the limit random variable.

REMARK 3.4.   The main goal of this work is to provide results for a system of N interacting reinforced stochastic processes. Therefore, we take $N \geq 2$ everywhere. However, it is worth to note that Theorems 3.1, 3.5 and 3.6 hold true for $N = 1$ without any changes; while, when $N = 1$, Theorem 3.2(a) holds true in both cases $1/2 < \gamma < 1$ and $\gamma = 1$. The proofs are analogous to the ones given in the sequel (with some simplifications due to the fact that $N = 1$).

4. Proofs.   This section contains all the proofs of the results presented in the previous Section 3.

4.1. *Preliminary relations and basic idea.*   We start by recalling that, given the eigenstructure of W described in Section 2, the matrix $\mathbf{u}_1\mathbf{v}_1^\top$ has real entries and the following relations hold:

$$(4.1) \quad V^\top\mathbf{u}_1 = U^\top\mathbf{v}_1 = \mathbf{0}, \qquad V^\top U = U^\top V = I \quad \text{and} \quad I = \mathbf{u}_1\mathbf{v}_1^\top + UV^\top,$$

which implies that the matrix $UV^\top$ has real entries [notice that in (4.1) the identity matrices have different dimensions]. Moreover, denoting by D the diagonal matrix whose elements are $\lambda_j \in \mathrm{Sp}(W) \setminus \{1\}$, we can decompose the matrix $W^\top$ as follows:

$$(4.2) \qquad\qquad W^\top = \mathbf{u}_1\mathbf{v}_1^\top + UDV^\top.$$

Now, in order to understand the asymptotic behavior of the stochastic process $(\mathbf{Z}_n)_n$, let us express the dynamics (2.1) as follows:

$$(4.3) \qquad\qquad \mathbf{Z}_{n+1} - \mathbf{Z}_n = -r_n(I - W^\top)\mathbf{Z}_n + r_n\Delta\mathbf{M}_{n+1},$$

where $\Delta\mathbf{M}_{n+1} = (\mathbf{X}_{n+1} - W^\top\mathbf{Z}_n)$ is a martingale increment with respect to the filtration $\mathcal{F} := (\mathcal{F}_n)_n$. It follows that:

(a) since $\mathbf{v}_1^\top W^\top = (W\mathbf{v}_1)^\top = \mathbf{v}_1^\top$, we have $\mathbf{v}_1^\top(I - W^\top) = \mathbf{0}$ and so, from (4.3), we deduce that the stochastic process $(\mathbf{v}_1^\top\mathbf{Z}_n)_n$ is a bounded real martingale;

(b) by (4.1), we have $\mathbf{Z}_n - \mathbf{u}_1(\mathbf{v}_1^\top \mathbf{Z}_n) = UV^\top \mathbf{Z}_n$ and so the dynamics of this multi-dimensional real stochastic process can be easily obtained from (4.3).

Hence, the basic idea is to decompose $\mathbf{Z}_n$ into two terms, establish the corresponding asymptotic results for each term separately and then combine them together to characterize the asymptotic behavior of $\mathbf{Z}_n$. More precisely, the process $\mathbf{Z}_n$ can be decomposed as follows:

$$\mathbf{Z}_n = \widetilde{Z}_n \mathbf{1} + \widehat{\mathbf{Z}}_n = \mathbf{u}_1 \sqrt{N} \widetilde{Z}_n + \widehat{\mathbf{Z}}_n,$$

(4.4)

$$\text{where } \begin{cases} \widetilde{Z}_n = N^{-1/2} \mathbf{v}_1^\top \mathbf{Z}_n, \\ \widehat{\mathbf{Z}}_n = \mathbf{Z}_n - \mathbf{1}\widetilde{Z}_n = (I - \mathbf{u}_1 \mathbf{v}_1^\top)\mathbf{Z}_n = UV^\top \mathbf{Z}_n. \end{cases}$$

Then the asymptotic behavior of the stochastic process $(\mathbf{Z}_n)_n$ is obtained by establishing the asymptotic behavior of $(\widetilde{Z}_n)_n$ and $(\widehat{\mathbf{Z}}_n)_n$.

REMARK 4.1.   In the particular case of W doubly stochastic, we have $\mathbf{v}_1 = \mathbf{u}_1 = N^{-1/2}\mathbf{1}$. As a consequence, we have

$$\widetilde{Z}_n = N^{-1}\mathbf{1}^\top \mathbf{Z}_n = N^{-1} \sum_{j=1}^{N} Z_{n,j},$$

which represents the average of the stochastic processes in the network, and $\widehat{\mathbf{Z}}_n = (I - N^{-1}\mathbf{1}\mathbf{1}^\top)\mathbf{Z}_n$. Notice that the assumed normalization $W^\top \mathbf{1} = \mathbf{1}$ implies that symmetric matrices W are also doubly stochastic. Therefore, the above equalities hold for any undirected graph for which W is obviously symmetric by definition.

4.2. *Proof of Theorem* 3.1 (*synchronization*).   By decomposition (4.4), that is,

$$\mathbf{Z}_n = \widetilde{Z}_n \mathbf{1} + \widehat{\mathbf{Z}}_n,$$

the proof of Theorem 3.1 follows by establishing the following two results:

(i) $\widetilde{Z}_n \xrightarrow{\text{a.s.}} Z_\infty$,

(ii) $\widehat{\mathbf{Z}}_n \xrightarrow{\text{a.s.}} \mathbf{0}$.

Concerning part (i), let us consider the real-valued stochastic process $(\widetilde{Z}_n)_n$ defined for any $n \geq 0$ as $\widetilde{Z}_n = N^{-1/2}\mathbf{v}_1^\top \mathbf{Z}_n$. Since all the elements of $\mathbf{v}_1$ are positive and since (2.5) holds, the elements of $N^{-1/2}\mathbf{v}_1$ can be seen as the weights of a convex combination, and hence $\min_j\{Z_{n,j}\} \leq \widetilde{Z}_n \leq \max_j\{Z_{n,j}\}$ for any n, which implies $0 \leq \widetilde{Z}_n \leq 1$. Moreover, it is easy to see that $(\widetilde{Z}_n)_n$ is an $\mathcal{F}$-martingale, since from (4.3) its dynamics can be expressed as follows:

$$(4.5) \qquad \widetilde{Z}_{n+1} - \widetilde{Z}_n = N^{-1/2} r_n (\mathbf{v}_1^\top \Delta \mathbf{M}_{n+1}).$$

Hence, we immediately get

$$(4.6) \qquad\qquad\qquad \widetilde{Z}_n \xrightarrow{\text{a.s.}} Z_\infty,$$

where Z_∞ is a random variable with values in $[0, 1]$. This concludes the proof of part (i).

Concerning part (ii), let us consider the multi-dimensional stochastic process $(\widehat{\mathbf{Z}}_n)_n$, with real entries, defined in (4.4). In order to find the dynamics of this process, we first observe that, by decomposition (4.4) and the fact that $W^\top \mathbf{u}_1 = (\mathbf{u}_1^\top W)^\top = \mathbf{u}_1$, we have

$$(I - W^\top)\mathbf{Z}_n = (I - W^\top)(\mathbf{u}_1\sqrt{N}\widetilde{Z}_n + \widehat{\mathbf{Z}}_n) = (I - W^\top)\widehat{\mathbf{Z}}_n$$

and so the dynamics (4.3) of $\mathbf{Z}_n$ can be rewritten as

(4.7) $$\mathbf{Z}_{n+1} - \mathbf{Z}_n = -r_n(I - W^\top)\widehat{\mathbf{Z}}_n + r_n\Delta\mathbf{M}_{n+1}.$$

Then, if we multiply the dynamics (4.7) by $UV^\top$ and use decomposition (4.2) and the relations (4.1), we obtain

$$\widehat{\mathbf{Z}}_{n+1} - \widehat{\mathbf{Z}}_n = -r_n[UV^\top - UV^\top(\mathbf{u}_1\mathbf{v}_1^\top + UDV^\top)]\widehat{\mathbf{Z}}_n + r_nUV^\top\Delta\mathbf{M}_{n+1}$$

(4.8) $$= -r_n(UV^\top - UDV^\top)\widehat{\mathbf{Z}}_n + r_nUV^\top\Delta\mathbf{M}_{n+1}$$

$$= -r_nU(I - D)V^\top\widehat{\mathbf{Z}}_n + r_nUV^\top\Delta\mathbf{M}_{n+1},$$

where I in (4.8) is a $(N-1) \times (N-1)$-identity matrix. We are now ready for proving that this multi-dimensional stochastic process converges a.s. to $\mathbf{0}$.

THEOREM 4.1. *We have*

$$\widehat{\mathbf{Z}}_n \xrightarrow{a.s.} \mathbf{0}.$$

PROOF. Let us consider the $(N-1)$-dimensional complex random vector defined as $\mathbf{Z}_{V,n} = V^\top\widehat{\mathbf{Z}}_n$. Since we have $\widehat{\mathbf{Z}}_n = U\mathbf{Z}_{V,n}$ by (4.1), it is enough to prove that $\mathbf{Z}_{V,n}$ converges almost surely to $\mathbf{0}$. To this purpose, we observe that the dynamics of $\mathbf{Z}_{V,n}$ can be obtained from (4.8) multiplying by $V^\top$:

$$\mathbf{Z}_{V,n+1} = (I - r_n(I - D))\mathbf{Z}_{V,n} + r_nV^\top\Delta\mathbf{M}_{n+1},$$

where I here indicates a $(N-1) \times (N-1)$-identity matrix. Hence, recalling that $E[\Delta\mathbf{M}_{n+1}|\mathcal{F}_n] = 0$, we obtain

$$E[\|\mathbf{Z}_{V,n+1}\|^2|\mathcal{F}_n] = E[\overline{\mathbf{Z}}_{V,n+1}^\top\mathbf{Z}_{V,n+1}|\mathcal{F}_n]$$

$$= \overline{\mathbf{Z}}_{V,n}^\top(I - r_n(I - \overline{D}))(I - r_n(I - D))\mathbf{Z}_{V,n}$$

$$+ r_n^2E[\Delta\mathbf{M}_{n+1}^\top\overline{V}V^\top\Delta\mathbf{M}_{n+1}|\mathcal{F}_n]$$

$$= \overline{\mathbf{Z}}_{V,n}^\top\mathbf{Z}_{V,n} - r_n\overline{\mathbf{Z}}_{V,n}^\top(2I - \overline{D} - D)\mathbf{Z}_{V,n} + r_n^2\xi_n,$$

where $(\xi_n)_n$ is a suitable bounded sequence of $\mathcal{F}_n$-measurable random variables. Since $\mathcal{R}e(\lambda_j) < 1$ for any $\lambda_j \in \mathrm{Sp}(W) \setminus \{1\}$, the matrix $2I - (\overline{D} + D)$ is positive definite, and hence we can write

$$E[\|\mathbf{Z}_{V,n+1}\|^2|\mathcal{F}_n] \leq \|\mathbf{Z}_{V,n}\|^2 + O(r_n^2).$$

Since $\sum_n r_n^2 < +\infty$ for $1/2 < \gamma \leq 1$, we can conclude that the real stochastic process $(\|\mathbf{Z}_{V,n}\|^2)_n$ is a positive almost supermartingale and so (see [42]) it converges almost surely (and in mean since it is also bounded). In order to prove that the limit is zero, it is enough to prove that $E[\|\mathbf{Z}_{V,n}\|^2]$ converges to zero. To this end, we observe that, from the above computations, we obtain

$$E[\|\mathbf{Z}_{V,n+1}\|^2] = E[\overline{\mathbf{Z}}_{V,n}^\top(I - r_n(I - \overline{D}))(I - r_n(I - D))\mathbf{Z}_{V,n}]$$
$$+ r_n^2 E[\Delta\mathbf{M}_{n+1}^\top \overline{V} V^\top \Delta\mathbf{M}_{n+1}]$$
$$\leq E[\overline{\mathbf{Z}}_{V,n}^\top(I - r_n(I - \overline{D}))(I - r_n(I - D))\mathbf{Z}_{V,n}] + C_1 r_n^2$$

for a suitable constant $C_1 \geq 0$. Then we note that the elements of the diagonal matrix above can be written as follows:

$$[(I - r_n(I - \overline{D}))(I - r_n(I - D))]_{jj} = 1 - 2r_n(1 - \mathcal{R}e(\lambda_j)) + r_n^2|1 - \lambda_j|^2.$$

Hence, setting $a_j = 1 - \mathcal{R}e(\lambda_j)$ and $a^* = \min_j\{a_j\} = 1 - \mathcal{R}e(\lambda^*)$ [we recall that λ^* indicates an eigenvalue belonging to $\lambda_{\max}(D)$], we have that

$$E[\overline{\mathbf{Z}}_{V,n}^\top(I - r_n(I - \overline{D}))(I - r_n(I - D))\mathbf{Z}_{V,n}]$$
$$\leq \sum_{j=2}^N (1 - 2a_j r_n)E[\overline{Z}_{V,n}^j Z_{V,n}^j] + C_2 r_n^2$$
$$\leq (1 - 2a^* r_n)E[\|\mathbf{Z}_{V,n}\|^2] + C_2 r_n^2$$

for a suitable constant $C_2 \geq 0$. Then, setting $x_n := E[\|\mathbf{Z}_{V,n}\|^2]$, we can write

$$x_{n+1} \leq (1 - 2a^* r_n)x_n + (C_1 + C_2)r_n^2.$$

Since $\mathcal{R}e(\lambda^*) < 1$, we have $a^* > 0$, which implies $\lim_n x_n = 0$ (see [21]). The proof is thus concluded. $\square$

Note that, by the synchronization result given in Theorem 3.1, we can state that

$$E[(\Delta\mathbf{M}_{n+1})(\Delta\mathbf{M}_{n+1})^\top|\mathcal{F}_n] \xrightarrow{\text{a.s.}} Z_\infty(1 - Z_\infty)I.$$

Indeed, since $\{X_{n+1,j} : j = 1, \ldots, N\}$ are conditionally independent given $\mathcal{F}_n$, we have

$$(4.9) \qquad E[\Delta M_{n+1,h}\Delta M_{n+1,j}|\mathcal{F}_n] = 0 \qquad \text{for } h \neq j;$$

while, for each j, we have

$$(4.10) \qquad E[(\Delta M_{n+1,j})^2|\mathcal{F}_n] = \left(\sum_{h=1}^N w_{h,j} Z_{n,h}\right)\left(1 - \sum_{h=1}^N w_{h,j} Z_{n,h}\right).$$

From this last equality, using synchronization and the normalization $W^\top \mathbf{1} = \mathbf{1}$, we immediately obtain

$$(4.11) \qquad E[(\Delta M_{n+1,j})^2|\mathcal{F}_n] \xrightarrow{\text{a.s.}} Z_\infty(1 - Z_\infty).$$

4.3. *A CLT for* $\widetilde{Z}_n$. The following result gives a central limit theorem for the real-valued stochastic process $(\widetilde{Z}_n)_n$.

THEOREM 4.2. *For* $1/2 < \gamma \leq 1$, *we have*

$$(4.12) \qquad n^{\gamma-\frac{1}{2}}(\widetilde{Z}_n - Z_\infty) \longrightarrow \mathcal{N}\big(0, \widetilde{\sigma}_\gamma^2 Z_\infty(1 - Z_\infty)\big) \qquad stably,$$

where $\widetilde{\sigma}_\gamma^2$ *is defined as in* (3.1) *(also for* $\gamma = 1$*). The above convergence is also in the sense of the almost sure conditional convergence w.r.t.* $\mathcal{F} = (\mathcal{F}_n)_n$.

PROOF. We want to apply Theorem B.3. Let us consider, for each $n \geq 1$ the filtration $(\mathcal{F}_{n,h})_h$ and the process $(L_{n,h})_h$ defined by

$$\mathcal{F}_{n,0} = \mathcal{F}_{n,1} = \mathcal{F}_n, \qquad L_{n,0} = L_{n,1} = 0$$

and, for $h \geq 2$,

$$\mathcal{F}_{n,h} = \mathcal{F}_{n+h-1}, \qquad L_{n,h} = n^{\gamma-\frac{1}{2}}(\widetilde{Z}_n - \widetilde{Z}_{n+h-1}).$$

By (4.5) and (4.6), the process $(L_{n,h})_h$ is a martingale w.r.t. $(\mathcal{F}_{n,h})_h$ which converges (for $h \to +\infty$) a.s. and in L^1 to the random variable $L_{n,\infty} = n^{\gamma-\frac{1}{2}}(Z_n - Z_\infty)$. In addition, the increment $Y_{n,j} = L_{n,j} - L_{n,j-1}$ is equal to zero for $j = 1$ and, for $j \geq 2$, it coincides with a random variable of the form $n^{\gamma-\frac{1}{2}}(\widetilde{Z}_k - \widetilde{Z}_{k+1})$ with $k \geq n$. Therefore, again by (4.5), we have

$$\sum_{j \geq 1} Y_{n,j}^2 = n^{2\gamma-1} \sum_{k \geq n} (\widetilde{Z}_k - \widetilde{Z}_{k+1})^2 = N^{-1} n^{2\gamma-1} \sum_{k \geq n} r_k^2 (\mathbf{v}_1^\top \Delta \mathbf{M}_{k+1})^2$$

$$\xrightarrow{\text{a.s.}} \frac{c^2}{N} \frac{\|\mathbf{v}_1\|^2}{(2\gamma - 1)} Z_\infty(1 - Z_\infty),$$

where the a.s. convergence follows by applying [22], Lemma 4.1, and by noticing that (4.11) implies

$$E\big[(\mathbf{v}_1^\top \Delta \mathbf{M}_{n+1})^2 | \mathcal{F}_n\big] = \sum_{j=1}^N v_{1,j}^2 E\big[(\Delta M_{n+1,j})^2 | \mathcal{F}_n\big]$$

$$\xrightarrow{\text{a.s.}} \sum_{j=1}^N v_{1,j}^2 Z_\infty(1 - Z_\infty) = \|\mathbf{v}_1\|^2 Z_\infty(1 - Z_\infty).$$

Finally, again by (4.5), we have

$$Y_n^* = \sup_{j \geq 1} |Y_{n,j}| = n^{\gamma-\frac{1}{2}} \sup_{k \geq n} |\widetilde{Z}_k - \widetilde{Z}_{k+1}| \leq \sup_{k \geq n} k^{\gamma-\frac{1}{2}} r_k \longrightarrow 0.$$

Hence, if in Theorem B.3 we take $k_n = 1$ for each n and $\mathcal{U} = \bigvee_n \mathcal{F}_n$, then the proof is concluded. $\square$

4.4. *Proofs of Theorem* 3.5 *and Theorem* 3.6 (*results on the distribution of* Z_∞). The proof of Theorem 3.5 is a consequence of the almost sure conditional convergence in Theorem 4.2, exactly as shown in [21].

To the proof of Theorem 3.6, we premise the following lemma.

LEMMA 4.1. *If condition* (3.7) *holds, then we have*

$$E[\widetilde{Z}_n(1 - \widetilde{Z}_n)] > 0 \qquad \forall n \geq 0.$$

PROOF. For convenience, set $x_n := E[\widetilde{Z}_n(1 - \widetilde{Z}_n)]$. We recall that $\widetilde{Z}_n = N^{-1/2}\mathbf{v}_1^\top \mathbf{Z}_n$, where $\mathbf{v}_1$ is such that

$$v_{1,j} > 0 \qquad \forall j \quad \text{and} \quad N^{-1/2}\sum_{j=1}^{N} v_{1,j} = 1.$$

Hence, under assumption (3.7), we immediately get $x_0 > 0$.

Now, we recall that $(\widetilde{Z}_n)_n$ is a bounded martingale which satisfies (4.5), that is,

$$\widetilde{Z}_n = (1 - r_{n-1})\widetilde{Z}_{n-1} + r_{n-1}N^{-1/2}\mathbf{v}_1^\top \mathbf{X}_n$$

with $E[N^{-1/2}\mathbf{v}_1^\top \mathbf{X}_n | \mathcal{F}_{n-1}] = \widetilde{Z}_{n-1}$. Therefore, we have $x_n = (E[\widetilde{Z}_0] - E[\widetilde{Z}_n^2])$ for each n and

$$\begin{aligned}
\widetilde{Z}_n^2 &= (1 - r_{n-1})^2 \widetilde{Z}_{n-1}^2 + 2(1 - r_{n-1})r_{n-1}\widetilde{Z}_{n-1}N^{-1/2}\mathbf{v}_1^\top \mathbf{X}_n \\
&\quad + r_{n-1}^2 \big(N^{-1/2}\mathbf{v}_1^\top \mathbf{X}_n\big)^2 \\
&\leq (1 - r_{n-1})^2 \widetilde{Z}_{n-1}^2 + 2(1 - r_{n-1})r_{n-1}\widetilde{Z}_{n-1}N^{-1/2}\mathbf{v}_1^\top \mathbf{X}_n \\
&\quad + r_{n-1}^2 \big(N^{-1/2}\mathbf{v}_1^\top \mathbf{X}_n\big).
\end{aligned}$$

Taking the conditional expectation given $\mathcal{F}_{n-1}$, we get

$$E[\widetilde{Z}_n^2 | \mathcal{F}_{n-1}] \leq (1 - r_{n-1}^2)\widetilde{Z}_{n-1}^2 + r_{n-1}^2 \widetilde{Z}_{n-1},$$

which implies

$$\begin{aligned}
E[\widetilde{Z}_n^2] &\leq (1 - r_{n-1}^2)E[\widetilde{Z}_{n-1}^2] + r_{n-1}^2 E[\widetilde{Z}_{n-1}] \\
&= (1 - r_{n-1}^2)E[\widetilde{Z}_{n-1}^2] + r_{n-1}^2 E[\widetilde{Z}_0].
\end{aligned}$$

Therefore, we can conclude by an induction argument on n. Indeed, if $x_{n-1} > 0$, that is, $E[\widetilde{Z}_{n-1}^2] < E[\widetilde{Z}_0]$, then from the above inequality, since $(1 - r_{n-1}^2) > 0$ by assumption, we obtain $E[\widetilde{Z}_n^2] < E[\widetilde{Z}_0]$, that is, $x_n > 0$. $\square$

We are now ready to prove Theorem 3.6.

PROOF OF THEOREM 3.6. We recall that Z_∞ takes values in [0, 1] and $(\widetilde{Z}_n)_n$ is a bounded martingale which converges a.s. to Z_∞. Therefore, in particular, setting $\widetilde{z}_0 := E[\widetilde{Z}_0]$, we have

$$E[Z_\infty] = E[\widetilde{Z}_n] = \widetilde{z}_0 \quad \forall n \quad \text{and} \quad \text{Var}[Z_\infty] = \lim_{n\to\infty} \text{Var}[\widetilde{Z}_n].$$

Now, as in the proof of the previous lemma, we set

$$(4.13) \qquad x_n := E\big[\widetilde{Z}_n(1 - \widetilde{Z}_n)\big] = \widetilde{z}_0 - \widetilde{z}_0^2 - \text{Var}[\widetilde{Z}_n]$$

and we can state that

$$P\big(Z_\infty \in \{0, 1\}\big) = 1 \quad \text{if and only if} \quad E\big[Z_\infty(1 - Z_\infty)\big] = \lim_n x_n = 0.$$

Thus, it is enough to prove that assumption (3.7) implies $\lim_n x_n > 0$. To this purpose, we observe that, by (4.5), we have

$$
\begin{aligned}
x_{n+1} &= \widetilde{z}_0 - \widetilde{z}_0^2 - \text{Var}[\widetilde{Z}_{n+1}] \\
&= \widetilde{z}_0 - \widetilde{z}_0^2 - E\big[\text{Var}[\widetilde{Z}_{n+1}|\mathcal{F}_n]\big] - \text{Var}\big[E[\widetilde{Z}_{n+1}|\mathcal{F}_n]\big] \\
(4.14) \qquad &= \widetilde{z}_0 - \widetilde{z}_0^2 - \frac{r_n^2}{N} E\big[E[(\mathbf{v}_1^\top \Delta\mathbf{M}_{n+1})^2|\mathcal{F}_n]\big] - \text{Var}[\widetilde{Z}_n] \\
&= x_n - \frac{r_n^2}{N} E\big[E[(\mathbf{v}_1^\top \Delta\mathbf{M}_{n+1})^2|\mathcal{F}_n]\big].
\end{aligned}
$$

Setting $\mathbf{Y}_n = E[\mathbf{X}_n|\mathcal{F}_{n-1}] = W^\top \mathbf{Z}_n$ (whose components obviously belong to [0, 1]) and recalling (4.9) and (4.10), we obtain

$$(4.15) \qquad E\big[(\mathbf{v}_1^\top \Delta\mathbf{M}_{n+1})^2|\mathcal{F}_n\big] = \sum_{j=1}^N v_{1,j}^2 Y_{n,j}(1 - Y_{n,j}).$$

Now, notice that

$$N^{-1/2}\mathbf{v}_1^\top \mathbf{Y}_n = N^{-1/2}\mathbf{v}_1^\top W^\top \mathbf{Z}_n = N^{-1/2}(W\mathbf{v}_1)^\top \mathbf{Z}_n = N^{-1/2}\mathbf{v}_1^\top \mathbf{Z}_n = \widetilde{Z}_n,$$

and so, for any $j = 1, \ldots, N$,

$$(4.16) \qquad N^{-1/2} v_{1,j} Y_{n,j} = \widetilde{Z}_n - N^{-1/2}\sum_{h\neq j} v_{1,h} Y_{n,h} \le \widetilde{Z}_n.$$

Analogously, notice that

$$N^{-1/2}\mathbf{v}_1^\top (\mathbf{1} - \mathbf{Y}_n) = \big(N^{-1/2}\mathbf{v}_1^\top \mathbf{1}\big) - \big(N^{-1/2}\mathbf{v}_1^\top \mathbf{Y}_n\big) = 1 - \widetilde{Z}_n$$

and so, for any $j = 1, \ldots, N$,

$$(4.17) \quad N^{-1/2} v_{1,j}(1 - Y_{n,j}) = (1 - \widetilde{Z}_n) - N^{-1/2}\sum_{h\neq j} v_{1,h}(1 - Y_{n,h}) \le 1 - \widetilde{Z}_n.$$

Then, combining (4.16) and (4.17), we get for any $j = 1, \ldots, N$

$$v_{1,j}^2 Y_{n,j}(1 - Y_{n,j}) \le N\widetilde{Z}_n(1 - \widetilde{Z}_n),$$

and hence, recalling (4.13), (4.14) and (4.15), we obtain

$$x_{n+1} \ge x_n - Nr_n^2 E[\widetilde{Z}_n(1 - \widetilde{Z}_n)] = (1 - Nr_n^2)x_n.$$

Finally, taking $\bar{n}$ such that $Nr_n^2 < 1$ for any $n \ge \bar{n}$, we find

$$x_{n+1} \ge x_{\bar{n}} \prod_{m=\bar{n}}^{n} (1 - Nr_m^2).$$

Hence, since $x_{\bar{n}} > 0$ by the previous lemma and $\sum_n r_n^2 < +\infty$ for $1/2 < \gamma \le 1$, we can conclude that $\lim_n x_n > 0$. $\square$

4.5. *A CLT for* $\widehat{\mathbf{Z}}_n$. The following result provides a central limit theorem for the multi-dimensional real stochastic process $(\widehat{\mathbf{Z}}_n)_n$.

THEOREM 4.3.   *We have*:

(a) *If* $1/2 < \gamma < 1$, *then*

(4.18) $$n^{\frac{\gamma}{2}}\widehat{\mathbf{Z}}_n \longrightarrow \mathcal{N}(\mathbf{0}, Z_\infty(1 - Z_\infty)\widehat{\Sigma}_\gamma) \qquad stably,$$

where $\widehat{\Sigma}_\gamma$ *is defined in* (3.4).

(b) *If* $\gamma = 1$ *and* $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$, *then*

(4.19) $$\sqrt{n}\widehat{\mathbf{Z}}_n \longrightarrow \mathcal{N}(\mathbf{0}, Z_\infty(1 - Z_\infty)\widehat{\Sigma}_1) \qquad stably,$$

where $\widehat{\Sigma}_1$ *is defined in* (3.2).

(c) *If* $\gamma = 1$, $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$ *and* (2.3) *holds*, *then*

(4.20) $$\sqrt{\frac{n}{\ln(n)}}\widehat{\mathbf{Z}}_n \longrightarrow \mathcal{N}(\mathbf{0}, Z_\infty(1 - Z_\infty)\widehat{\Sigma}_1^*) \qquad stably,$$

where $\widehat{\Sigma}_1^*$ *is defined in* (3.3).

PROOF.   Set $\alpha_j = 1 - \lambda_j = a_j + ib_j$ with $\lambda_j \in \mathrm{Sp}(W) \setminus \{1\}$. Remember that $a_j > 0$ for each j since $\mathcal{R}e(\lambda_j) < 1$ for each j. Moreover, recall the definition of the matrices U, V and D given in Section 2 and in Section 4.1.

From dynamics (4.8), we get

$$\begin{aligned}\widehat{\mathbf{Z}}_{n+1} &= [I - r_n U(I - D)V^\top]\widehat{\mathbf{Z}}_n + r_n UV^\top \Delta\mathbf{M}_{n+1} \\ &= U[I - r_n(I - D)]V^\top\widehat{\mathbf{Z}}_n + r_n UV^\top \Delta\mathbf{M}_{n+1},\end{aligned}$$

where the identity matrices used above have different dimensions and the last equality holds because relations (4.1) imply $UV^\top \widehat{\mathbf{Z}}_n = UV^\top \mathbf{Z}_n = \widehat{\mathbf{Z}}_n$. Therefore, if we take m_0 large enough such that $a_j r_n < 1$ for $n \geq m_0$ and all j, we can write

$$(4.21) \qquad \widehat{\mathbf{Z}}_{n+1} = C_{m_0,n}\widehat{\mathbf{Z}}_{m_0} + \sum_{k=m_0}^{n} C_{k+1,n} r_k UV^\top \Delta\mathbf{M}_{k+1},$$

with

$$C_{k+1,n} = \prod_{m=k+1}^{n} \{U[I - r_m(I - D)]V^\top\}.$$

For the sequel, it is important to note that $C_{k+1,n}$ is a real matrix since, by (4.1) and (4.2) it is equivalent to a product of real matrices, that is, $(UV^\top) - r_m(UV^\top + \mathbf{u}_1\mathbf{v}_1^\top - W^\top) = (UV^\top) - r_m(I - W^\top)$. Moreover, using relations (4.1) again, we get

$$(4.22) \qquad C_{k+1,n} = U A_{k+1,n} V^\top,$$

where $A_{k+1,n}$ is the diagonal matrix given by

$$[A_{k+1,n}]_{j,j} = \begin{cases} \displaystyle\prod_{m=k+1}^{n} (1 - \alpha_j r_m) & \text{for } m_0 - 1 \leq k \leq n - 1, \\ 1 & \text{for } k = n. \end{cases}$$

Observe that we have

$$[A_{k+1,n}]_{j,j} = \frac{p_{n,j}}{p_{k,j}} = \frac{\ell_{k,j}}{\ell_{n,j}} \qquad \text{for } m_0 - 1 \leq k \leq n,$$

with

$$p_{m_0-1,n} = \ell_{m_0-1,n} = 1,$$

$$p_{k,j} = \prod_{m=m_0}^{k} (1 - \alpha_j r_m), \qquad \ell_{k,j} = p_{k,j}^{-1} \qquad \text{for } m_0 \leq k \leq n.$$

Finally, notice that, since $C_{k+1,n}UV^\top = C_{k+1,n}$ by relations (4.1) and (4.22), we can rewrite (4.21) as

$$\widehat{\mathbf{Z}}_{n+1} = C_{m_0,n}\widehat{\mathbf{Z}}_{m_0} + \sum_{k=m_0}^{n} \mathbf{T}_{n,k} \qquad \text{where } \mathbf{T}_{n,k} = r_k C_{k+1,n} \Delta\mathbf{M}_{k+1}.$$

We will establish the asymptotic behavior of $\widehat{\mathbf{Z}}_n$ by studying separately the terms $C_{m_0,n}\widehat{\mathbf{Z}}_{m_0}$ and $\sum_{k=m_0}^{n} \mathbf{T}_{n,k}$.

Concerning the first term, note that by (A.5) in Lemma A.4, we have that, for any $\varepsilon \in (0, 1)$,

$$|C_{m_0,n}\widehat{\mathbf{Z}}_{m_0}| = O(|p_n^*|) = \begin{cases} O\left(\exp\left[-(1-\varepsilon)\dfrac{ca^*}{1-\gamma}n^{1-\gamma}\right]\right) & \text{if } 1/2 < \gamma < 1, \\ O(n^{-(1-\varepsilon)ca^*}) & \text{if } \gamma = 1, \end{cases}$$

where the symbol $*$ refers to quantities a_j and $p_{n,j}$ corresponding to $\lambda_j = \lambda^* \in \lambda_{\max}(D)$. Therefore, for the case (a) (i.e., $1/2 < \gamma < 1$) and (b) [i.e., $\gamma = 1$ and $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$], we have

$$|C_{m_0,n}\widehat{\mathbf{Z}}_{m_0}| = o(n^{-\gamma/2}).$$

Indeed, this fact follows immediately for $1/2 < \gamma < 1$ and, for $\gamma = 1$ one has to note that, since we assume $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$, that is $ca^* > 1/2$, we can choose ε small enough so that $(1 - \varepsilon)ca^* > 1/2$. Moreover, for the case (c) [i.e., $\gamma = 1$ and $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$, i.e., $ca^* = 1/2$], since we assume condition (2.3), by (A.8) in Lemma A.4, we have

$$|C_{m_0,n}\widehat{\mathbf{Z}}_{m_0}| = O(|p_n^*|) = O(n^{-ca^*}) = O(n^{-\frac{1}{2}}).$$

Therefore, if we set

$$t_n = \begin{cases} n^{\frac{\gamma}{2}} & \text{for case (a),} \\ n^{\frac{1}{2}} & \text{for case (b),} \\ (n/\ln(n))^{\frac{1}{2}} & \text{for case (c),} \end{cases}$$

then we obtain $t_n|C_{m_0,n}\widehat{\mathbf{Z}}_{m_0}| \to 0$ almost surely.

We now focus on the asymptotic behavior of the second term. Specifically, we aim at proving that $t_n \sum_{k=m_0}^n \mathbf{T}_{n,k}$ converges stably to a suitable Gaussian kernel. For this purpose, we set $\mathcal{G}_{n,k} = \mathcal{F}_{k+1}$, and consider Theorem B.1 (recall that $\mathbf{T}_{n,k}$ are real random vectors). Given the fact that condition (c1) of Theorem B.1 is obviously satisfied, we will check conditions (c2) and (c3).

Regarding condition (c2), we observe that

$$\sum_{k=m_0}^n (t_n \mathbf{T}_{n,k})(t_n \mathbf{T}_{n,k})^\top$$

$$= t_n^2 \sum_{k=m_0}^n r_k^2 C_{k+1,n}(\Delta \mathbf{M}_{k+1})(\Delta \mathbf{M}_{k+1})^\top C_{k+1,n}^\top$$

$$= U\left(t_n^2 \sum_{k=m_0}^n r_k^2 A_{k+1,n} V^\top (\Delta \mathbf{M}_{k+1})(\Delta \mathbf{M}_{k+1})^\top V A_{k+1,n}\right) U^\top.$$

Therefore, it is enough to study the convergence of

$$t_n^2 \sum_{k=m_0}^n r_k^2 A_{k+1,n} V^\top (\Delta \mathbf{M}_{k+1})(\Delta \mathbf{M}_{k+1})^\top V A_{k+1,n}.$$

To this purpose, we set $B_{k+1,h,j} = [V^\top (\Delta \mathbf{M}_{k+1})(\Delta \mathbf{M}_{k+1})^\top V]_{h,j}$ and observe that an element of the above matrix is of the form

$$t_n^2 \sum_{k=m_0}^n r_k^2 [A_{k+1,n}]_{h,h} B_{k+1,h,j} [A_{k+1,n}]_{j,j}$$

$$= t_n^2 p_{n,h} p_{n,j} \sum_{k=m_0}^{n-1} r_k^2 \ell_{k,h} \ell_{k,j} B_{k+1,h,j} + t_n^2 r_n^2 B_{n+1,h,j},$$

where $t_n^2 r_n^2 B_{n+1,h,j} = O(t_n^2 r_n^2) \to 0$. We now fix h and j and apply Lemma A.3 to the first addend in the above equality. Indeed, this quantity can be written as $v_n \sum_{k=m_0}^{n-1} Y_k / c_k v_k$, where

$$Y_n = B_{n+1,h,j}, \qquad c_n = \frac{1}{t_n^2 r_n^2} > 0 \quad \text{and} \quad v_n = t_n^2 p_{n,h} p_{n,j} \in \mathbb{C} \setminus \{0\}$$

satisfy the assumptions of Lemma A.3. More precisely, setting $\mathcal{H}_n = \mathcal{F}_{n+1}$, by (4.11), we have

$$E[Y_n|\mathcal{H}_{n-1}] = E[B_{n+1,h,j}|\mathcal{F}_n] = [V^\top E[(\Delta \mathbf{M}_{n+1})(\Delta \mathbf{M}_{n+1})^\top |\mathcal{F}_n]V]_{h,j}$$

$$\xrightarrow{\text{a.s.}} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty (1 - Z_\infty)$$

and, moreover, we have

$$\sum_n \frac{E[|Y_n|^2]}{c_n^2} = \sum_n E[|Y_n|^2] r_n^4 t_n^4 = \sum_n r_n^4 O(n^{2\gamma}) = \sum_n O(1/n^{2\gamma}) < +\infty.$$

In addition, as we have observed above, $|v_n| = t_n^2 |p_{n,h} p_{n,j}| = t_n^2 O(|p_n^*|^2) \to 0$ and, by (A.11) in Lemma A.5 and (A.18) in Lemma A.6, we have

$$\lim_n v_n \sum_{k=m_0}^{n-1} \frac{1}{c_k v_k}$$

$$= \begin{cases} \dfrac{c}{\alpha_h + \alpha_j} & \text{if } 1/2 < \gamma < 1, \\[2mm] \dfrac{c^2}{c(\alpha_h + \alpha_j) - 1} & \text{if } \gamma = 1, c(a_h + a_j) > 1, \\[2mm] 0 & \text{if } \gamma = 1, c(a_h + a_j) = 1, c(\alpha_h + \alpha_j) \neq 1 \\ & \text{and (2.3) holds,} \\[2mm] c^2 & \text{if } \gamma = 1, c(\alpha_h + \alpha_j) = 1 \text{ and (2.3) holds.} \end{cases}$$

Finally, we have $|v_n|\sum_{k=m_0}^{n-1}\frac{1}{c_k|v_k|}=O(1)$ by (A.12) in Lemma A.5 and (A.19) in Lemma A.6 (with $u=1$), and, using (A.16) and (A.22) in the Appendix, we get

$$c_n|v_n|\left|\frac{1}{v_n}-\frac{1}{v_{n-1}}\right|=\frac{1}{r_n^2|\ell_{n,h}\ell_{n,j}|}\left|\frac{\ell_{n,h}\ell_{n,j}}{t_n^2}-\frac{\ell_{n-1,h}\ell_{n-1,j}}{t_{n-1}^2}\right|=O(1).$$

Hence, recalling Remark A.2, also the last condition required in Lemma A.3 is verified.

Regarding condition (c3), we observe that, using the inequalities

$$|\mathbf{T}_{n,k}|=r_k|C_{k+1,n}\Delta\mathbf{M}_{k+1}|\leq r_k|U||A_{k+1,n}||V^\top||\Delta\mathbf{M}_{k+1}|\leq Kr_k|A_{k+1,n}|,$$

with a suitable constant K, we find for any $u>1$

$$\left(\sup_{m_0\leq k\leq n}|t_n\mathbf{T}_{n,k}|\right)^{2u}\leq t_n^{2u}\sum_{k=m_0}^{n-1}|\mathbf{T}_{n,k}|^{2u}+t_n^{2u}|\mathbf{T}_{n,n}|^{2u}$$

$$=t_n^{2u}O\left(|p_n^*|^{2u}\sum_{k=m_0}^{n-1}r_k^{2u}|\ell_k^*|^{2u}\right)+t_n^{2u}O(r_n^{2u}),$$

where, for the last equality, we have used (A.8) and (A.9) in Lemma A.4. Now, by (A.12) in Lemma A.5 and (A.19) in Lemma A.6 (with $\alpha_1=\alpha_2=\alpha^*=1-\lambda^*$ and $u>1$), we have

$$O\left(|p_n^*|^{2u}\sum_{k=m_0}^{n-1}r_k^{2u}|\ell_k^*|^{2u}\right)$$

$$=\begin{cases}O(n^{-\gamma(2u-1)}) & \text{if } 1/2<\gamma<1,\\ O(n^{-(2u-1)}) & \text{if } \gamma=1, 2uca^*>2u-1,\\ O(n^{-u}) & \text{if } \gamma=1, 2ca^*=1 \text{ and (2.3) holds.}\end{cases}$$

Therefore, for cases (a) and (c), it is immediate to obtain

$$t_n^{2u}O\left(|p_n^*|^{2u}\sum_{k=m_0}^{n-1}r_k^{2u}|\ell_k^*|^{2u}\right)+t_n^{2u}O(r_n^{2u})\longrightarrow 0$$

for any $u>1$; while in case (b) in order to have the above convergence to zero, we have to choose $u>1$ such that $2uca^*>2u-1$, that is, $2u(ca^*-1)+1>0$. This choice is always possible: indeed, or $ca^*-1\geq 0$ and so we can take any $u>1$, or $ca^*-1<0$ and we have to take $u\in(1,(2-2ca^*)^{-1})$ [note that $(2-2ca^*)^{-1}>1$ since $2ca^*>1$ by assumption]. As a consequence of the above convergence to zero, we obtain condition (c3) of Theorem B.1.

Summing up, all the conditions required by Theorem B.1 are satisfied and so we can apply this theorem and obtain the stable convergence of $t_n\sum_{k=m_0}^n\mathbf{T}_{n,k}$ to the Gaussian kernel with zero mean and random variance defined in Theorem 4.3 for each of the three considered cases. $\square$

PROOF OF THEOREM 3.4.   First, consider case (a), that is, $1/2 < \gamma < 1$, and recall the definition of $\widehat{S}_\gamma$ in (3.4). Then, since $l_j := (1 - \lambda_j)$, for $\lambda_j \in \mathrm{Sp}(W) \setminus \{1\}$, have positive real parts by the assumptions on W, we have

$$[\widehat{S}_\gamma]_{h,j} = (\mathbf{v}_h^\top \mathbf{v}_j)\frac{c}{l_h + l_j} = (\mathbf{v}_h^\top \mathbf{v}_j)c \int_0^\infty \exp[-u(l_h + l_j)]\,du.$$

Then, setting $L := (I - D)$ and $M(u) := U \exp(-uL)V^\top$ for $u \in (0, +\infty)$, we can write

$$\widehat{\Sigma}_\gamma = U\widehat{S}_\gamma U^\top = c \int_0^\infty M(u)M^\top(u)\,du.$$

Notice that, for any $u \in (0, +\infty)$, the matrix $M(u)$ has real entries since, by (4.1) and (4.2), we have

$$M(u) = U \sum_{k=0}^{+\infty} \frac{(-uL)^k}{k!}V^\top = UV^\top + \sum_{k=1}^{+\infty}\frac{(-uULV^\top)^k}{k!}$$

$$= \exp[-u(I - W^\top)] - \mathbf{u}_1\mathbf{v}_1^\top.$$

Moreover, for any $u \in (0, +\infty)$, the matrix $M(u)$ has rank $(N - 1)$ and $M^\top(u)\mathbf{v}_1 = \mathbf{0}$ by (4.1). Therefore, for any $u \in (0, +\infty)$, the matrix $M(u)M^\top(u)$ is a positive semidefinite real matrix with rank $(N - 1)$ and $M(u)M^\top(u)\mathbf{v}_1 = \mathbf{0}$ (see [30], Observation 7.1.8). These facts imply the first part of statement (a). For the second part, denoting by $\mathbf{e}_j$ the vector such that $e_{j,j} = 1$ and $e_{j,h} = 0$ for all $h \neq j$, we have for $h \neq j$

$$\Sigma_{\gamma,h,j} = (\mathbf{e}_h - \mathbf{e}_j)^\top \widehat{\Sigma}_\gamma (\mathbf{e}_h - \mathbf{e}_j)$$

and so $\Sigma_{\gamma,h,j} = 0$ if and only if $M(u)^\top(\mathbf{e}_h - \mathbf{e}_j) = 0$ for almost every $u \in (0, +\infty)$. But this is not possible since $\mathrm{Ker}(M(u)^\top)$ is generated by $\mathbf{v}_1$, which has all the entries strictly greater than zero as pointed out in Section 2. This concludes the proof of case (a).

The proof of case (b) is analogous, by setting $l_j := (1 - \lambda_j - (2c)^{-1})$ for $\lambda_j \in \mathrm{Sp}(W) \setminus \{1\}$, which have positive real parts by condition $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$, and $L := (I - D - I(2c)^{-1})$.

For the proof of case (c), that is, $\gamma = 1$ and $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$, let q, with $1 \leq q \leq (N - 1)$, be the number of distinct eigenvalues $\lambda_j = a_j + ib_j \in \mathrm{Sp}(W) \setminus \{1\}$ and, for any $1 \leq h \leq q$, let U_h and V_h be the submatrices of U and V whose columns are, respectively, the left and the right eigenvectors associated to λ_h. Then, by the definition of $\widehat{S}_1^*$ in (3.3), we have

$$\widehat{\Sigma}_1^* = U\widehat{S}_1^* U^\top = \sum_{1 \leq h,j \leq q} U_h V_h^\top V_j U_j^\top \mathbb{1}_{\{\lambda_h + \lambda_j = 2 - c^{-1}\}}.$$

Then, since

$$\{\lambda_h + \lambda_j = 2 - c^{-1}\} = (\{a_h = 1 - (2c)^{-1}\} \cap \{a_j = 1 - (2c)^{-1}\} \cap \{b_h = -b_j\}),$$

setting the $N \times N$-matrix $M_h := U_h V_h^\top$ for any $1 \leq h \leq q$ and denoting by p, with $1 \leq p \leq q$, the number of distinct eigenvalues $\lambda_j \in A^*$, we can write

$$\sum_{1 \leq h, j \leq q} M_h M_j^\top \mathbb{1}_{\{\lambda_h + \lambda_j = 2 - c^{-1}\}} = \sum_{1 \leq h, j \leq p} M_h M_j^\top \mathbb{1}_{\{b_h = -b_j\}} = \sum_{1 \leq h \leq p} M_h M_{j(h)}^\top,$$

where $j(h)$ indicates the index j, with $1 \leq j \leq p$, such that $b_j = -b_h$. Notice that, since W has real entries, for any nonreal $\lambda_h \in \mathrm{Sp}(W)$, there exists $\lambda_j \in \mathrm{Sp}(W)$ such that $\lambda_j = \overline{\lambda}_h$; moreover, $\overline{\mathbf{u}}_h$ and $\overline{\mathbf{v}}_h$ are respectively left and right eigenvectors associated to λ_j. Hence, denoting by T the nonsingular matrix such that $U_j = \overline{U}_h T$ and $V_j = \overline{V}_h (T^\top)^{-1}$, we have that

$$M_j = U_j V_j^\top = \overline{U}_h T T^{-1} \overline{V}_h^\top = \overline{U}_h \overline{V}_h^\top = \overline{M}_h.$$

Thus, we have

$$\sum_{1 \leq h \leq p} M_h M_{j(h)}^\top = \sum_{1 \leq h \leq p} M_h \overline{M}_h^\top,$$

which is a positive semidefinite matrix of rank m^* (see [30], Observation 7.1.8).

Concerning the second part of case (c), since

$$\Sigma_{1,h,j}^* = (\mathbf{e}_h - \mathbf{e}_j)^\top \widehat{\Sigma}_1^* (\mathbf{e}_h - \mathbf{e}_j),$$

we have that $\Sigma_{1,h,j}^* = 0$ if and only if $(\mathbf{e}_h - \mathbf{e}_j) \in \mathrm{Ker}(\widehat{\Sigma}_1^*)$. Now, notice that $\mathrm{Ker}(\widehat{\Sigma}_1^*) = \bigcap_{1 \leq k \leq p} \mathrm{Ker}(U_k^\top)$, and hence $\mathrm{Ker}(\widehat{\Sigma}_1^*)$ is generated by $\{\mathbf{v}_j : \lambda_j \notin A^*\}$. Finally, since the following decomposition holds:

$$(\mathbf{e}_h - \mathbf{e}_j) = \sum_{k=1}^N (\mathbf{u}_k^\top (\mathbf{e}_h - \mathbf{e}_j)) \mathbf{v}_k = \sum_{k=1}^N (u_{k,h} - u_{k,j}) \mathbf{v}_k,$$

it is enough to have $u_{k,h} \neq u_{k,j}$ for some k such that $\lambda_k \in A^*$ to prove that $\Sigma_{1,h,j}^* > 0$. Analogously, we can prove that $[\widehat{\Sigma}_1^*]_{jj} > 0$ when $u_{h,j} \neq 0$ for some h such that $\lambda_h \in A^*$. This concludes the proof of case (c). $\quad \square$

4.6. *Proofs of Theorem* 3.2 *and Theorem* 3.3 (*CLTs for* $\mathbf{Z}_n$).   Let us remind the decomposition (4.4), that is,

$$\mathbf{Z}_n = \widetilde{Z}_n \mathbf{1} + \widehat{\mathbf{Z}}_n.$$

Hence, the asymptotic behavior of the process $(\mathbf{Z}_n)_n$ can be obtained by combining the asymptotic results concerning $(\widetilde{Z}_n)_n$ and $(\widehat{\mathbf{Z}}_n)_n$ established in the previous subsections. As we have already seen, we have the almost sure synchronization, that is,

$$\mathbf{Z}_n \xrightarrow{\text{a.s.}} Z_\infty \mathbf{1}.$$

Moreover, from Theorem 4.2, we easily obtain for $1/2 < \gamma \leq 1$

$$n^{\gamma - \frac{1}{2}}(\widetilde{Z}_n - Z_\infty)\mathbf{1} \longrightarrow \mathcal{N}(\mathbf{0}, \widetilde{\sigma}_\gamma^2 Z_\infty(1 - Z_\infty)\mathbf{1}\mathbf{1}^\top) \qquad \text{stably in the strong sense,}$$

and we recall the central limit theorem for the multi-dimensional process $\widehat{\mathbf{Z}}_n$ presented in Theorem 4.3.

Hence, for Theorem 3.2(a), we observe that

$$n^{\gamma - 1/2}(\mathbf{Z}_n - Z_\infty\mathbf{1}) = n^{\gamma - 1/2}(\widetilde{Z}_n - Z_\infty)\mathbf{1} + \frac{1}{n^{(1-\gamma)/2}}(n^{\gamma/2}\widehat{\mathbf{Z}}_n),$$

where the first term converges stably to a Gaussian kernel and the second one converges in probability to zero.

For Theorem 3.2(b), we observe that

$$\sqrt{n}(\mathbf{Z}_n - Z_\infty\mathbf{1}) = \sqrt{n}(\widetilde{Z}_n - Z_\infty)\mathbf{1} + \sqrt{n}\widehat{\mathbf{Z}}_n,$$

where the first term converges to a Gaussian kernel stably in the strong sense and the second one converges stably to a Gaussian kernel. Since $\widehat{\mathbf{Z}}_n$ is $\mathcal{F}_n$-measurable, by applying Theorem B.2, we can conclude.

For Theorem 3.2(c), we observe that

$$\frac{\sqrt{n}}{\sqrt{\ln(n)}}(\mathbf{Z}_n - Z_\infty\mathbf{1}) = \left(\frac{1}{\sqrt{\ln(n)}}\right)\sqrt{n}(\widetilde{Z}_n - Z_\infty)\mathbf{1} + \frac{\sqrt{n}}{\sqrt{\ln(n)}}\widehat{\mathbf{Z}}_n,$$

where the first term converges in probability to zero and the second one converges stably to a Gaussian kernel. Thus, Theorem 3.2 is proven.

Finally, we observe that

$$Z_{n,h} - Z_{n,j} = \widehat{Z}_{n,h} - \widehat{Z}_{n,j}.$$

Therefore, Theorem 3.3 immediately follows from the central limit theorem for the N-dimensional process $(\widehat{\mathbf{Z}}_n)_n$.

5. Examples of weighted adjacency matrices.

In this section, we analyze in detail the results presented in Section 3 for some interesting examples of weighted adjacency matrices.

5.1. *"Mean-field" interaction.* This kind of interaction can be expressed in terms of a particular weighted adjacency matrix W as follows: for any $1 \leq h$, $j \leq N$,

$$(5.1) \qquad w_{h,j} = \frac{\alpha}{N} + \delta_{h,j}(1 - \alpha) \qquad \text{with } \alpha \in [0, 1],$$

where $\delta_{h,j}$ is equal to 1 when $h = j$ and to 0 otherwise. Note that W in (5.1) is irreducible for $\alpha > 0$. Since W is doubly stochastic, we have (see Remark 4.1) $\mathbf{v}_1 = \mathbf{u}_1 = N^{-1/2}\mathbf{1}$ and so (i) the random variable $\widetilde{Z}_n$ coincides with the average of the processes $Z_{n,j}$, that is, $N^{-1}\mathbf{1}^\top\mathbf{Z}_n$, (ii) $\widehat{\mathbf{Z}}_n = (I - N^{-1}\mathbf{1}\mathbf{1}^\top)\mathbf{Z}_n$ and

(iii) $\widetilde{\sigma}_\gamma^2 = \frac{c^2}{N(2\gamma-1)}$ for $1/2 < \gamma \leq 1$. Furthermore, we have $\lambda_j = 1 - \alpha$ for all $\lambda_j \in \mathrm{Sp}(W) \setminus \{1\}$, and consequently, the conditions $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$ or $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$ required in the previous results when $\gamma = 1$ correspond to the conditions $2c\alpha > 1$ or $2c\alpha = 1$. Finally, since W is also symmetric, we have $U = V$ and so $U^\top U = V^\top V = I$ and $UU^\top = VV^\top = I - N^{-1}\mathbf{11}^\top$. We thus obtain:

(a) for $1/2 < \gamma < 1$, $\widehat{S}_\gamma = \frac{c}{2\alpha}I$ and $\widehat{\Sigma}_\gamma = \frac{c}{2\alpha}(I - N^{-1}\mathbf{11}^\top)$;

(b) for $\gamma = 1$ and $2c\alpha > 1$, $\widehat{S}_1 = \frac{c^2}{2c\alpha-1}I$ and $\widehat{\Sigma}_1 = \frac{c^2}{2c\alpha-1}(I - N^{-1}\mathbf{11}^\top)$;

(c) for $\gamma = 1$ and $2c\alpha = 1$, $\widehat{S}_1^* = c^2 I$ and $\widehat{\Sigma}_1^* = c^2(I - N^{-1}\mathbf{11}^\top)$.

Therefore, our theorems contain as particular cases part of the results proven in [21, 22, 25]. However, differently from these papers, we do not deal with the cases $2c\alpha < 1$ or $\gamma \leq 1/2$, which are still interesting for synchronization phenomena but lead to quite different asymptotic results. We have already discussed the case $\gamma \leq 1/2$ in Section 2 and, regarding the case $\gamma = 1$ and $0 < 2c\alpha < 1$, we recall that in [22] it has been determined the rate of synchronization, but not the asymptotic distribution.

5.2. *"Cycle" interaction.* Another possible scenario consists in a graph in which the vertices form a circle and each one influences only the vertex at his right side. This interaction can be modeled by using the adjacency matrix W defined as follows: for any $1 \leq h \leq (N-1)$ and $1 \leq j \leq N$ we have

$$(5.2) \qquad w_{h,j} = \begin{cases} 1 & \text{if } j = h+1, \\ 0 & \text{otherwise}, \end{cases}$$

while we have $w_{N,1} = 1$ for $h = N$ and $w_{N,j} = 0$ for any $2 \leq j \leq N$. Since W is again doubly stochastic, we have $\mathbf{u}_1 = \mathbf{v}_1 = N^{-1/2}\mathbf{1}$, which implies (i) $\widetilde{Z}_n = N^{-1}\mathbf{1}^\top \mathbf{Z}_n$, (ii) $\widehat{\mathbf{Z}}_n = (I - N^{-1}\mathbf{11}^\top)\mathbf{Z}_n$ and (iii) $\widetilde{\sigma}_\gamma^2 = \frac{c^2}{N(2\gamma-1)}$ for $1/2 < \gamma \leq 1$ as in Section 5.1. Moreover, it is easy to verify that in this case the eigenvalues of W are $\lambda_1 = 1$ and $\lambda_j = \exp[i(j-1)2\pi/N]$, for $j = 2, \ldots, N$. Hence, since in this case $\mathcal{R}e(\lambda^*) = \cos(2\pi/N)$, conditions $\mathcal{R}e(\lambda^*) < 1 - (2c)^{-1}$ or $\mathcal{R}e(\lambda^*) = 1 - (2c)^{-1}$ required in the previous results when $\gamma = 1$ correspond to the conditions $2c(1 - \cos(2\pi/N)) > 1$ or $2c(1 - \cos(2\pi/N)) = 1$. Moreover, for each $\lambda_h \in \mathrm{Sp}(W) \setminus \{1\}$, the jth element of the corresponding left and right eigenvectors are, respectively, $u_{h,j} = N^{-1/2}\exp[-i(h-1)j2\pi/N]$ and $v_{h,j} = N^{-1/2}\exp[i(h-1)j2\pi/N]$. Therefore, since we have the analytic expressions of U and V, it is possible to compute the asymptotic variance-covariance matrices according to the size N of the network and their eigenvalues and eigenvectors. For instance, for $N = 4$ we have:

(a) for $1/2 < \gamma < 1$, the nonzero eigenvalues of $\widehat{\Sigma}_\gamma$ are $c/2, c/2, c/4$, with the corresponding eigenvectors $(-1, 0, 1, 0)$, $(0, -1, 0, 1)$, $(-1, 1, -1, 1)$;

(b) for $\gamma = 1$ and $c > 1/2$, the nonzero eigenvalues of $\widehat{\Sigma}_1$ are $c^2(2c-1)^{-1}$, $c^2(2c-1)^{-1}$, $c^2(4c-1)^{-1}$, with the corresponding eigenvectors $(-1,0,1,0)$, $(0,-1,0,1)$, $(-1,1,-1,1)$;

(c) for $\gamma = 1$ and $c = 1/2$, the nonzero eigenvalue of $\widehat{\Sigma}_1^*$ is $1/4$ with multiplicity two and the corresponding eigenvectors are $(-1,0,1,0)$, $(0,-1,0,1)$.

5.3. *"Special vertex" case.*

In the previous two examples, the matrix W is doubly stochastic (also symmetric in the first example). As a different situation, we may consider the case in which there exists a "special vertex" whose influence on the graph is different with respect to the one of all the other elements in the system. This interactive structure can be expressed in terms of a particular adjacency matrix defined as follows:

$$(5.3) \qquad W = \mathbf{a}_p \mathbf{1}^\top \qquad \text{with } \mathbf{a}_p := \left(p, \frac{1-p}{N-1}, \dots, \frac{1-p}{N-1} \right)^\top,$$

where $0 < p < 1$ is a weight that represents how much any vertex of the system is influenced by the "special vertex". Notice that $\sum_{j=1}^N a_{p,j} = 1$ for any $0 < p < 1$. Moreover, we have $\mathbf{v}_1 = \mathbf{a}_p N^{1/2}$ and hence $UV^\top = I - \mathbf{u}_1 \mathbf{v}_1^\top = I - \mathbf{1}\mathbf{a}_p^\top$ and

$$\tilde{\sigma}_\gamma^2 = \frac{c^2}{N} \frac{\|\mathbf{v}_1\|^2}{(2\gamma-1)} = \frac{c^2}{(2\gamma-1)} \|\mathbf{a}_p\|^2$$

$$= \frac{c^2}{(2\gamma-1)} \left(p^2 + \frac{(1-p)^2}{N-1} \right) \qquad \text{for } 1/2 < \gamma \leq 1.$$

Furthermore, since $\mathrm{Sp}(W) \setminus \{1\}$ coincides with the eigenvalue 0 with multiplicity $(N-1)$, conditions $\lambda^* < 1 - (2c)^{-1}$ or $\lambda^* = 1 - (2c)^{-1}$ required in the previous results when $\gamma = 1$ correspond to the conditions $c > 1/2$ or $c = 1/2$ and, setting

$$A_p := UV^\top (UV^\top)^\top = (I - \mathbf{1}\mathbf{a}_p^\top)(I - \mathbf{a}_p \mathbf{1}^\top) = I + \|\mathbf{a}_p\|^2 \mathbf{1}\mathbf{1}^\top - (\mathbf{1}\mathbf{a}_p^\top + \mathbf{a}_p \mathbf{1}^\top),$$

we have:

(a) $\widehat{\Sigma}_\gamma = \frac{c}{2} A_p$ for $1/2 < \gamma < 1$;
(b) $\widehat{\Sigma}_1 = \frac{c^2}{2c-1} A_p$, for $\gamma = 1$ and $c > 1/2$;
(c) $\widehat{\Sigma}_1^* = \frac{1}{4} A_p$, for $\gamma = 1$ and $c = 1/2$.

In order to highlight the role of the "special vertex" in the synchronization of the system, let us set the initial state of the stochastic processes at the vertices as follows: $Z_0^1 = z_1$ for the "special vertex" and $Z_0^2 = \cdots = Z_0^N = z_2$ for the other vertices of the graph, with $z_1 \neq z_2$. This may represent a situation in which initially the "special agent" has an inclination z_1 that is different from the rest of the population which is settled on another inclination z_2. Since $(\tilde{Z}_n)_n$ is a martingale, we have that $E[Z_\infty] = E[\tilde{Z}_0] = N^{-1/2} \mathbf{v}_1^\top \mathbf{Z}_0$, which in this case reduces to

$$E[Z_\infty] = z_1 p + z_2(1-p).$$

Then the expected limiting inclination, that is, $E[Z_\infty]$, is strongly related to the influence that the "special vertex" exercises on the rest of the vertices (which is ruled by the parameter p). For instance, consider the following cases:

(i) If $p \simeq 1$, then we have $E[Z_\infty] \simeq z_1$ regardless the value of z_2; this reflects a situation in which the "special vertex" is very charismatic in the system and he leads the other elements to synchronize on average towards his initial inclination.

(ii) If $p = 1/N$ with N large, then we have $E[Z_\infty] \simeq z_2$ regardless the value of z_1; this reflects a situation in which the "diversity" of the "special vertex" is dispersed because of the large number of individuals in the population, and so the expected limiting inclination is close to the initial inclination of the majority of the system.

6. Statistical inference.   First of all, we observe that by means of the central limit theorem for $\widetilde{Z}_n = N^{-1/2} \mathbf{v}_1^\top \mathbf{Z}_n$ presented in Theorem 4.2, it is possible to construct, for each $1/2 < \gamma \leq 1$, asymptotic confidence intervals for Z_∞, that is, the limit random variable at which all the stochastic processes $\{(Z_{n,j})_n : 1 \leq j \leq N\}$ converge. Specifically, an asymptotic confidence interval for Z_∞ with approximate level $(1 - \theta)$ is the following:

$$CI_{1-\theta}(Z_\infty) := \big(\widetilde{Z}_n - \widetilde{\sigma}_\gamma \sqrt{\widetilde{Z}_n(1 - \widetilde{Z}_n)} n^{-(\gamma - 1/2)} z_\theta;$$

$$\widetilde{Z}_n + \widetilde{\sigma}_\gamma \sqrt{\widetilde{Z}_n(1 - \widetilde{Z}_n)} n^{-(\gamma - 1/2)} z_\theta\big),$$

where z_θ is such that $\mathcal{N}(0, 1)(z_\theta, +\infty) = \theta/2$.

Note that, in order to compute the above confidence interval, we need to know $\mathbf{v}_1$ (as well as N, c and γ). Nevertheless, it is not required to know the whole weighted adjacency matrix W. For example, for doubly stochastic matrices, the vector $\mathbf{v}_1$ is known (see Remark 4.1).

We now focus on the inferential problem of testing the hypothesis that the network is characterized by a given weighted adjacency matrix W_0, that is, $H_0 : W = W_0$, using the multi-dimensional stochastic process $(\mathbf{Z}_n)_n$ observed at the vertices. Since the distribution of Z_∞ is unknown, we propose a test statistics whose limit does not involve Z_∞. The parameters N, c and γ are again considered known.

First, we need to introduce some notation. Given a $N \times N$ positive semidefinite matrix Σ of rank $1 \leq r \leq (N - 1)$ and having spectral decomposition $\Sigma = O \Lambda O^\top$ (more precisely, Λ is the diagonal matrix containing the eigenvalues of Σ and the columns of O form a corresponding orthonormal basis of right eigenvectors), we denote by L the diagonal matrix such that

$$[L]_{h,j} = \begin{cases} \lambda_j^{-1/2} & \text{if } h = j \text{ and } \lambda_j > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and by H the $r \times N$-matrix such that

$$[H]_{h,j} = \begin{cases} 1 & \text{if } h = j \text{ and } 1 \leq h \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Then:

(a) when $1/2 < \gamma < 1$, take $\Sigma = \widehat{\Sigma}_\gamma$ with rank $r = (N-1)$ and set $O_\gamma = O$, $L_\gamma = L$, $H_\gamma = H$ and $M_\gamma = H_\gamma L_\gamma O_\gamma^\top$;

(b) when $\gamma = 1$ and $\lambda^* < 1 - (2c)^{-1}$, take $\Sigma = \widehat{\Sigma}_1$ with rank $r = (N-1)$ and set $O_1 = O$, $L_1 = L$, $H_1 = H$ and $M_1 = H_1 L_1 O_1^\top$;

(c) when $\gamma = 1$ and $\lambda^* = 1 - (2c)^{-1}$, take $\Sigma = \widehat{\Sigma}_1^*$ with rank r equal to the cardinality m^* of the set

$$A^* = \{\lambda_j \in \mathrm{Sp}(W) : \mathcal{R}e(\lambda_j) = 1 - (2c)^{-1}\},$$

defined in (3.6) and set $O_1^* = O$, $L_1^* = L$, $H_1^* = H$ and $M_1^* = H_1^* L_1^* (O_1^*)^\top$.

Fixed the weighted adjacency matrix assumed under H_0, that is, W_0, we can compute for it the vector $\mathbf{v_1}$ and the matrices U and V as defined in Section 2. Hence, we can obtain under H_0 the real process $\widetilde{Z}_n = N^{-1/2}(\mathbf{v}_1^\top \mathbf{Z}_n)$ and the multi-dimensional process $\widehat{\mathbf{Z}}_n = (I - N^{-1/2}\mathbf{1}\mathbf{v}_1^\top)\mathbf{Z}_n = UV^\top\mathbf{Z}_n$. Then, using (4.6), (4.18), (4.19) and applying Lemma A.7, we have under H_0 that:

(a) for $1/2 < \gamma < 1$,

$$(6.1) \qquad \mathbf{T}_{\gamma,n} := n^{\gamma/2}[\widetilde{Z}_n(1 - \widetilde{Z}_n)]^{-1/2} M_\gamma UV^\top \mathbf{Z}_n,$$

(b) for $\gamma = 1$ and $\lambda^* < 1 - (2c)^{-1}$,

$$(6.2) \qquad \mathbf{T}_{1,n} := n^{1/2}[\widetilde{Z}_n(1 - \widetilde{Z}_n)]^{-1/2} M_1 UV^\top \mathbf{Z}_n,$$

are asymptotically normal distributed with covariance matrix equal to the $(N - 1) \times (N - 1)$ identity matrix (in the above formulas the matrices M_γ and M_1 are those related to $\widehat{\Sigma}_\gamma$ and $\widehat{\Sigma}_1$, respectively, computed for W_0). Hence, both the test statistics $\|\mathbf{T}_{\gamma,n}\|^2$ and $\|\mathbf{T}_{1,n}\|^2$ are asymptotically chi-squared distributed with $(N - 1)$ degrees of freedom. In the case (c), that is, $\gamma = 1$ and $\lambda^* = 1 - (2c)^{-1}$, using (4.6), (4.20) and applying Lemma A.7, we have under H_0 that

$$(6.3) \qquad \mathbf{T}_{1,n}^* := \sqrt{\frac{n}{\ln(n)}}[\widetilde{Z}_n(1 - \widetilde{Z}_n)]^{-1/2} M_1^* UV^\top \mathbf{Z}_n$$

is asymptotically normal distributed with covariance matrix equal to the $m^* \times m^*$ identity matrix (in the above formula the matrix M_1^* is the one related to $\widehat{\Sigma}_1^*$ computed for W_0), and hence the test statistics $\|\mathbf{T}_{1,n}^*\|^2$ is asymptotically chi-squared distributed with m^* degrees of freedom. These results let us construct asymptotic

critical regions for testing any W_0. The performance in terms of power of these inferential procedures is strongly related to the considered adjacency matrix W_1 belonging to the alternative hypothesis H_1. As an example, let us consider case (a), since (b) and (c) are analogous. First, note that the vector $\mathbf{v}_1$, computed under $H_0 : W = W_0$, may not be the eigenvector of W_1 associated to $\lambda_1 = 1$, and so we may have under H_1 that $\widetilde{Z}_n \neq N^{-1/2}(\mathbf{v}_1^\top \mathbf{Z}_n)$. However, by (2.5) and Theorem 3.1, we still have that $N^{-1/2}(\mathbf{v}_1^\top \mathbf{Z}_n) \xrightarrow{\text{a.s.}} N^{-1/2}(\mathbf{v}_1^\top \mathbf{1})Z_\infty = Z_\infty$, which implies that in (6.1) $[\widetilde{Z}_n(1 - \widetilde{Z}_n)]^{-1/2} \xrightarrow{\text{a.s.}} [Z_\infty(1 - Z_\infty)]^{-1/2}$ remains valid under H_1. Analogously, note that the columns of U and V, computed under $H_0 : W = W_0$, may not be the eigenvectors of W_1 associated to $\mathrm{Sp}(W_1) \setminus \{1\}$. However, by (4.1), (4.2) and (4.4), we still have that $UV^\top \mathbf{Z}_n = UV^\top \widehat{\mathbf{Z}}_n$ holds under H_1. As a consequence, under H_1 we have that $\mathbf{T}_{\gamma,n}$ is asymptotically normal distributed with covariance matrix $M_\gamma UV^\top \widehat{\Sigma}_\gamma VU^\top M_\gamma^\top$, where $\widehat{\Sigma}_\gamma$ is here computed using the eigenstructure of W_1, while M_γ is related to $\widehat{\Sigma}_\gamma$ computed for W_0. Consequently, the distance between the identity matrix I and the matrix $M_\gamma UV^\top \widehat{\Sigma}_\gamma VU^\top M_\gamma^\top$ describes the relation between the asymptotic distribution of $\|\mathbf{T}_{\gamma,n}\|^2$ under H_0 and the one under H_1, which determines the power of the test. For instance, note that $E[\|\mathbf{T}_{\gamma,n}\|^2] = (n - 1)$ under H_0, while $E[\|\mathbf{T}_{\gamma,n}\|^2]$ is equal to the trace of $M_\gamma UV^\top \widehat{\Sigma}_\gamma VU^\top M_\gamma^\top$ under H_1.

We now apply these testing procedures to the meaningful examples of weighted adjacency matrices considered in Section 5.

6.1. *"Mean-field" interaction.* Consider the family of weighted adjacency matrices $\{W_\alpha; \alpha \in (0, 1]\}$ defined in (5.1). It may be of interest to test whether the unknown parameter α can be assumed to be equal to a specific value $\alpha_0 \in (0, 1]$, that is,

$$H_0 : W = W_{\alpha_0} \quad \text{vs} \quad H_1 : W = W_\alpha \qquad \text{for some } \alpha \in (0, 1] \setminus \{\alpha_0\}.$$

In this case, assuming $2c\alpha_0 \geq 1$ when $\gamma = 1$, by the results presented in Section 5.1, using $\mathbf{v}_1$ and $U = V$ computed for W_{α_0}, we have:

(a) for $1/2 < \gamma < 1$, $\mathbf{T}_{\gamma,n} = n^{\gamma/2}[\widetilde{Z}_n(1 - \widetilde{Z}_n)]^{-1/2}\sqrt{\frac{2\alpha_0}{c}}U^\top \mathbf{Z}_n$;
(b) for $\gamma = 1$ and $2c\alpha_0 > 1$, $\mathbf{T}_{1,n} = n^{1/2}[\widetilde{Z}_n(1 - \widetilde{Z}_n)]^{-1/2}\frac{\sqrt{2c\alpha_0 - 1}}{c}U^\top \mathbf{Z}_n$;
(c) for $\gamma = 1$ and $2c\alpha_0 = 1$, $\mathbf{T}_{1,n}^* = \sqrt{\frac{n}{\ln(n)}}[\widetilde{Z}_n(1 - \widetilde{Z}_n)]^{-1/2}\frac{1}{c}U^\top \mathbf{Z}_n$,

where $\widetilde{Z}_n = N^{-1}\mathbf{1}^\top \mathbf{Z}_n$. Under H_0, we have $\|\mathbf{T}_{\gamma,n}\|^2, \|\mathbf{T}_{1,n}\|^2, \|\mathbf{T}_{1,n}^*\|^2 \overset{d}{\sim} \chi^2_{N-1}$. Concerning the distribution of the test statistics for $\alpha \neq \alpha_0$, notice that the eigenvectors of W do not depend on α and so U is the same for any α. Therefore, for any fixed $\alpha \in (0, 1] \setminus \{\alpha_0\}$, under the hypothesis $\{W = W_\alpha\} \subset H_1$ we have:

(a) for $1/2 < \gamma < 1$, $\|\mathbf{T}_{\gamma,n}\|^2 \overset{d}{\sim} (\frac{\alpha_0}{\alpha})\chi^2_{N-1}$;

(b) for $\gamma = 1$ and $2c\alpha_0 > 1$, $\|\mathbf{T}_{1,n}\|^2 \overset{d}{\sim} (\frac{2c\alpha_0-1}{2c\alpha-1})\chi^2_{N-1}$ if $2c\alpha > 1$, and $\|\mathbf{T}_{1,n}\|^2 \overset{P}{\to} +\infty$ if $2c\alpha = 1$;

(c) for $\gamma = 1$ and $2c\alpha_0 = 1$, $\|\mathbf{T}^*_{1,n}\|^2 \overset{P}{\to} 0$ for $2c\alpha > 1$.

6.2. *"Cycle" interaction.* We could test whether the weighted adjacency matrix is the one, say W_0, defined in (5.2). Then we consider the following hypothesis test:

$$H_0 : W = W_0 \quad \text{vs} \quad H_1 : W \neq W_0.$$

Once obtained the eigenstructure of $\widehat{\Sigma}_\gamma$, $\widehat{\Sigma}_1$ and $\widehat{\Sigma}^*_1$, we can define $\mathbf{T}_{\gamma,n}$, $\mathbf{T}_{1,n}$ and $\mathbf{T}^*_{1,n}$ as in (6.1), (6.2) and (6.3), respectively, and under H_0 we have that:

(a) for $1/2 < \gamma < 1$, $\|\mathbf{T}_{\gamma,n}\|^2 \overset{d}{\sim} \chi^2_{N-1}$;

(b) for $\gamma = 1$ and $2c(1 - \cos(2\pi/N)) > 1$, $\|\mathbf{T}_{1,n}\|^2 \overset{d}{\sim} \chi^2_{N-1}$;

(c) for $\gamma = 1$ and $2c(1 - \cos(2\pi/N)) = 1$, $\|\mathbf{T}^*_{1,n}\|^2 \overset{d}{\sim} \chi^2_2$.

6.3. *"Special vertex" case.* We could test whether there is a "special vertex" in the network, that is, the weighted adjacency matrix is the one, say W_p, defined in (5.3), and so in this case the considered hypothesis test is the following:

$$H_0 : W = W_p \quad \text{vs} \quad H_1 : W \neq W_p.$$

From Section 5.3, we get that $\widehat{\Sigma}_\gamma = \frac{c}{2}A_p$, $\widehat{\Sigma}_1 = \frac{c^2}{2c-1}A_p$ and $\widehat{\Sigma}^*_1 = \frac{1}{4}A_p$, where $A_p = (I - \mathbf{1}\mathbf{a}_p^\top)(I - \mathbf{a}_p\mathbf{1}^\top)$. Hence, since A_p has rank $(N-1)$, we have under H_0 that $\|\mathbf{T}_{\gamma,n}\|^2$, $\|\mathbf{T}_{1,n}\|^2$ and $\|\mathbf{T}^*_{1,n}\|^2$ defined as in (6.1), (6.2) and (6.3) are all asymptotically chi-squared distributed with $(N-1)$ degrees of freedom. Note that, differently from the "mean-field" case, the eigenvectors of $\widehat{\Sigma}_\gamma$, $\widehat{\Sigma}_1$ and $\widehat{\Sigma}^*_1$ here change with p. Hence, in this case the power of the test may be more efficiently investigated through a numerical study on the distance between the matrices I and $M_\gamma U V^\top \widehat{\Sigma}_\gamma V U^\top M_\gamma^\top$, as discussed above in the general framework.

7. Variants. We can consider the following two variants.

7.1. *The case of a "forcing input".* As in [21], we can consider the following variant:

$$(7.1) \qquad \mathbf{Z}_{n+1} = (1 - r_n)\mathbf{Z}_n + r_n[\rho\mathbf{X}_{n+1} + (1 - \rho)q\mathbf{1}],$$

where $E[\mathbf{X}_n|\mathcal{F}_{n-1}] = W^\top\mathbf{Z}_{n-1}$, $\rho \in [0, 1[$ and $q \in [0, 1]$. The assumptions on W and (r_n) are the same as in the previous sections. (Here, we exclude the case $\rho = 1$ since it corresponds to the model studied in the previous sections.) Note that this variant of the model contains the classical Friedman's urn [indeed, if a is

the number of added balls with the same color of the extracted one and b is the number of added balls with opposite color, with $a \geq b$, then it is enough to take $\rho = (a - b)/(a + b)$ and $q = 1/2$].

With the same notation as before, we consider the decomposition (4.4). In particular, setting $\widetilde{Z}_n = N^{-1/2}\mathbf{v}_1^\top \mathbf{Z}_n$, we obtain the dynamics

$$\widetilde{Z}_{n+1} - \widetilde{Z}_n = -(1 - \rho)r_n(\widetilde{Z}_n - q) + \rho r_n N^{-1/2}(\mathbf{v}_1^\top \Delta \mathbf{M}_{n+1}),$$

where $\Delta \mathbf{M}_{n+1} = \mathbf{X}_{n+1} - W^\top \mathbf{Z}_n$. Therefore, we have

$$\widetilde{Z}_{n+1} - q = \big(1 - (1 - \rho)r_n\big)(\widetilde{Z}_n - q) + \rho r_n N^{-1/2}(\mathbf{v}_1^\top \Delta \mathbf{M}_{n+1})$$

and so

$$(\widetilde{Z}_{n+1} - q)^2 = \big(1 - 2r_n(1 - \rho)\big)(\widetilde{Z}_n - q)^2 + r_n^2\big[(1 - \rho)^2(\widetilde{Z}_n - q)^2$$
$$+ \rho^2 N^{-1}(\mathbf{v}_1^\top \Delta \mathbf{M}_{n+1})^2\big].$$

It follows

$$E\big[(\widetilde{Z}_{n+1} - q)^2|\mathcal{F}_n\big] = \big(1 - 2r_n(1 - \rho)\big)(\widetilde{Z}_n - q)^2$$
$$+ r_n^2\Big\{(1 - \rho)^2(\widetilde{Z}_n - q)^2 + \frac{\rho^2}{N}E\big[(\mathbf{v}_1^\top \Delta \mathbf{M}_{n+1})^2|\mathcal{F}_n\big]\Big\}$$
$$\leq (\widetilde{Z}_n - q)^2 + r_n^2\xi_n.$$

Since $(\xi_n)_n$ is a bounded sequence of $\mathcal{F}_n$-measurable random variables and $\sum_n r_n^2 < +\infty$ for $1/2 < \gamma \leq 1$, we have that $((\widetilde{Z}_n - q)^2)_n$ is a positive almost supermartingale and so it converges almost surely (and also in mean since it is bounded). On the other hand, from the above computations, we also get

$$E\big[(\widetilde{Z}_{n+1} - q)^2\big] = \big(1 - 2r_n(1 - \rho)\big)E\big[(\widetilde{Z}_n - q)^2\big] + r_n^2 E[\xi_n]$$

and so, since $\rho < 1$, by [21], Lemma A.1, we can conclude that $\lim_n E[(\widetilde{Z}_n - q)^2] = 0$. Therefore, we obtain

$$\widetilde{Z}_n \xrightarrow{\text{a.s.}} q.$$

Regarding $\widehat{\mathbf{Z}}_n = \mathbf{Z}_n - \mathbf{1}\widetilde{Z}_n = UV^\top \mathbf{Z}_n$, we can prove again that it converges almost surely to $\mathbf{0}$. Indeed, if we set $\mathbf{Z}_{V,n} = V^\top \widehat{\mathbf{Z}}_n = V^\top \mathbf{Z}_n$, that is $\widehat{\mathbf{Z}}_n = U\mathbf{Z}_{V,n}$ by (4.1), we get from (7.1)

$$\mathbf{Z}_{V,n+1} = (1 - r_n)\mathbf{Z}_{V,n} + r_n\big[\rho V^\top \Delta \mathbf{M}_{n+1} + \rho V^\top W^\top \mathbf{Z}_n + (1 - \rho)q V^\top \mathbf{1}\big]$$
$$= (1 - r_n)\mathbf{Z}_{V,n}$$
$$+ r_n\big[\rho V^\top \Delta \mathbf{M}_{n+1} + \rho V^\top(\mathbf{u}_1\mathbf{v}_1^\top + UDV^\top)\mathbf{Z}_n + (1 - \rho)q N^{1/2} V^\top \mathbf{u}_1\big]$$
$$= (1 - r_n)\mathbf{Z}_{V,n} + r_n\rho D\mathbf{Z}_{V,n} + r_n\rho V^\top \Delta \mathbf{M}_{n+1}$$
$$= \big[I - r_n(I - \rho D)\big]\mathbf{Z}_{V,n} + r_n\rho V^\top \Delta \mathbf{M}_{n+1},$$

where I here denotes the $(N-1) \times (N-1)$ identity matrix. Arguing as in the proof of Theorem 4.1, we can obtain that $(\|\mathbf{Z}_{V,n}\|^2)_n$ is a positive almost supermartingale, which satisfies

$$x_{n+1} \leq (1 - 2a^* r_n) x_n + C r_n^2$$

with $x_n = E[\|\mathbf{Z}_{V,n}\|^2]$, $a^* = 1 - \rho \mathcal{R}e(\lambda^*)$ and C a suitable constant. Since $\rho \mathcal{R}e(\lambda^*) < 1$, we have $a^* > 0$, which implies $\lim_n x_n = 0$ and so $\|\mathbf{Z}_{V,n}\|^2 \to 0$ almost surely, that is, $\mathbf{Z}_{V,n} \to \mathbf{0}$ almost surely.

Summing up, also for the considered variant, we have an almost sure synchronization, that is all the stochastic processes $\{(Z_{n,j})_n : j \in V\}$ converge almost surely to the same limit, but in this case *the limit is the constant "forcing input" q*. It is interesting to observe that this occurs for any weighted adjacency matrix W satisfying the required assumptions. It is also worthwhile to note that, in this case, for the above computations, we do not need the condition $\mathcal{R}e(\lambda^*) < 1$ since $\rho < 1$ automatically implies $\rho \mathcal{R}e(\lambda^*) < 1$ when $\mathcal{R}e(\lambda^*) \leq 1$.

We refer to [21] for some functional central limit theorems in the case of $\rho < 1$ and the mean-field interaction.

7.2. *The case of a reducible weighted adjacency matrix.*

We now present an extension of the theory presented in this paper to the case of a *reducible* weighted adjacency matrix (see [2] for a similar approach to systems of interacting generalized Friedman's urns).

We here consider a particular decomposition of W that individuates subgraphs composed by processes $(Z_{n,k})_n$ that evolve with different behaviors. The same decomposition has been applied to the interacting matrix in [2], and it is typically applied to the transition matrices in the context of discrete time-homogeneous Markov chains (see [39]) to characterize the state space. More precisely, denoting by m, with $1 \leq m \leq N$, the multiplicity of the eigenvalue 1 of W, that is, $\lambda_1 = \cdots = \lambda_m = 1$, the reducible matrix W can be decomposed as follows (see [39], Example 1.2.2):

$$(7.2) \qquad W = \begin{bmatrix} W_1 & 0 & \cdots & 0 & W_{1,f} \\ 0 & W_2 & \cdots & 0 & W_{2,f} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & W_m & W_{m,f} \\ 0 & 0 & \cdots & 0 & W_f \end{bmatrix},$$

where:

  (i) $\{W_j; 1 \leq j \leq m\}$ are irreducible $N_j \times N_j$-matrices with $\lambda_{\max}(W_j) = 1$ that identify the recurrent communicating classes in the state space;
  (ii) (if $\sum_{j=1}^m N_j \leq N - 1$) W_f is a $N_f \times N_f$-matrix with $\lambda_{\max}(W_f) < 1$ that contains all the transient communicating classes in the state space;
  (iii) (if $\sum_{j=1}^m N_j \leq N - 1$) $\{W_{j,f}; 1 \leq j \leq m\}$ are $N_j \times N_f$-matrices.

Obviously, when $\sum_{j=1}^{m} N_j = N$ we have $N_f = 0$, and hence the elements in $\{W_{j,f}; 1 \le j \le m\}$ and W_f do not exist. This occurs when all the classes are closed and recurrent, and hence the state space can be partitioned into irreducible and disjoint subspaces. In the particular case of W irreducible considered previously in the paper, there is only one closed and recurrent class, and hence $m = 1$, $N_1 = N$ and $N_f = 0$.

The structure of W given in (7.2) leads to a natural decomposition of the graph in different subgraphs $\{G_j; 1 \le j \le m\}$ associated to the submatrices $\{W_j; 1 \le j \le m\}$ and G_f associated to W_f. Specifically, from (7.2) we can deduce that, for each $1 \le j \le m$, the vertices in G_j are not influenced by the vertices in the rest of the network, and hence the dynamics of the processes in G_j can be fully established by considering only the correspondent irreducible submatrix W_j (see [2] for further details). Then, applying the results presented in this paper to each subgraph G_j, it is possible to show that all the processes positioned at the vertices in the same G_j synchronize, that is they all converge almost surely to the same random limit. Concerning the subgraph G_f, the weighted adjacency matrix in (7.2) shows that its vertices are influenced by the vertices in $\{G_j; 1 \le j \le m\}$. Therefore, applying similar arguments to the ones presented above, it is possible to establish that the processes positioned at the vertices in G_f converge almost surely to convex combinations of the limits of the processes positioned at the vertices in $\{G_j; 1 \le j \le m\}$, where the weights of such combinations are related to the matrices $\{W_{j,f}; 1 \le j \le m\}$ and W_f.

We now formalize the considerations illustrated above. To this end, let us first define the following quantities:

(1) for any $k \in \{1, \ldots, N\}$, let j_k be the index in $\{1, \ldots, m, f\}$ such that $k \in G_{j_k}$, that is, j_k denotes the index of the subgraph containing the vertex k;

(2) for any $k \in \{1, \ldots, N\}$, let $i_k := k - \sum_{m=1}^{j_k - 1} N_m$ [with $\sum_{m=1}^{0} N_m = 0$ and $(f - 1) = m$], that is, i_k denotes the position of the vertex k within the subgraph G_{j_k} it belongs to;

(3) for any $j \in \{1, \ldots, m\}$, let $\mathbf{f}_j \in \mathbb{R}_+^{N_f}$ and $\mathbf{a}_j \in \mathbb{R}_+^{N}$ be defined as follows:

$$\mathbf{f}_j := (I - W_f^\top)^{-1} W_{j,f}^\top \mathbf{1} \quad \text{and} \quad [\mathbf{a}_j]_k = \begin{cases} 1 & \text{if } k \in G_j, \\ [\mathbf{f}_j]_{i_k} & \text{if } k \in G_f, \\ 0 & \text{otherwise;} \end{cases}$$

(4) for any $j \in \{1, \ldots, m\}$, let $\mathbf{v}_{(j)} \in \mathbb{R}_+^{N_j}$ be the right eigenvector of W_j associated to the eigenvalue 1 such that $\mathbf{1}^\top \mathbf{v}_{(j)} = \|\mathbf{a}_j\| = \sqrt{N_j + \|\mathbf{f}_j\|^2}$.

Hence, the m left and right eigenvectors of W associated to the eigenvalue 1 can be taken as follows: $\mathbf{u}_j = \mathbf{a}_j / \|\mathbf{a}_j\|$ and

$$[\mathbf{v}_j]_k = \begin{cases} [\mathbf{v}_{(j)}]_{i_k} & \text{if } k \in G_j, \\ 0 & \text{otherwise,} \end{cases}$$

so that $\|\mathbf{u}_j\| = 1$ and relations (2.4) hold, that is, $\mathbf{u}_j^\top \mathbf{v}_j = 1$ and, for any $1 \le j \ne h \le m$, $\mathbf{u}_j^\top \mathbf{v}_h = 0$. Moreover, denoting by D the diagonal matrix with the $(N - m)$ eigenvalues $\lambda_j \in \mathrm{Sp}(W) \setminus \{1\}$, and by U and V the matrices whose columns are the corresponding left and right eigenvectors, respectively, we have that relations (4.1) hold and the decomposition of $W^\top$ as in (4.2) is here obtained as follows:

$$W^\top = \sum_{j=1}^m \mathbf{u}_j \mathbf{v}_j^\top + U D V^\top = \sum_{j=1}^m \frac{1}{\|\mathbf{a}_j\|} \mathbf{a}_j \mathbf{v}_j^\top + U D V^\top.$$

Hence, the decomposition of the process $\mathbf{Z}_n$ in (4.4) is here replaced by the following:

$$(7.3) \qquad \mathbf{Z}_n = \sum_{j=1}^m \mathbf{a}_j \widetilde{Z}_{n,j} + \widehat{\mathbf{Z}}_n = \sum_{j=1}^m \mathbf{u}_j \|\mathbf{a}_j\| \widetilde{Z}_{n,j} + \widehat{\mathbf{Z}}_n,$$

where

$$\begin{cases} \text{for any } j \in \{1, \dots, m\}, & \widetilde{Z}_{n,j} := \|\mathbf{a}_j\|^{-1} \mathbf{v}_j^\top \mathbf{Z}_n, \\ \widehat{\mathbf{Z}}_n = \mathbf{Z}_n - \sum_{j=1}^m \mathbf{a}_j \widetilde{Z}_{n,j} = \left(I - \sum_{j=1}^m \mathbf{u}_j \mathbf{v}_j^\top \right) \mathbf{Z}_n = U V^\top \mathbf{Z}_n. \end{cases}$$

For any $1 \le j \le m$, let us consider the real-valued stochastic process $(\widetilde{Z}_{n,j})_n$. Since all the elements of $\mathbf{v}_{(j)}$ are positive and $\mathbf{1}^\top \mathbf{v}_{(j)} = \|\mathbf{a}_j\|$, the elements of the vector $\|\mathbf{a}_j\|^{-1} \mathbf{v}_j$ can be seen as the weights of a convex combination, so that $0 \le \widetilde{Z}_{n,j} \le 1$ for any n. Moreover, it is easy to see that it is an $\mathcal{F}$-martingale. Indeed, from (4.3) we obtain that its dynamics can be expressed as follows:

$$\widetilde{Z}_{n+1,j} - \widetilde{Z}_{n,j} = \|\mathbf{a}_j\|^{-1} r_n (\mathbf{v}_j^\top \Delta \mathbf{M}_{n+1}),$$

where $\Delta \mathbf{M}_{n+1} = \mathbf{X}_{n+1} - W^\top \mathbf{Z}_n$. Hence, we immediately get

$$\widetilde{Z}_{n,j} \xrightarrow{\text{a.s.}} Z_{\infty,j},$$

where $Z_{\infty,j}$ is a random variable with values in $[0, 1]$.

Arguing as in the proof of Theorem 4.1, we can obtain that the process $(\widehat{\mathbf{Z}}_n)_n$ converges to zero a.s., and hence by (7.3) we obtain

$$(7.4) \qquad \mathbf{Z}_n = \sum_{j=1}^m \mathbf{a}_j \widetilde{Z}_{n,j} + \widehat{\mathbf{Z}}_n \xrightarrow{\text{a.s.}} \mathbf{Z}_\infty := \sum_{j=1}^m \mathbf{a}_j Z_{\infty,j},$$

that is,

$$Z_{n,k} \xrightarrow{\text{a.s.}} \sum_{j=1}^m [\mathbf{a}_j]_k Z_{\infty,j}.$$

To interpret the convergence result expressed in (7.4), denote, for each $j \in \{1, \dots, m, f\}$, the N_j-dimensional vector $\mathbf{Z}_{n(j)} := (Z_{n,k}; k \in G_j)^\top$ composed by the processes positioned at the vertices in the same subgraph G_j, and note that:

- if $k \in G_j$ with $1 \leq j \leq m$, then $[\mathbf{a}_j]_k = 1$ and $[\mathbf{a}_h]_k = 0$ for any $h \neq j$, and hence we have

$$\mathbf{Z}_{n(j)} \xrightarrow{\text{a.s.}} \mathbf{Z}_{\infty(j)} := \mathbf{1} Z_{\infty,j};$$

this means that all the processes $(Z_{n,k})_n$ positioned at the vertices in the same subgraph G_j, with $1 \leq j \leq m$, synchronize, that is, they all converge almost surely to the same random limit $Z_{\infty,j}$;
- if $k \in G_f$, then $[\mathbf{a}_j]_k = [\mathbf{f}_j]_{i_k}$, and hence

$$\mathbf{Z}_{n(f)} \xrightarrow{\text{a.s.}} \mathbf{Z}_{\infty(f)} := \sum_{j=1}^m \mathbf{f}_j Z_{\infty,j} = (I - W_f^\top)^{-1} \sum_{j=1}^m W_{j,f}^\top \mathbf{Z}_{\infty(j)}.$$

Moreover, notice that, since $\sum_{j=1}^m (W_{j,f}^\top \mathbf{1}) = (I - W_f^\top)\mathbf{1}$, we have

$$\sum_{j=1}^m \mathbf{f}_j = (I - W_f^\top)^{-1} \sum_{j=1}^m (W_{j,f}^\top \mathbf{1}) = \mathbf{1},$$

and so we can interpret $\{\mathbf{f}_j; 1 \leq j \leq m\}$ as the weights of a convex combination and, consequently, the processes $(Z_{n,k})_n$ positioned at the vertices $k \in G_f$ do not synchronize but each of them converges to a suitable weighted average of the m random variables $\{Z_{\infty,j}; 1 \leq j \leq m\}$.

Therefore, the random vector $\mathbf{Z}_\infty$ can be decomposed as

$$\mathbf{Z}_\infty = (\mathbf{Z}_{\infty(1)}, \dots, \mathbf{Z}_{\infty(m)}, \mathbf{Z}_{\infty(f)})^\top.$$

We conclude observing that it is possible to realize a more complete analysis of the random limit of the processes $(Z_{n,k})_n$ positioned at the vertices in the subgraph G_f. Indeed, using that $W^\top \mathbf{a}_j = \mathbf{a}_j$ for any $j \in \{1, \dots, m\}$ (which follows from $\mathbf{u}_j = \mathbf{a}_j / \|\mathbf{a}_j\|$ and $W^\top \mathbf{u}_j = \mathbf{u}_j$) and (7.4), we obtain the following relation on $\mathbf{Z}_\infty$:

$$\begin{aligned} W^\top \mathbf{Z}_\infty = W^\top \left(\sum_{j=1}^m \mathbf{a}_j Z_{\infty,j} \right) &= \sum_{j=1}^m (W^\top \mathbf{a}_j) Z_{\infty,j} \\ &= \sum_{j=1}^m \mathbf{a}_j Z_{\infty,j} = \mathbf{Z}_\infty. \end{aligned}$$

(7.5)

The above relation suggests that a more detailed decomposition of W may be useful to better characterize the elements in $\mathbf{Z}_\infty$, and in particular in $\mathbf{Z}_{\infty(f)}$. To this end, we recall the decomposition of W in (7.2) and we express W_f and $\{W_{j,f}; 1 \leq j \leq m\}$ as follows:

(7.6) $$W_f = \begin{bmatrix} W_{m+1}^{(f)} & W_{m+1,m+2}^{(f)} & \cdots & W_{m+1,m+m_f}^{(f)} \\ 0 & W_{m+2}^{(f)} & \cdots & W_{m+2,m+m_f}^{(f)} \\ \vdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & W_{m+m_f}^{(f)} \end{bmatrix},$$

where:

(iv) $\{W_h^{(f)}; m+1 \le h \le m+m_f\}$ are irreducible $N_h \times N_h$-matrices with $\lambda_{\max}(W_h^{(f)}) < 1$, that identify the transient communicating classes in the state space;

(v) $\{W_{l,h}^{(f)}; m+1 \le l \le m+m_f-1, m+2 \le h \le m+m_f\}$ are $N_l \times N_h$-matrices;

$$(7.7) \qquad \begin{bmatrix} W_{1,f} \\ W_{2,f} \\ \vdots \\ W_{m,f} \end{bmatrix} = \begin{bmatrix} W_{1,m+1}^{(f)} & W_{1,m+2}^{(f)} & \cdots & & W_{1,m+m_f}^{(f)} \\ W_{2,m+1}^{(f)} & W_{2,m+2}^{(f)} & \cdots W_{2,m+m_f}^{(f)} & \\ \vdots & \vdots & \cdots & & \vdots \\ W_{m,m+1}^{(f)} & W_{m,m+2}^{(f)} & \cdots & & W_{m,m+m_f}^{(f)} \end{bmatrix},$$

where

(vi) $\{W_{j,h}^{(f)}; 1 \le j \le m, m+1 \le h \le m+m_f\}$ are $N_j \times N_h$-matrices.

Analogously to what we have done in the previous part, the full structure of W obtained combining (7.2), (7.6) and (7.7) leads to a complete decomposition of the graph in different subgraphs: $\{G_j; 1 \le j \le m\}$ associated to the submatrices $\{W_j; 1 \le j \le m\}$ considered in the previous part, and $\{G_h; m+1 \le h \le m+m_f\}$ associated to the submatrices $\{W_h^{(f)}; m+1 \le h \le m+m_f\}$, where naturally $G_h \subset G_f$ for any $m+1 \le h \le m+m_f$. Then, for each $h \in \{m+1, \dots, m+m_f\}$, we denote by $\mathbf{Z}_{n(h)} := (Z_{n,k}; k \in G_h)^\top$ the N_h-dimensional vector composed by the processes positioned at the vertices in the same subgraph G_h. Hence, setting $\mathbf{Z}_{\infty(h)} := \text{a.s.-}\lim_n \mathbf{Z}_{n(h)}$ and decomposing $\mathbf{Z}_\infty$ as $\mathbf{Z}_\infty = (\mathbf{Z}_{\infty(1)}, \dots, \mathbf{Z}_{\infty(m+m_f)})^\top$, by (7.5), we get for each $G_h \subset G_f$ the relation

$$\mathbf{Z}_{\infty(h)} = \sum_{j=1}^m (W_{j,h})^\top \mathbf{Z}_{\infty(j)} + \sum_{l=m+1}^{h-1} (W_{l,h}^{(f)})^\top \mathbf{Z}_{\infty(l)} + (W_h^{(f)})^\top \mathbf{Z}_{\infty(h)},$$

which implies

$$\mathbf{Z}_{\infty(h)} := (I - (W_h^{(f)})^\top)^{-1} \left(\sum_{j=1}^m (W_{j,h})^\top \mathbf{Z}_{\infty(j)} + \sum_{l=m+1}^{h-1} (W_{l,h}^{(f)})^\top \mathbf{Z}_{\infty(l)} \right).$$

APPENDIX A: SOME TECHNICAL RESULTS

Throughout the sequel, given (a_n), (b_n) two sequences of real numbers with $b_n \ge 0$, the notation $a_n = O(b_n)$ means $|a_n| \le Cb_n$ for a suitable constant $C > 0$ and n large enough. Therefore, if we also have $a_n^{-1} = O(b_n^{-1})$, then $C'b_n \le |a_n| \le Cb_n$ for suitable constants $C, C' > 0$ and n large enough. Given (z_n), (z_n') two sequences of complex numbers, with $z_n' \ne 0$, the notation $z_n \sim zz_n'$, with $z \in \mathbb{C}$, $z \ne 0$, means $\lim_n z_n/z_n' = z$ and the notation $z_n = o(z_n')$ means $\lim_n z_n/z_n' = 0$.

A.1. Asymptotic results for sums of complex numbers. We start recalling Toeplitz lemma (see [35]), from which we get useful corollaries employed in our proofs.

LEMMA A.1 (Toeplitz lemma). *Let $\{x_{n,k} : 1 \le k \le k_n\}$ be a triangular array of real numbers with $k_n \uparrow +\infty$ and such that*:

(i) $\lim_n x_{n,k} = 0$ *for each fixed k*;
(ii) $\lim_n \sum_{k=1}^{k_n} x_{n,k} = 1$;
(iii) $\sum_{k=1}^{k_n} |x_{n,k}| = O(1)$.

Let $(y_n)_n$ be a sequence of real numbers with $\lim_n y_n = y \in \mathbb{R}$. Then we have $\lim_n \sum_{k=1}^{k_n} x_{n,k} y_k = y$.

REMARK A.1. If in the above lemma we replace condition (ii) by $\lim_n \sum_{k=1}^{k_n} x_{n,k} = 0$, we get $\lim_n \sum_{k=1}^{k_n} x_{n,k} y_k = 0$. Indeed, applying Lemma A.1 to $\widetilde{x}_{n,k} = x_{n,k} - (k_n)^{-1}$, we find

$$\lim_n \sum_{k=1}^{k_n} \left(x_{n,k} - \frac{1}{k_n} \right) y_k = \lim_n \sum_{k=1}^{k_n} \widetilde{x}_{n,k} y_k = y.$$

Hence, since $\lim_n \sum_{k=1}^{k_n} y_k / k_n = y$ (again by Lemma A.1), we finally get $\lim_n \sum_{k=1}^{k_n} x_{n,k} y_k = 0$.

From Lemma A.1, we can easily get the following corollary.

COROLLARY A.1. *Let $(x_n)_n$, $(x'_n)_n$ and $(c_n)_n$ be three sequences of real numbers such that $x'_n > 0$, $c_n \ge 0$, $x_n \sim x x'_n$ with $x \in (0, +\infty)$ and $\lim_n c_n = 0$. Suppose to have $\lim_n c_n \sum_{k=1}^{n} x_k = s \in \{0, 1\}$, then $\lim_n c_n \sum_{k=1}^{n} x'_k = s/x$.*

PROOF. By assumption, taking $\varepsilon \in (0, x)$, we have $x_n > (x - \varepsilon) x'_n > 0$ for $n \ge \bar{n}$ with a suitable $\bar{n}$. Moreover, since $c_n \to 0$, we have $\lim_n c_n \sum_{k=1}^{n} x'_k = \lim_n c_n \sum_{k=\bar{n}}^{n} x'_k$. Therefore, without loss of generality, we can suppose $x_n > 0$ for each n. Hence, if $s = 1$, it is enough to apply Lemma A.1 with $x_{n,k} = c_n x_k$, $y_n = x'_n / x_n$, $y = x^{-1}$; if $s = 0$, it is enough to apply Remark A.1 to $x_{n,k} = c_n x_k$. □

The following lemma extends the Toeplitz lemma and Remark A.1 to complex numbers.

LEMMA A.2 (Generalized Toeplitz lemma). *Let $\{z_{n,k} : 1 \le k \le k_n\}$ be a triangular array of complex numbers such that*:

(i) $\lim_n z_{n,k} = 0$ *for each fixed k*;

(ii) $\lim_n \sum_{k=1}^{k_n} z_{n,k} = s \in \{0, 1\}$;

(iii) $\sum_{k=1}^{k_n} |z_{n,k}| = O(1)$.

Let $(w_n)_n$ be a sequence of complex numbers with $\lim_n w_n = w \in \mathbb{C}$. Then we have $\lim_n \sum_{k=1}^{k_n} z_{n,k} w_k = s w$.

PROOF. Set $z_{n,k} = a_{n,k} + i b_{n,k}$, $w_n = c_n + i d_n$ and $w = c + id$. By assumption (i), we have $\lim_n a_{n,k} = 0$ and $\lim_n b_{n,k} = 0$, for each fixed k, and, by assumption (ii), we have $\lim_n \sum_{k=1}^{k_n} a_{n,k} = s$ and $\lim_n \sum_{k=1}^{k_n} b_{n,k} = 0$. Applying Lemma A.1 to $a_{n,k}$, we easily get $\lim_n \sum_{k=1}^{k_n} a_{n,k} c_k = sc$ and $\lim_n \sum_{k=1}^{k_n} a_{n,k} d_k = sd$. Then, applying Remark A.1 to $b_{n,k}$, we find $\lim_n \sum_{k=1}^{k_n} b_{n,k} c_k = 0$ and $\lim_n \sum_{k=1}^{k_n} b_{n,k} d_k = 0$. Therefore, we have

$$\sum_{k=1}^{k_n} z_{n,k} w_k = \sum_{k=1}^{k_n} a_{n,k} c_k - \sum_{k=1}^{k_n} b_{n,k} d_k + i \sum_{k=1}^{k_n} a_{n,k} d_k + \sum_{k=1}^{k_n} b_{n,k} c_k$$

$$\longrightarrow s(c + id) = sw. \qquad \square$$

As before, from this lemma, we can easily get the following corollaries.

COROLLARY A.2. *Let $(z_n)_n$, $(v_n)_n$ and $(w_n)_n$ be three sequences of complex numbers such that $\lim_n v_n = 0$ and $\lim_n w_n = w \neq 0$. Set $z'_n = z_n w_n$ and suppose to have $\lim_n v_n \sum_{k=1}^{n} z_k = s \in \{0, 1\}$ and $|v_n| \sum_{k=1}^{n} |z_k| = O(1)$ or, equivalently, $|v_n| \sum_{k=1}^{n} |z'_k| = O(1)$. Then $\lim_n v_n \sum_{k=1}^{n} z'_k = sw$.*

PROOF. It is enough to apply Lemma A.2 to $z_{n,k} = v_n z_k$ and w_n. To this purpose, note that, by assumption, taking $\varepsilon \in (0, |w|)$ (note that $|w| > 0$ by assumption), we have $0 < |w| - \varepsilon \leq |w_n| \leq |w| + \varepsilon$ for $n \geq \bar{n}$ with $\bar{n}$ large enough. Therefore, by the relation $z'_k = z_k w_k$, we can affirm that

$$|v_n| \sum_{k=\bar{n}}^{n} \frac{|z'_k|}{|w| + \varepsilon} \leq |v_n| \sum_{k=\bar{n}}^{n} |z_k| \leq |v_n| \sum_{k=\bar{n}}^{n} \frac{|z'_k|}{|w| - \varepsilon}$$

and so the two conditions $|v_n| \sum_{k=1}^{n} |z_k| = O(1)$ and $|v_n| \sum_{k=1}^{n} |z'_k| = O(1)$ are equivalent since $|v_n| \to 0$. $\square$

COROLLARY A.3 (Generalized Kronecker lemma). *Let (v_n) and (z_k) be two sequences of complex numbers such that*

$$v_n \neq 0, \qquad \lim_n v_n = 0, \qquad |v_n| \sum_{k=1}^{n} \left| \frac{1}{v_k} - \frac{1}{v_{k-1}} \right| = O(1)$$

and $\sum_n z_n$ is convergent. Then

$$\lim_n v_n \sum_{k=1}^n \frac{z_k}{v_k} = 0.$$

PROOF.   Set $w_n = \sum_{j=n}^{+\infty} z_j$ and observe that, since $\sum_n z_n$ is convergent, we have $\lim_n w_n = w = 0$ and, moreover, we can write

$$v_n \sum_{k=1}^n \frac{z_k}{v_k} = v_n \sum_{k=1}^n \frac{w_k - w_{k+1}}{v_k}$$

$$= v_n \left[\sum_{k=2}^n \left(\frac{1}{v_k} - \frac{1}{v_{k-1}} \right) w_k + \frac{w_1}{v_1} - \frac{w_{n+1}}{v_n} \right]$$

$$= v_n \sum_{k=2}^n \left(\frac{1}{v_k} - \frac{1}{v_{k-1}} \right) w_k + v_n \frac{w_1}{v_1} - w_{n+1}.$$

The second and the third term obviously converge to zero. In order to prove that the first term converges to zero, it is enough to apply Lemma A.2 to w_n and $z_{n,k} = v_n (\frac{1}{v_k} - \frac{1}{v_{k-1}})$.   □

The above corollary is useful to get the following result for complex random variables (that extends the second part of Lemma 2 in [10] and Lemma 4.1(a) in [22] concerning the real case).

LEMMA A.3.   *Let $\mathcal{H} = (\mathcal{H}_n)_n$ be an increasing filtration and (Y_n) a $\mathcal{H}$-adapted sequence of complex random variables such that $E[Y_n|\mathcal{H}_{n-1}] \to Y$ almost surely. Moreover, let (c_n) be a sequence of strictly positive real numbers such that $\sum_n E[|Y_n|^2]/c_n^2 < +\infty$ and let (v_n) be a sequence of complex numbers such that $v_n \neq 0$ and*

(A.1) $$\lim_n v_n = 0, \qquad \lim_n v_n \sum_{k=1}^n \frac{1}{c_k v_k} = \eta \in \mathbb{C},$$

(A.2) $$|v_n| \sum_{k=1}^n \frac{1}{c_k |v_k|} = O(1), \qquad |v_n| \sum_{k=1}^n \left| \frac{1}{v_k} - \frac{1}{v_{k-1}} \right| = O(1).$$

Then $v_n \sum_{k=1}^n Y_k/(c_k v_k) \xrightarrow{a.s.} \eta Y$.

PROOF.   Let A be an event such that $P(A) = 1$ and $\lim_n E[Y_n|\mathcal{H}_{n-1}](\omega) = Y(\omega)$ for each $\omega \in A$. Fix $\omega \in A$ and set $w_n = E[Y_n|\mathcal{H}_{n-1}](\omega)$ and $w = Y(\omega)$. If $\eta \neq 0$, applying Lemma A.2 to $z_{n,k} = v_n/(c_k v_k \eta)$, $s = 1$ and w_n, we obtain

$$\lim_n v_n \sum_{k=1}^n \frac{E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k v_k \eta} = Y(\omega).$$

If $\eta = 0$, applying Lemma A.2 to $z_{n,k} = v_n/(c_k v_k)$, $s = 0$ and w_n, we obtain

$$\lim_n v_n \sum_{k=1}^{n} \frac{E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k v_k} = 0.$$

Therefore, for both cases, we have

$$v_n \sum_{k=1}^{n} \frac{E[Y_k|\mathcal{H}_{k-1}]}{c_k v_k} \xrightarrow{\text{a.s.}} \eta Y.$$

Now, consider the martingale (M_n) defined by

$$M_n = \sum_{k=1}^{n} \frac{Y_k - E[Y_k|\mathcal{H}_{k-1}]}{c_k}.$$

It is bounded in L^2 since $\sum_{k=1}^{n} \frac{E[|Y_k|^2]}{c_k^2} < +\infty$ by assumption and so it is almost surely convergent, that means

$$\sum_k \frac{Y_k(\omega) - E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k} < +\infty$$

for $\omega \in B$ with $P(B) = 1$. Therefore, fixing $\omega \in B$ and setting $z_k = \frac{Y_k(\omega) - E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k}$, by Corollary A.3, we get

$$\lim_n v_n \sum_{k=1}^{n} \frac{Y_k(\omega) - E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k v_k} = 0$$

and so

$$v_n \sum_{k=1}^{n} \frac{Y_k - E[Y_k|\mathcal{H}_{k-1}]}{c_k v_k} \xrightarrow{\text{a.s.}} 0.$$

In order to conclude, it is enough to observe that

$$v_n \sum_{k=1}^{n} \frac{Y_k}{c_k v_k} = v_n \sum_{k=1}^{n} \frac{Y_k - E[Y_k|\mathcal{H}_{k-1}]}{c_k v_k} + v_n \sum_{k=1}^{n} \frac{E[Y_k|\mathcal{H}_{k-1}]}{c_k v_k}. \qquad \square$$

REMARK A.2. It is useful to note that:

- The relations $|v_n| \sum_{k=1}^{n} \frac{1}{c_k|v_k|} = O(1)$ and $c_n|v_n||\frac{1}{v_n} - \frac{1}{v_{n-1}}| = O(1)$ imply conditions (A.2). Indeed, we have

$$|v_n| \sum_{k=1}^{n} \left|\frac{1}{v_k} - \frac{1}{v_{k-1}}\right| = |v_n| \sum_{k=1}^{n} \frac{1}{c_k|v_k|} c_k|v_k| \left|\frac{1}{v_k} - \frac{1}{v_{k-1}}\right|.$$

- Whenever (v_n) is a decreasing sequence of positive real numbers (the case of the classical Kronecker lemma), conditions (A.1) obviously entail conditions (A.2).

We conclude this subsection recalling the following well-known relations for $a \in \mathbb{R}$:

(A.3)
$$\sum_{k=1}^{n} \frac{1}{k^{1-a}} = \begin{cases} O(1) & \text{for } a < 0, \\ \ln(n) + O(1) & \text{for } a = 0, \\ a^{-1}n^a + O(1) & \text{for } 0 < a \le 1, \\ a^{-1}n^a + O(n^{a-1}) & \text{for } a > 1. \end{cases}$$

A.2. Asymptotic results for products of complex numbers. We now present the framework for the results of this subsection. Fix $1/2 < \gamma \le 1$ and $c > 0$, and consider a sequence $(r_n)_n$ of real numbers such that $0 \le r_n < 1$ and

(A.4)
$$r_n \sim \frac{c}{n^\gamma}.$$

Obviously, we have $r_n > 0$ for n large enough and so, in the sequel, without loss of generality, we will assume $0 < r_n < 1$ for all n.

Let $\alpha_1 = a_1 + i b_1 \in \mathbb{C}$ and $\alpha_2 = a_2 + i b_2 \in \mathbb{C}$ with $a_1, a_2 > 0$. Denote by $m_0 \ge 2$ an integer such that $\max(a_1, a_2) < r_m^{-1}$ for all $m \ge m_0$ and define for $n \ge m_0$ and $j = 1, 2$,

$$p_{n,j} = \prod_{m=m_0}^{n} (1 - \alpha_j r_m) \quad \text{and} \quad \ell_{n,j} = p_{n,j}^{-1}.$$

Then, inspired by the computation done in [21], we can prove the following technical results.

LEMMA A.4.   *For $j = 1, 2$ and for any $\varepsilon \in (0, 1)$, we have that*

(A.5)
$$|p_{n,j}| = \begin{cases} O\left(\exp\left[-(1-\varepsilon)\dfrac{ca_j}{1-\gamma}n^{1-\gamma}\right]\right) & \text{for } 1/2 < \gamma < 1, \\ O(n^{-(1-\varepsilon)ca_j}) & \text{for } \gamma = 1 \end{cases}$$

and

(A.6)
$$|\ell_{n,j}| = \begin{cases} O\left(\exp\left[(1+\varepsilon)\dfrac{ca_j}{1-\gamma}n^{1-\gamma}\right]\right) & \text{for } 1/2 < \gamma < 1, \\ O(n^{(1+\varepsilon)ca_j}) & \text{for } \gamma = 1. \end{cases}$$

Moreover, if we replace (A.4) *with the condition*

(A.7)
$$n^\gamma r_n - c = O(n^{-\gamma}),$$

we have that

(A.8)
$$|p_{n,j}| = \begin{cases} O\left(\exp\left[-\dfrac{ca_j}{1-\gamma}n^{1-\gamma}\right]\right) & \text{for } 1/2 < \gamma < 1, \\ O(n^{-ca_j}) & \text{for } \gamma = 1 \end{cases}$$

and

$$(A.9) \qquad |\ell_{n,j}| = \begin{cases} O\left(\exp\left[\dfrac{ca_j}{1-\gamma}n^{1-\gamma}\right]\right) & \text{for } 1/2 < \gamma < 1, \\ O(n^{ca_j}) & \text{for } \gamma = 1. \end{cases}$$

PROOF. Consider $j = 1, 2$. We can easily write $p_{n,j} = p^*_{n,j} q_{n,j}$, where

$$p^*_{n,j} = \prod_{m=m_0}^{n} (1 - a_j r_m) \quad \text{and} \quad q_{n,j} = \prod_{m=m_0}^{n} \left(1 - i\frac{b_j r_m}{1 - a_j r_m}\right).$$

We now observe that

$$|q_{n,j}|^2 = \prod_{m=m_0}^{n} \left(1 + \frac{b_j^2 r_m^2}{(1 - a_j r_m)^2}\right) = \exp\left[\sum_{m=m_0}^{n} \ln\left(1 + \frac{b_j^2 r_m^2}{(1 - a_j r_m)^2}\right)\right],$$

and using the inequalities $-x \leq \ln(1 + x) \leq x$ for $x \geq 0$, we have that

$$\exp\left[-b_j^2 \sum_{m=m_0}^{n} \frac{r_m^2}{(1 - a_j r_m)^2}\right] \leq |q_{n,j}|^2 \leq \exp\left[b_j^2 \sum_{m=m_0}^{n} \frac{r_m^2}{(1 - a_j r_m)^2}\right].$$

Hence, since the series $\sum_m \frac{r_m^2}{(1 - a_j r_m)^2}$ is convergent for $1/2 < \gamma \leq 1$, we have $p_{n,j} = O(|p^*_{n,j}|)$ and $\ell_{n,j} = O(|\ell^*_{n,j}|)$ with $\ell^*_{n,j} = 1/p^*_{n,j}$. Therefore, it is enough to study

$$p^*_{n,j} = \exp\left(\sum_{k=m_0}^{n} \ln(1 - a_j r_k)\right) \quad \text{and} \quad \ell^*_{n,j} = \exp\left(-\sum_{k=m_0}^{n} \ln(1 - a_j r_k)\right).$$

Recalling the inequalities $\ln(1 - x) \leq -x$ and $-\ln(1 - x) \leq x + x^2$ for $0 \leq x \leq 1/2$ and the fact that the series $\sum_k r_k^2$ is convergent for $1/2 < \gamma \leq 1$, we get

$$p^*_{n,j} = O\left(\exp\left(-a_j \sum_{k=m_0}^{n} r_k\right)\right),$$

$$\ell^*_{n,j} = O\left(\exp\left(a_j \sum_{k=m_0}^{n} r_k\right)\right).$$

We now take into account the decomposition $\exp(a_j \sum_{k=m_0}^{n} r_k) = s^*_{n,j} t^*_{n,j}$, where

$$s^*_{n,j} = \exp\left(a_j c \sum_{k=m_0}^{n} k^{-\gamma}\right) \quad \text{and} \quad t^*_{n,j} = \exp\left(a_j \sum_{k=m_0}^{n} (r_k - ck^{-\gamma})\right).$$

Now, since by condition (A.4), for any $\varepsilon \in (0, 1)$ we have $|r_k - ck^{-\gamma}| \leq \varepsilon c k^{-\gamma}$ for k large enough (depending on ε), we obtain

$$p^*_{n,j} = O((s^*_{n,j})^{-(1-\varepsilon)}) \quad \text{and} \quad \ell^*_{n,j} = O((s^*_{n,j})^{1+\varepsilon}).$$

Then (A.5) and (A.6) follow by noticing that, by means of (A.3), we have

$$
s_{n,j}^* =
\begin{cases}
O\left(\exp\left(\dfrac{a_j c}{1-\gamma} n^{1-\gamma}\right)\right) & \text{if } 1/2 < \gamma < 1, \\
O(n^{a_j c}) & \text{if } \gamma = 1
\end{cases}
\quad \text{and}
$$

(A.10)

$$
(s_{n,j}^*)^{-1} =
\begin{cases}
O\left(\exp\left(\dfrac{-a_j c}{1-\gamma} n^{1-\gamma}\right)\right) & \text{if } 1/2 < \gamma < 1, \\
O(n^{-a_j c}) & \text{if } \gamma = 1.
\end{cases}
$$

Finally, by condition (A.7) and since the series $\sum_k O(k^{-2\gamma})$ is convergent, we have $t_{n,j}^* = O(1)$ and $(t_{n,j}^*)^{-1} = O(1)$, which imply $p_{n,j}^* = O((s_{n,j}^*)^{-1})$ and $\ell_{n,j}^* = O(s_{n,j}^*)$. Then result (A.8) follows by applying (A.10). $\quad\square$

LEMMA A.5. *We have that*

$$
\lim_n n^\gamma p_{n,1} p_{n,2} \sum_{k=m_0}^{n} r_k^2 \ell_{k,1} \ell_{k,2}
$$

(A.11)

$$
=
\begin{cases}
\dfrac{c}{\alpha_1 + \alpha_2} & \text{if } 1/2 < \gamma < 1, \\[2mm]
\dfrac{c^2}{c(\alpha_1 + \alpha_2) - 1} & \text{if } \gamma = 1, c(a_1 + a_2) > 1
\end{cases}
$$

and, for any $u \geq 1$, *when* $1/2 < \gamma < 1$ *or when* $\gamma = 1$ *and* $uc(a_1 + a_2) > (2u - 1)$, *we have*

(A.12) $$|p_{n,1}|^u |p_{n,2}|^u \sum_{k=m_0}^{n} r_k^{2u} |\ell_{k,1}|^u |\ell_{k,2}|^u = O(n^{-\gamma(2u-1)}).$$

PROOF.   Let us start with observing that relations (A.5) imply in particular

(A.13) $$\lim_n n^\gamma |p_{n,1}| |p_{n,2}| = 0.$$

Indeed, this fact follows immediately for $1/2 < \gamma < 1$ and, for $\gamma = 1$ one has to note that, since we assume $c(a_1 + a_2) > 1$, we can choose ε small enough so that $c(1 - \varepsilon)(a_1 + a_2) > 1$. Now, fix $k \geq 2$ and let us define the following quantity:

$$
\begin{aligned}
D_{\gamma,k} &= \frac{1}{k^\gamma} \ell_{k,1} \ell_{k,2} - \frac{1}{(k-1)^\gamma} \ell_{k-1,1} \ell_{k-1,2} \\
&= \left(\frac{1}{k^\gamma} - \frac{1}{(k-1)^\gamma}\right) \ell_{k-1,1} \ell_{k-1,2} + \frac{1}{k^\gamma} (\ell_{k,1} \ell_{k,2} - \ell_{k-1,1} \ell_{k-1,2}) \\
&= \ell_{k,1} \ell_{k,2} \left[\left(\frac{1}{k^\gamma} - \frac{1}{(k-1)^\gamma}\right) \frac{\ell_{k-1,1} \ell_{k-1,2}}{\ell_{k,1} \ell_{k,2}} + \frac{1}{k^\gamma} \left(1 - \frac{\ell_{k-1,1} \ell_{k-1,2}}{\ell_{k,1} \ell_{k,2}}\right) \right].
\end{aligned}
$$

Then we observe the following:

$$\left(\frac{1}{k^\gamma} - \frac{1}{(k-1)^\gamma}\right) = -\frac{\gamma}{k^{1+\gamma}} + O\left(\frac{1}{k^{2+\gamma}}\right)$$

(A.14)

$$= -\frac{\gamma}{k^{1+\gamma}} + o\left(\frac{1}{k^{1+\gamma}}\right) \qquad \text{for } k \to +\infty$$

and

(A.15) $\qquad \dfrac{\ell_{k-1,1}\ell_{k-1,2}}{\ell_{k,1}\ell_{k,2}} = (1 - \alpha_1 r_k)(1 - \alpha_2 r_k) = 1 + \alpha_1\alpha_2 r_k^2 - (\alpha_1 + \alpha_2)r_k.$

Now, by using (A.14) and (A.15) in the above expression of $D_{\gamma,k}$, and recalling (A.4), we have for $k \to +\infty$

$$D_{\gamma,k} = \ell_{k,1}\ell_{k,2}\left[-\frac{\gamma}{k^{\gamma+1}}(1 - \alpha_1 r_k)(1 - \alpha_2 r_k) + \frac{1}{k^\gamma}(-\alpha_1\alpha_2 r_k^2 + (\alpha_1 + \alpha_2)r_k)\right]$$

$$+ o\left(\frac{\ell_{k,1}\ell_{k,2}}{k^{1+\gamma}}\right)$$

$$= \ell_{k,1}\ell_{k,2}\left[\frac{r_k}{k^\gamma}(\alpha_1 + \alpha_2) - \frac{\gamma}{k^{\gamma+1}}\right] + o\left(\frac{\ell_{k,1}\ell_{k,2}}{k^{1+\gamma}}\right)$$

$$= \begin{cases} \dfrac{(\alpha_1 + \alpha_2)r_k}{k^\gamma}\ell_{k,1}\ell_{k,2} + o(r_k^2\ell_{k,1}\ell_{k,2}) \\ \qquad \text{if } 1/2 < \gamma < 1, \\ \left(\dfrac{(\alpha_1 + \alpha_2)r_k}{k} - \dfrac{1}{k^2}\right)\ell_{k,1}\ell_{k,2} + o(r_k^2\ell_{k,1}\ell_{k,2}) \\ \qquad \text{if } \gamma = 1, c(\alpha_1 + \alpha_2) \neq 1, \end{cases}$$

that is,

(A.16) $\qquad D_{\gamma,k} \sim \begin{cases} \dfrac{(\alpha_1 + \alpha_2)}{c}r_k^2\ell_{k,1}\ell_{k,2} & \text{if } 1/2 < \gamma < 1, \\ \dfrac{c(\alpha_1 + \alpha_2) - 1}{c^2}r_k^2\ell_{k,1}\ell_{k,2} & \text{if } \gamma = 1, c(\alpha_1 + \alpha_2) \neq 1. \end{cases}$

Note that, when $\gamma = 1$, the condition $c(a_1 + a_2) > 1$ implies that $c(\alpha_1 + \alpha_2) \neq 1$ that ensures $D_{1,k} \sim r_k^2\ell_{k,1}\ell_{k,2}$. Now, we want to apply Corollary A.2 with

$$z_n = D_{\gamma,n}, \qquad v_n = n^\gamma p_{n,1} p_{n,2}, \qquad w_n = \frac{r_n^2\ell_{n,1}\ell_{n,2}}{D_{\gamma,n}},$$

$$w = \begin{cases} \dfrac{c}{(\alpha_1 + \alpha_2)} & \text{if } 1/2 < \gamma < 1, \\ \dfrac{c^2}{c(\alpha_1 + \alpha_2) - 1} & \text{if } \gamma = 1, c(a_1 + a_2) > 1. \end{cases}$$

Indeed, $\lim_n v_n = 0$ by (A.13), $\lim_n w_n = w \neq 0$ by (A.16),

$$v_n \sum_{k=m_0}^{n} z_k = n^\gamma p_{n,1} p_{n,2} \sum_{k=m_0}^{n} D_{\gamma,k}$$

$$= n^\gamma p_{n,1} p_{n,2} \left(\frac{\ell_{n,1}\ell_{n,2}}{n^\gamma} - \frac{\ell_{m_0-1,1}\ell_{m_0-1,2}}{(m_0-1)^\gamma} \right) \longrightarrow 1$$

by (A.13) and $z'_n = z_n w_n = r_n^2 \ell_{n,1} \ell_{n,2}$. Finally, in order to apply Corollary A.2, it remains to prove that $|v_n| \sum_{k=m_0}^{n} |z'_k| = O(1)$. In order to do this, we apply Corollary A.1 to

$$x_n = \frac{1}{n^\gamma} |\ell_{n,1}||\ell_{n,2}| - \frac{1}{(n-1)^\gamma}|\ell_{n-1,1}||\ell_{n-1,2}|, \qquad x'_n = r_n^2 |\ell_{n,1}||\ell_{n,2}| > 0,$$

$$c_n = n^\gamma |p_{n,1}||p_{n,2}|.$$

Indeed, we have $\lim_n c_n \sum_{k=m_0}^{n} x_k = 1$, and since

$$\frac{|\ell_{k-1,1}||\ell_{k-1,2}|}{|\ell_{k,1}||\ell_{k,2}|} = |1 + \alpha_1\alpha_2 r_k^2 - (\alpha_1+\alpha_2)r_k| = 1 - (a_1+a_2)r_k + O(r_k^2),$$

by computations similar to the ones done above, we can obtain

$$x_n \sim \begin{cases} \dfrac{(a_1+a_2)}{c}x'_n & \text{if } 1/2 < \gamma < 1, \\[2ex] \dfrac{c(a_1+a_2)-1}{c^2}x'_n & \text{if } \gamma = 1, c(a_1+a_2) > 1, \end{cases}$$

where both constants belong to $(0, +\infty)$. Therefore, $c_n \sum_{k=m_0}^{n} x'_k$ converges and so it is bounded. Hence, we have verified all the conditions required by Corollary A.2 and so we can conclude that we have $\lim_n v_n \sum_{k=m_0}^{n} z'_k = w$, that is, (A.11).

Regarding (A.12), we have already considered the case $u = 1$, which is related to $c_n \sum_{k=m_0}^{n} x'_k$. Similarly, in order to prove (A.12) for $u > 1$, we use

$$\frac{|\ell_{k-1,1}|^u|\ell_{k-1,2}|^u}{|\ell_{k,1}|^u|\ell_{k,2}|^u} = |1 + \alpha_1\alpha_2 r_k^2 - (\alpha_1+\alpha_2)r_k|^u = 1 - u(a_1+a_2)r_k + O(r_k^2)$$

and apply Corollary A.1 again. Indeed, with computations similar to the one done before, we obtain

$$\frac{|\ell_{k,1}|^u|\ell_{k,2}|^u}{k^{\gamma(2u-1)}} - \frac{|\ell_{k-1,1}|^u|\ell_{k-1,2}|^u}{(k-1)^{\gamma(2u-1)}}$$

$$\sim \begin{cases} \dfrac{u(a_1+a_2)}{c^{2u-1}}r_k^{2u}|\ell_{k,1}|^u|\ell_{k,2}|^u \\ \qquad \text{if } 1/2 < \gamma < 1, \\[3ex] \dfrac{uc(a_1+a_2)-(2u-1)}{c^{2u}}r_k^{2u}|\ell_{k,1}|^u|\ell_{k,2}|^u \\ \qquad \text{if } \gamma = 1, uc(a_1+a_2) > 2u-1, \end{cases}$$

where both constants belong to $(0, +\infty)$, and so we have

$$\lim_n n^{\gamma(2u-1)} |p_{n,1}|^u |p_{n,2}|^u \sum_{k=m_0}^{n} r_k^{2u} |\ell_{k,1}|^u |\ell_{k,2}|^u$$

$$= C(\gamma, u) \lim_n n^{\gamma(2u-1)} |p_{n,1}|^u |p_{n,2}|^u$$

$$\times \sum_{k=m_0}^{n} \frac{|\ell_{k,1}|^u |\ell_{k,2}|^u}{k^{\gamma(2u-1)}} - \frac{|\ell_{k-1,j}|^u |\ell_{k-1,2}|^u}{(k-1)^{\gamma(2u-1)}}$$

$$= C(\gamma, u)$$

for a suitable constant $C(\gamma, u) \in (0, +\infty)$. $\quad\square$

REMARK A.3. We note that, if $\gamma = 1$ and (A.7) holds, then we can add to (A.12) the following:

(A.17)
$$|p_{n,1}|^u |p_{n,2}|^u \sum_{k=m_0}^{n} r_k^{2u} |\ell_{k,1}|^u |\ell_{k,2}|^u$$

$$= \begin{cases} O\big(\ln(n)/n^{uc(a_1+a_2)}\big) & \text{if } uc(a_1+a_2) = (2u-1), \\ O\big(n^{-uc(a_1+a_2)}\big) & \text{if } uc(a_1+a_2) < (2u-1). \end{cases}$$

Indeed, by means of (A.8) and (A.9) in Lemma A.4, we have

$$|p_{n,1}|^u |p_{n,2}|^u \sum_{k=m_0}^{n} r_k^{2u} |\ell_{k,1}|^u |\ell_{k,2}|^u$$

$$= O\big(n^{-uc(a_1+a_2)}\big) \sum_{k=m_0}^{n} O\big(k^{uc(a_1+a_2)-2u}\big)$$

$$= O\big(n^{-uc(a_1+a_2)}\big) \sum_{k=m_0}^{n} O\left(\frac{1}{k^{1-(uc(a_1+a_2)-2u+1)}} \right).$$

LEMMA A.6. Let $\gamma = 1$, $c(a_1 + a_2) = 1$ and replace condition (A.4) by (A.7). Then we have

(A.18)
$$\lim_n \frac{n}{\ln(n)} p_{n,1} p_{n,2} \sum_{k=m_0}^{n} r_k^2 \ell_{k,1} \ell_{k,2} = \begin{cases} 0 & \text{if } b_1 + b_2 \neq 0, \\ c^2 & \text{if } b_1 + b_2 = 0 \end{cases}$$

and

(A.19)
$$|p_{n,1}|^u |p_{n,2}|^u \sum_{k=m_0}^{n} r_k^{2u} |\ell_{k,1}|^u |\ell_{k,2}|^u = \begin{cases} O\big(\ln(n)/n\big) & \text{for } u = 1, \\ O\big(n^{-u}\big) & \text{for } u > 1. \end{cases}$$

PROOF.   First, note that (A.18) for the case $b_1 + b_2 \neq 0$ can be established using the computations done for the proof of Lemma A.5 with $\gamma = 1$. Indeed, we can apply Corollary A.2 with

$$z_n = D_{1,n}, \qquad v_n = \frac{n}{\ln(n)} p_{n,1} p_{n,2}, \qquad w_n = \frac{r_n^2 \ell_{n,1} \ell_{n,2}}{D_{1,n}},$$

$$z_n' = z_n w_n = r_n^2 \ell_{n,1} \ell_{n,2}, \qquad w = \frac{c^2}{c(\alpha_1 + \alpha_2) - 1}.$$

In fact, by assumptions (A.7) and $c(a_1 + a_2) = 1$, we have

(A.20)
$$\lim_n \frac{n}{\ln(n)} |p_{n,1}||p_{n,2}| = 0$$

since (A.8) in Lemma A.4 and, moreover, we have $\lim_n w_n = w \neq 0$ by (A.16) since $c(\alpha_1 + \alpha_2) \neq 1$, and $|v_n| \sum_{k=m_0}^{n} |z_k'| = O(1)$ by (A.17) with $u = 1$, and finally, we have $\lim_n v_n \sum_{k=m_0}^{n} z_k = 0$.

We now focus on the case $b_1 + b_2 = 0$. Fix $k \geq 2$ and let us define the following quantity:

$$\begin{aligned}
D_{\ln,k} &= \frac{\ln(k)}{k} \ell_{k,1} \ell_{k,2} - \frac{\ln(k-1)}{k-1} \ell_{k-1,1} \ell_{k-1,2} \\
&= \left(\frac{\ln(k)}{k} - \frac{\ln(k-1)}{k-1} \right) \ell_{k-1,1} \ell_{k-1,2} + \frac{\ln(k)}{k} (\ell_{k,1} \ell_{k,2} - \ell_{k-1,1} \ell_{k-1,2}) \\
&= \ell_{k,1} \ell_{k,2} \Bigg[\left(\frac{\ln(k)}{k} - \frac{\ln(k-1)}{k-1} \right) \frac{\ell_{k-1,1} \ell_{k-1,2}}{\ell_{k,1} \ell_{k,2}} \\
&\quad + \frac{\ln(k)}{k} \left(1 - \frac{\ell_{k-1,1} \ell_{k-1,2}}{\ell_{k,1} \ell_{k,2}} \right) \Bigg].
\end{aligned}$$

We observe that for $k \to +\infty$

(A.21)
$$\begin{aligned}
\left(\frac{\ln(k)}{k} - \frac{\ln(k-1)}{k-1} \right) &= -\frac{\ln(k)}{k(k-1)} - \frac{\ln(1 - k^{-1})}{k-1} \\
&= -\frac{\ln(k)}{k^2} + \frac{1}{k^2} + O\left(\frac{\ln(k)}{k^3} \right) \\
&= -\frac{\ln(k)}{k^2} + \frac{1}{k^2} + o\left(\frac{1}{k^2} \right).
\end{aligned}$$

Now, by using (A.15) and (A.21) in the expression of $D_{\ln,k}$, and recalling (A.7), we have that

$$\begin{aligned}
D_{\ln,k} &= \ell_{k,1} \ell_{k,2} \Bigg[\left(-\frac{\ln(k)}{k^2} + \frac{1}{k^2} \right)(1 - \alpha_1 r_k)(1 - \alpha_2 r_k) \\
&\quad + \frac{\ln(k)}{k} (-\alpha_1 \alpha_2 r_k^2 + (\alpha_1 + \alpha_2) r_k) \Bigg] + o\left(\frac{\ell_{k,1} \ell_{k,2}}{k^2} \right)
\end{aligned}$$

$$= \ell_{k,1}\ell_{k,2}\left[\frac{r_k \ln(k)}{k}(\alpha_1 + \alpha_2) - \frac{\ln(k)}{k^2} + \frac{1}{k^2}\right] + o\left(\frac{\ell_{k,1}\ell_{k,2}}{k^2}\right)$$

$$= \left[\ln(k)\left(\frac{(\alpha_1 + \alpha_2)r_k}{k} - \frac{1}{k^2}\right) + \frac{1}{k^2}\right]\ell_{k,1}\ell_{k,2} + o\left(\frac{\ell_{k,1}\ell_{k,2}}{k^2}\right).$$

Then, since the equalities $c(a_1 + a_2) = 1$ and $b_1 + b_2 = 0$ imply $c(\alpha_1 + \alpha_2) = 1$, and recalling (A.7), we obtain

(A.22)
$$D_{\ln,k} = \frac{1}{k^2}\ell_{k,1}\ell_{k,2} + o\left(\frac{\ell_{k,1}\ell_{k,2}}{k^2}\right)$$
$$= \frac{1}{k^2}\ell_{k,1}\ell_{k,2} + o\left(r_k^2\ell_{k,1}\ell_{k,2}\right) \sim \frac{1}{c^2}r_k^2\ell_{k,1}\ell_{k,2}.$$

Now, we want to apply Corollary A.2 with

$$z_n = D_{\ln,n}, \qquad v_n = \frac{n}{\ln(n)}p_{n,1}p_{n,2}, \qquad w_n = \frac{r_n^2\ell_{n,1}\ell_{n,2}}{D_{\ln,n}},$$
$$z_n' = z_n w_n = r_n^2\ell_{n,1}\ell_{n,2}, \qquad w = c^2.$$

Indeed, $\lim_n v_n = 0$ by (A.20), $\lim_n w_n = w \neq 0$ by (A.22), $|v_n|\sum_{k=m_0}^n |z_k'| = O(1)$ by (A.17) (with $u = 1$) since $c(a_1 + a_2) = 1$ by assumption,

$$\lim_n v_n \sum_{k=m_0}^n z_k = \lim_n \frac{n}{\ln(n)}p_{n,1}p_{n,2}\sum_{k=m_0}^n D_{\ln,k}$$
$$= \frac{n}{\ln(n)}p_{n,1}p_{n,2}\left(\frac{\ln(n)\ell_{n,1}\ell_{n,2}}{n} - \frac{\ln(m_0 - 1)\ell_{m_0-1,1}\ell_{m_0-1,2}}{(m_0 - 1)}\right)$$
$$\longrightarrow 1$$

by (A.20). Hence, all the conditions required by Corollary A.2 hold and so we can conclude that we have $\lim_n v_n \sum_{k=m_0}^n z_k' = w$, that is, (A.18) for $b_1 + b_2 = 0$.

Finally, relations (A.19) follows from (A.17) using the assumption that $c(a_1 + a_2) = 1$. $\quad\square$

A.3. A result for Gaussian random vectors. The following result is about the standardization of Gaussian random vectors with singular covariance matrix.

LEMMA A.7. *Let X be a random vector with distribution $\mathcal{N}_N(\mathbf{0}, \Sigma)$ and consider the spectral decomposition $\Sigma = O\Lambda O^\top$ (more precisely, Λ is the diagonal matrix containing the eigenvalues of Σ and the columns of O form a corresponding orthonormal basis of right eigenvectors). Let $1 \leq r < N$ be the rank of Σ, define the diagonal matrix L as follows*:

$$[L]_{h,j} = \begin{cases} \lambda_j^{-1/2} & \text{if } h = j \text{ and } \lambda_j > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and denote by H the $r \times N$-matrix such that

$$[H]_{h,j} = \begin{cases} 1 & \text{if } h = j \text{ and } 1 \leq h \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Then, setting $M = HLO^{\top}$ and $Y = MX$, the distribution of Y is $\mathcal{N}_r(\mathbf{0}, I)$.

PROOF.   It is immediate to see that Y is a Gaussian vector since it is a linear transformation of the Gaussian vector X. Then the result follows by noticing that

$$\text{Cov}(Y) = M\Sigma M^{\top} = HL(O^{\top}\Sigma O)LH^{\top} = H(L\Lambda L)H^{\top} = I. \qquad \square$$

APPENDIX B: STABLE CONVERGENCE AND ITS VARIANTS

We recall here some basic definitions and results. For more details, we refer the reader to [18, 20, 23, 29] and the references therein.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let S be a Polish space, endowed with its Borel σ-field. A *kernel* on S, or a random probability measure on S, is a collection $K = \{K(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ-field of S such that, for each bounded Borel real function f on S, the map

$$\omega \mapsto Kf(\omega) = \int f(x)K(\omega)(dx)$$

is $\mathcal{A}$-measurable. Given a sub-σ-field $\mathcal{H}$ of $\mathcal{A}$, a kernel K is said $\mathcal{H}$-measurable if all the above random variables Kf are $\mathcal{H}$-measurable.

On $(\Omega, \mathcal{A}, P)$, let (Y_n) be a sequence of S-valued random variables, let $\mathcal{H}$ be a sub-σ-field of $\mathcal{A}$ and let K be a $\mathcal{H}$-measurable kernel on S. Then we say that Y_n converges $\mathcal{H}$-*stably* to K, and we write $Y_n \longrightarrow K$ $\mathcal{H}$-stably, if

$$P(Y_n \in \cdot | H) \overset{\text{weakly}}{\longrightarrow} E[K(\cdot)|H] \qquad \text{for all } H \in \mathcal{H} \text{ with } P(H) > 0.$$

In the case when $\mathcal{H} = \mathcal{A}$, we simply say that Y_n converges *stably* to K and we write $Y_n \longrightarrow K$ stably. Clearly, if $Y_n \longrightarrow K$ $\mathcal{H}$-stably, then Y_n converges in distribution to the probability distribution $E[K(\cdot)]$. Moreover, the $\mathcal{H}$-stable convergence of Y_n to K can be stated in terms of the following convergence of conditional expectations:

$$\text{(B.1)} \qquad\qquad E[f(Y_n)|\mathcal{H}] \overset{\sigma(L^1, L^{\infty})}{\longrightarrow} Kf$$

for each bounded continuous real function f on S.

In [23], the notion of $\mathcal{H}$-stable convergence is firstly generalized in a natural way replacing in (B.1) the single sub-σ-field $\mathcal{H}$ by a collection $\mathcal{G} = (\mathcal{G}_n)$ (called conditioning system) of sub-σ-fields of $\mathcal{A}$ and then it is strengthened by substituting the convergence in $\sigma(L^1, L^{\infty})$ by the one in probability (i.e., in L^1, since f is

bounded). Hence, according to [23], we say that Y_n converges to K *stably in the strong sense*, with respect to $\mathcal{G} = (\mathcal{G}_n)$, if

$$(B.2) \qquad E[f(Y_n)|\mathcal{G}_n] \xrightarrow{P} Kf$$

for each bounded continuous real function f on S.

Finally, a strengthening of the stable convergence in the strong sense can be naturally obtained if in (B.2) we replace the convergence in probability by the almost sure convergence: given a conditioning system $\mathcal{G} = (\mathcal{G}_n)$, we say that Y_n converges to K in the sense of the *almost sure conditional convergence*, with respect to $\mathcal{G}$, if

$$E[f(Y_n)|\mathcal{G}_n] \xrightarrow{\text{a.s.}} Kf$$

for each bounded continuous real function f on S. Evidently, this last type of convergence can be reformulated using the conditional distributions. Indeed, if K_n denotes a version of the conditional distribution of Y_n given $\mathcal{G}_n$, then the random variable $K_n f$ is a version of the conditional expectation $E[f(Y_n)|\mathcal{G}_n]$ and so we can say that Y_n converges to K in the sense of the almost sure conditional convergence, with respect to $\mathcal{F}$, if, for almost every ω in Ω, the probability measure $K_n(\omega)$ converges weakly to $K(\omega)$. The almost sure conditional convergence has been introduced in [18] and, subsequently, employed by others in the urn model literature (e.g., [5, 45]).

We now conclude this section with some convergence results that we need in our proofs.

From [24], Proposition 3.1, we can get the following result.

THEOREM B.1. *Let $(\mathbf{T}_{n,k})_{n\geq 1, 1\leq k\leq k_n}$ be a triangular array of d-dimensional real random vectors, such that, for each fixed n, the finite sequence $(\mathbf{T}_{n,k})_{1\leq k\leq k_n}$ is a martingale difference array with respect to a given filtration $(\mathcal{G}_{n,k})_{k\geq 0}$. Moreover, let $(t_n)_n$ be a sequence of real numbers and assume that the following conditions hold:*

(c1) $\mathcal{G}_{n,k} \subseteq \mathcal{G}_{n+1,k}$ *for each n and $1 \leq k \leq k_n$;*

(c2) $\sum_{k=1}^{k_n} (t_n \mathbf{T}_{n,k})(t_n \mathbf{T}_{n,k})^\top = t_n^2 \sum_{k=1}^{k_n} \mathbf{T}_{n,k} \mathbf{T}_{n,k}^\top \xrightarrow{P} \Sigma$, *where Σ is a random positive semidefinite matrix;*

(c3) $\sup_{1\leq k\leq k_n} |t_n \mathbf{T}_{n,k}| \xrightarrow{L^1} 0$.

Then $t_n \sum_{k=1}^{k_n} \mathbf{T}_{n,k}$ converges stably to the Gaussian kernel $\mathcal{N}(\mathbf{0}, \Sigma)$.

The following result combines together a stable convergence and a stable convergence in the strong sense.

THEOREM B.2 ([10], Lemma 1).   *Suppose that C_n and D_n are S-valued random variables that M and N are kernels on S, and that $\mathcal{G} = (\mathcal{G}_n)_n$ is an (increasing) filtration satisfying for all n:*

$$\sigma(C_n) \subseteq \mathcal{G}_n \quad and \quad \sigma(D_n) \subseteq \sigma\left(\bigcup_n \mathcal{G}_n\right).$$

If C_n stably converges to M and D_n converges to N stably in the strong sense, with respect to $\mathcal{G}$, then

$$[C_n, D_n] \longrightarrow M \otimes N \qquad stably.$$

[Here, $M \otimes N$ is the kernel on $S \times S$ such that $(M \otimes N)(\omega) = M(\omega) \otimes N(\omega)$ for all ω.]

Given a conditioning system $\mathcal{G} = (\mathcal{G}_n)_n$, if $\mathcal{U}$ is a sub-σ-field of $\mathcal{A}$ such that, for each real integrable random variable Y, the conditional expectation $\mathrm{E}[Y|\mathcal{G}_n]$ converges almost surely to the conditional expectation $\mathrm{E}[Y|\mathcal{U}]$, then we shall briefly say that $\mathcal{U}$ is an *asymptotic σ-field* for $\mathcal{G}$. In order that there exists an asymptotic σ-field $\mathcal{U}$ for a given conditioning system $\mathcal{G}$, it is obviously sufficient that the sequence $(\mathcal{G}_n)_n$ is increasing or decreasing. (Indeed, we can take $\mathcal{U} = \bigvee_n \mathcal{G}_n$ in the first case and $\mathcal{U} = \bigcap_n \mathcal{G}_n$ in the second one.)

THEOREM B.3 ([18], Theorem A.1).   *On $(\Omega, \mathcal{A}, P)$, for each $n \geq 1$, let $(\mathcal{F}_{n,h})_{h\in\mathbb{N}}$ be a filtration and $(M_{n,h})_{h\in\mathbb{N}}$ a real martingale with respect to $(\mathcal{F}_{n,h})_{h\in\mathbb{N}}$, with $M_{n,0} = 0$, which converges in L^1 to a random variable $M_{n,\infty}$. Set*

$$X_{n,j} := M_{n,j} - M_{n,j-1} \qquad for\ j \geq 1, \qquad U_n := \sum_{j\geq 1} X_{n,j}^2,$$

$$X_n^* := \sup_{j\geq 1} |X_{n,j}|.$$

Further, let $(k_n)_{n\geq 1}$ be a sequence of strictly positive integers such that $k_n X_n^* \overset{a.s.}{\rightarrow} 0$ and let $\mathcal{U}$ be a sub-σ-field which is asymptotic for the conditioning system $\mathcal{G}$ defined by $\mathcal{G}_n = \mathcal{F}_{n,k_n}$. Assume that the sequence $(X_n^*)_n$ is dominated in L^1 and that the sequence $(U_n)_n$ converges almost surely to a positive real random variable U which is measurable with respect to $\mathcal{U}$.

Then, with respect to the conditioning system $\mathcal{G}$, the sequence $(M_{n,\infty})_n$ converges to the Gaussian kernel $\mathcal{N}(0, U)$ in the sense of the almost sure conditional convergence.

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