

SUBOPTIMAL SOLUTIONS TO TEAM OPTIMIZATION PROBLEMS WITH STOCHASTIC INFORMATION STRUCTURE*

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Abstract. Existence, uniqueness, and approximations of smooth solutions to team optimization problems with stochastic information structure are investigated. Suboptimal strategies made up of linear combinations of basis functions containing adjustable parameters are considered. Estimates of their accuracies are derived by combining properties of the unknown optimal strategies with tools from nonlinear approximation theory. The estimates are obtained for basis functions corresponding to sinusoids with variable frequencies and phases, Gaussians with variable centers and widths, and sigmoidal ridge functions. The theoretical results are applied to a problem of optimal production in a multidivisional firm, for which numerical simulations are presented.

Key words. information structure, team utility, infinite-dimensional programming (functional optimization), approximation schemes, suboptimal solutions, model complexity, curse of dimensionality, optimal production in a firm

AMS subject classifications. 90B50, 90B70, 90C15, 90C30, 91A12, 91A35, 91B06, 91B16, 91B38

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1. Introduction. Team theory investigates the way in which a group of *decision makers (DMs)*, each having at disposal some *information* (obtained, e.g., by measurement devices or exit polls) and various possibilities of *decisions*, coordinate their efforts to achieve a common goal, expressed via a *team utility function*. Decisions are generated by the DMs via *strategies*, on the basis of the information available to each of them and in the presence of uncertainties in the external world, which the DMs do not control.

In general one centralized DM, which maximizes a common goal relying on the whole available information, provides a better performance than a group of decentralized DMs, each one with partial information. However, often centralization is not feasible. For example, the DMs may have access to local information that cannot be exchanged instantaneously, or the cost of making the whole information available to a unique DM may be unacceptably high.

Teams cooperating to achieve a common goal model a variety of problems in economic systems, management science, and operations research. Team organizations abound in science, engineering, and everyday life: examples are communication and computer networks in geographical areas, production plants, energy distribution systems, traffic systems in large metropolitan regions, and freeway systems. For instance, a situation with a natural team formulation is represented by routing in packet-switching telecommunication networks. In this context, the DMs are the routers at the nodes and they choose their decisions on the basis of the respective routing strate-

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gies. The DMs do not possess common information on the state of the network (the state may be represented, e.g., by the lengths of the packet queues at the nodes and the delays in the links): each of them has a “personal” information but they share the common goal of minimizing the total time spent by the messages at the nodes and in the communication links.

In the team optimization problems that we address in this paper, the information of each DM depends on a random variable, called *state of the world*, and is independent of the decisions of the other DMs. These are called *static teams* (first investigated by Marschak and Radner [33, 34, 44]), in contrast to *dynamic teams* [4], where each DM’s information can be affected by previous decisions of other DMs. Many dynamic team optimization problems can be reformulated in terms of equivalent static ones [53].

Unfortunately, closed-form solutions to team optimization problems can be derived only under quite strong assumptions on the team utility function, on the way in which each DM’s information depends on the state of the world, and, for dynamic teams, on the decisions previously taken by the other DMs. Typically, closed-form solutions can be derived under the so-called *LQG hypotheses* [44] (i.e., linear information structure, concave quadratic team utility, and Gaussian random variables) and, in the dynamic case, under the hypothesis of *partially nested information* [20] (i.e., each DM can reconstruct the whole information known to the DMs that affects its own information). However, often these assumptions are too simplified or unrealistic; for example, this is the case when modeling a price as a Gaussian random variable [19], which takes negative values with nonzero probability.

In general, even in situations where optimal centralized strategies can be derived, computing the optimal decentralized ones may be an intractable problem [40, 52]. Thus, typically one has to search for suboptimal solutions. Toward this end, a fruitful approach consists in searching for them among linear combinations of a certain number of “basis elements,” corresponding to computational units with a simple structure (e.g., Gaussians or sines) containing some parameters to be optimized (e.g., centers and widths in Gaussians or frequencies and phases in sines) [58]. In doing this, it is important to choose the kind of computational units in order to avoid the so-called *curse of dimensionality* [7], i.e., an unmanageably fast (e.g., exponential) growth, with respect to the dimension of each DM’s information vector, of the minimum number of basis functions required to guarantee a desired accuracy of the suboptimal strategies. In the presence of large information vectors, the curse of dimensionality implies that a very large number of parameters has to be optimized in the computational units. Often, this makes the search for suboptimal solutions too computationally demanding.

In this paper, we consider static team optimization problems in which the information available to each DM is expressed via a probability density function. This is called a *stochastic information structure* [3], in contrast to the *deterministic information structure*, where the information that each DM has at its disposal is uniquely determined by the state of the world. The objectives of our work are the following: (i) deriving smoothness properties of the (unknown) optimal strategies, (ii) exploiting such properties to search for suboptimal strategies, in such a way to avoid the curse of dimensionality, and (iii) estimating their accuracies. Concerning (i), in [17] we investigated existence and Lipschitz continuity of the optimal strategies, whereas here we consider a higher degree of smoothness. Moreover, in [17] we did not address issues (ii) and (iii). A related work is [16], where we proved smoothness properties of the solutions and investigated suboptimal strategies for centralized T -stage deterministic optimization problems.

For static team optimization problems, we first derive conditions guaranteeing existence, uniqueness, and certain smoothness properties of the solutions. Then, we search for suboptimal strategies taking on the form of *variable-basis approximation schemes* [22, 27, 30], i.e., linear combinations of at most k elements from a set of basis functions that depend on some inner parameters, where k is large enough to provide accurate suboptimal solutions. The coefficients in the linear combinations and the parameters inside the basis functions can be optimized via nonlinear programming algorithms (see, e.g., [9]). Then, we investigate the accuracy of the suboptimal solutions by estimating the difference between the *value of the team*, i.e., the expected value of the team utility function when optimal strategies are used, and its expected value when the strategies are restricted to certain families of variable-basis functions with k elements in their expansions. For bases formed by sines with variable frequencies and phases, sigmoidals, and Gaussians with variable centers and widths, we derive estimates proportional to $k^{-1/2}$. Hence, for a desired accuracy ε , the minimum number of basis functions grows at most quadratically with $1/\varepsilon$, thus avoiding the curse of dimensionality. To the best of our knowledge, no theoretical estimates of the accuracy of suboptimal strategies having the form of variable-basis functions were previously derived for static team optimization problems. Finally, as an application of our results, we consider a problem of optimal production in a multidivisional firm, for which we present numerical simulations.

The paper is organized as follows. Section 2 introduces definitions and assumptions, formulates the family of team optimization problems that we address, and describes an instance of such problems, namely, optimal production in a multidivisional firm. Section 3 investigates existence and uniqueness of smooth optimal strategies. Section 4 estimates the accuracies of suboptimal strategies that can be obtained via variable-basis schemes. Section 5 applies the results to the optimal production problem described in section 2, for which numerical results are provided. Section 6 discusses other applications and consequences of our results and investigates possible extensions. All the proofs are detailed in section 7.

2. Problem formulation. The context in which we formalize the team optimization problem and derive our results is the following:

- *Static team of n DMs*, $i = 1, \dots, n$.
- $x \in X \subseteq \mathbb{R}^{d_0}$: random variable, called *state of the world*. The vector x models the uncertainties in the “external world,” which are not controlled by the DMs.
- $y_i \in Y_i \subseteq \mathbb{R}^{d_i}$: random variable representing the *information* that the DM i has about x .
- $s_i : Y_i \rightarrow A_i \subseteq \mathbb{R}^{l_i}$: *strategy* of the DM i .
- $a_i = s_i(y_i) \in A_i$: *decision* that the DM i takes on the basis of the information y_i .
- $u : X \times \prod_{i=1}^n Y_i \times \prod_{i=1}^n A_i \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$, where $N \triangleq d_0 + \sum_{i=1}^n (d_i + l_i)$: *team utility function*.

The DMs’ information on the state of the world x is modeled by an n -tuple of (possibly dependent) random variables $y_1 \in Y_1, \dots, y_n \in Y_n$. This is called a *stochastic information structure* [3], with a probability density function $\rho(x, y_1, \dots, y_n)$ on the set $X \times \prod_{i=1}^n Y_i$. Our model is a *static team*, as the joint probability density function depends only on the state of the world and the information y_1, \dots, y_n . When the decision of some DM can affect the information of other DMs, the team is called

dynamic. As noted in the introduction, many dynamic team optimization problems can be reformulated in terms of equivalent static ones.

We state the following problem, for which we suppose that the set of optimal strategies is nonempty; in the next section, we shall give conditions guaranteeing this. By $\mathcal{M}(Y_i, A_i)$ we denote the set of bounded and measurable functions from Y_i to A_i .

Problem TO (team optimization). *Given the joint probability density function $\rho(x, y_1, \dots, y_n)$ and the team utility function $u(x, \{y_i\}_{i=1}^n, \{a_i\}_{i=1}^n)$, find n strategies $s_1^\circ, \dots, s_n^\circ$ such that*

$$(1) \quad (s_1^\circ, \dots, s_n^\circ) \in \operatorname{argmax} \left\{ v(s_1, \dots, s_n) \mid s_i \in \mathcal{M}(Y_i, A_i), i = 1, \dots, n \right\},$$

where

$$(2) \quad v(s_1, \dots, s_n) \triangleq \mathbb{E}_{x, y_1, \dots, y_n} \left\{ u(x, \{y_i\}_{i=1}^n, \{s_i(y_i)\}_{i=1}^n) \right\}.$$

The quantity $v(s_1^\circ, \dots, s_n^\circ)$ is called *value of the team*.

We adopt the following notation and definitions. The symbol \mathcal{C} is used for the space of continuous functions, endowed with the supremum norm. For a positive integer m , by \mathcal{C}^m we denote the space of functions that are continuous together with their partial derivatives up to the order m . For $\Omega \subseteq \mathbb{R}^d$, a function $f : \Omega \rightarrow \mathbb{R}^s$ is *Lipschitz continuous on Ω with constant L* iff there exists $L > 0$ such that for every $z, w \in \Omega$ one has $\|f(z) - f(w)\| \leq L \|z - w\|$. For a convex set $\Omega \subseteq \mathbb{R}^d$ and a concave function $f : \Omega \rightarrow \mathbb{R}$, a vector $\alpha_z \in \mathbb{R}^d$ is a *supergradient* of f at $z \in \Omega$ iff for every $w \in \Omega$ one has $f(w) - f(z) \leq \alpha_z \cdot (w - z)$. For $\tau > 0$, a concave function f defined on a convex set $\Omega \subseteq \mathbb{R}^d$ is *strongly concave with constant τ* iff for every $z, w \in \Omega$ and every supergradient α_z of f at z one has $f(w) - f(z) \leq \alpha_z \cdot (w - z) - \tau \|w - z\|^2$ [36]. It is *separately strongly concave with constant τ* iff each function obtained by fixing each time all variables but one is strongly concave with constant τ . The strong concavity with constant τ is equivalent to the concavity of the function $f(\cdot) + \tau \|\cdot\|^2$. If $f \in \mathcal{C}^2$, then it is also equivalent to the condition

$$(3) \quad \sup_{z \in \Omega} \lambda_{\max}(\nabla^2 f(z)) \leq -2\tau,$$

where $\lambda_{\max}(\nabla^2 f(z))$ is the maximum eigenvalue of the Hessian $\nabla^2 f(z)$.

Assumption A1. The set X of the states of the world is compact, the sets Y_1, \dots, Y_n and A_1, \dots, A_n are compact, convex, and with nonempty interiors. For a positive integer $m \geq 2$, the team utility function u is of class \mathcal{C}^m on an open set containing $X \times \prod_{i=1}^n Y_i \times \prod_{i=1}^n A_i$ and ρ is a strictly positive probability density function on $X \times \prod_{i=1}^n Y_i$, which can be extended to a strictly positive function of class \mathcal{C}^m on an open set containing $X \times \prod_{i=1}^n Y_i$.

Assumption A2. There exists $\tau > 0$ such that the team utility function is separately strongly concave with constant τ .

According to Assumption A2, if one fixes all the arguments of the team utility function u except the decision variable a_i , then the resulting function of a_i is strongly concave with constant τ . For example, in economic problems this is motivated by the *law of diminishing returns*, i.e., the fact that typically the marginal productivity of an input diminishes if the amount of the output increases [34, pp. 99, 110].

Assumption A3. For every n -tuple $\{s_1, \dots, s_n\}$ of strategies and every $y_1 \in Y_1, \dots, y_n \in Y_n$, the sets

$$\begin{aligned} & \operatorname{argmax}_{a_1 \in A_1} \mathbb{E}_{x, y_2, \dots, y_n | y_1} \{u(x, \{y_i\}_{i=1}^n, a_1, \{s_i(y_i)\}_{i=2}^n)\} \\ & \quad \vdots \\ & \operatorname{argmax}_{a_n \in A_n} \mathbb{E}_{x, y_1, \dots, y_{n-1} | y_n} \{u(x, \{y_i\}_{i=1}^n, \{s_i(y_i)\}_{i=1}^{n-1}, a_n)\} \end{aligned}$$

have nonempty intersections with the interiors of A_1, \dots, A_n , respectively.

Assumption A3 guarantees for each DM the existence of an interior “person-by-person” optimal strategy, i.e., an optimal strategy when the strategies of all the other DMs are fixed.

Nontrivial instances of Problem TO for which Assumptions A1–A3 hold can be constructed in the following way. One takes as a departure point an instance of Problem TO in which there is no interaction among the DMs, i.e., whose utility function is of the form $u(x, y_1, \dots, y_n, a_1, \dots, a_n) = \sum_{i=1}^n u_i(x, y_1, \dots, y_n, a_i)$, so that the assumptions are easy to impose (e.g., by choosing functions u_i that are quadratic, strictly concave, and suitably penalized on the boundaries of the sets A_i). Then, one adds a “sufficiently small” interaction term $\beta u_{\text{int}}(x, y_1, \dots, y_n, a_1, \dots, a_n)$, where u_{int} is of class \mathcal{C}^m and $\beta > 0$ is “sufficiently small,” too, so that Assumptions A2 and A3 hold.

Among problems that do not satisfy at least one of Assumptions A1–A3 (e.g., compactness of the sets Y_i and A_i), we mention the linear-quadratic-Gaussian team and the linear-exponential-Gaussian team [25].

In the rest of this section we describe an instance of Problem TO, whose formulation is along the lines of [19, section 3]. It will be studied in detail in section 5, where we provide conditions under which it satisfies Assumptions A1–A3.

Example: Optimal production in a multidivisional firm. A firm consists of two autonomous divisions that produce two different goods in quantities $a_1 \in [0, a_{1,\max}]$ and $a_2 \in [0, a_{2,\max}]$, respectively. The goods are sold in two competitive markets at prices $\xi \in [\xi_{\min}, \xi_{\max}]$ and $\zeta \in [\zeta_{\min}, \zeta_{\max}]$, respectively. Because of random fluctuations in supply and demand, ξ and ζ are known exactly only when the goods are sold. Each division separately collects information about the market it sells in. The information $y_1 \in [\xi_{\min}, \xi_{\max}]^{d_1}$ available to the first division is represented by d_1 price forecasts of the good it produces and is exploited to decide, via the strategy $s_1(y_1)$, the produced amount a_1 . This is similar for the second division, whose information is given by the d_2 -tuple $y_2 \in [\zeta_{\min}, \zeta_{\max}]^{d_2}$ of price forecasts of the other good. So, $a_1 = s_1(y_1)$ and $a_2 = s_2(y_2)$. The firm’s revenue is $\xi a_1 + \zeta a_2$ and the total cost of producing the quantities a_1 and a_2 of goods is given by

$$c(a_1, a_2) \triangleq \frac{1}{2}c_{11}a_1^2 + c_{12}a_1a_2 + \frac{1}{2}c_{22}a_2^2,$$

where $c_{11}, c_{22} > 0$ and $c_{12} \neq 0$. The choice of a quadratic function may be motivated, for example, by a second-order local approximation of a nonquadratic one. As $c_{12} \neq 0$, in general the optimal choice of each division depends on the behavior of the other division.

The price levels ξ and ζ are modeled by real-valued random variables. For each realization of the prices ξ, ζ and price forecasts y_1, y_2 , the firm's net profit is given by

$$(4) \quad \begin{aligned} U(\xi, \zeta, s_1(y_1), s_2(y_2)) &\triangleq \xi s_1(y_1) + \zeta s_2(y_2) - \frac{1}{2} c_{11} s_1(y_1)^2 \\ &\quad - c_{12} s_1(y_1) s_2(y_2) - \frac{1}{2} c_{22} s_2(y_2)^2. \end{aligned}$$

The two divisions collaborate toward the maximization of the firm's expected net profit by choosing suitable production strategies. The optimal production levels can be found by solving the following static team optimization problem.

Problem OPMF (optimal production in a multidivisional firm). *Given a joint probability density function $\rho((\xi, \zeta), y_1, y_2)$ for the prices ξ, ζ , the price forecasts y_1, y_2 , and the firm's net profit $U(\xi, \zeta, s_1(y_1), s_2(y_2))$, find the production strategies $s_1^\circ(y_1), s_2^\circ(y_2)$ such that*

$$(s_1^\circ, s_2^\circ) \in \operatorname{argmax} \{v(s_1, s_2) \mid s_i \in \mathcal{M}(Y_i, A_i), i = 1, 2\},$$

where $v(s_1, s_2) \triangleq \mathbb{E}_{\xi, \zeta, y_1, y_2} \{U(\xi, \zeta, s_1(y_1), s_2(y_2))\}$.

Problem OPMF is an instance of Problem TO with a two-dimensional state of the world $x \triangleq (\xi, \zeta) \in [\xi_{\min}, \xi_{\max}] \times [\zeta_{\min}, \zeta_{\max}]$, a number $n = 2$ of DMs, the sets $Y_1 \triangleq [\xi_{\min}, \xi_{\max}]^{d_1}$, $Y_2 \triangleq [\zeta_{\min}, \zeta_{\max}]^{d_2}$, $A_1 \triangleq [0, a_{1, \max}]$, $A_2 \triangleq [0, a_{2, \max}]$, the joint probability density function $\rho(x, y_1, y_2) \triangleq \rho((\xi, \zeta), y_1, y_2)$, and the team utility $u(x, s_1(y_1), s_2(y_2)) \triangleq U(\xi, \zeta, s_1(y_1), s_2(y_2))$. The generalization of Problem OPMF to $n \geq 2$ divisions is straightforward.

In section 5, we shall specialize to Problem OPMF the smoothness properties of Problem TO that we shall derive in section 3 and the estimates of accuracy of suboptimal strategies that we shall obtain in section 4.

3. Smooth optimal strategies. In this section, we investigate existence and uniqueness of smooth optimal strategies for Problem TO. According to the next lemma, their search in the space $\mathcal{M}(Y_i, A_i)$ of bounded measurable functions from Y_i to A_i can be restricted within the space $\mathcal{C}(Y_i, A_i)$ of continuous functions from Y_i to A_i .

LEMMA 3.1. *Let Assumptions A1 and A2 hold. Then*

$$(5) \quad \begin{aligned} &\sup \left\{ v(s_1, \dots, s_n) \mid s_i \in \mathcal{M}(Y_i, A_i), i = 1, \dots, n \right\} \\ &= \sup \left\{ v(s_1, \dots, s_n) \mid s_i \in \mathcal{C}(Y_i, A_i), i = 1, \dots, n \right\}. \end{aligned}$$

The next theorem gives conditions guaranteeing that for a utility function of class \mathcal{C}^m , Problem TO has a solution made up of an n -tuple of strategies with a degree of smoothness that grows linearly with m . The theorem provides a higher degree of smoothness than [17, Theorem 1] and [24, Theorem 11, p. 162]. To this end, Assumption A3 plays a basic role. See section 6 for a discussion of some useful consequences of such a higher degree of smoothness.

THEOREM 3.2. *Let Assumptions A1–A3 hold. Then Problem TO has an n -tuple $(s_1^\circ, \dots, s_n^\circ)$ of optimal strategies with partial derivatives that are Lipschitz up to the order $m - 2$.*

Some estimates of the Lipschitz constants of the optimal strategies are given in section 6.

The next theorem states that under an additional condition, the optimal n -tuple of smooth strategies is unique. For simplicity, we consider $n = 2$ DMs; the extension to $n \geq 2$ DMs can be made taking the hint from [31, section 6].

THEOREM 3.3. *Let Assumptions A1–A3 hold with $n = 2$, u be a quadratic function with respect to a_1 and a_2 , and $\beta_{1,2}/(2\tau) < 1$, where*

$$\beta_{1,2} \triangleq \sqrt{d_1} \sqrt{d_2} \max_{(x, y_1, y_2) \in X \times Y_1 \times Y_2} \max_{q=1, \dots, l_1, r=1, \dots, l_2} \left| \frac{\partial^2}{\partial a_{1,q} \partial a_{2,r}} u(x, y_1, y_2, a_1, a_2) \right|.$$

Then, Problem TO has a unique optimal pair (s_1°, s_2°) of strategies in $\mathcal{C}^{m-2}(Y_1, A_1) \times \mathcal{C}^{m-2}(Y_2, A_2)$.

Another situation in which the optimal strategies are unique occurs when the team utility function u is strictly concave with respect to the decision variables. The following theorem states this fact.

THEOREM 3.4. *Let Assumptions A1–A3 hold and the team utility function $u(x, y_1, \dots, y_n, a_1, \dots, a_n)$ be strictly concave with respect to (a_1, \dots, a_n) . Then Problem TO has a unique optimal n -tuple $(s_1^\circ, \dots, s_n^\circ)$ of strategies in $\mathcal{C}^{m-2}(Y_1, A_1) \times \dots \times \mathcal{C}^{m-2}(Y_n, A_n)$.*

4. Accuracies of suboptimal strategies. Closed-form solutions to Problem TO can be found only in particular cases (see the introduction). In general, only suboptimal solutions can be obtained. One possible way to find them consists of using suitable *approximation schemes*, in which the search is restricted to strategies having a simple form, e.g., linear combinations of a certain number of *basis functions*.

Classical approximation schemes in a normed function space \mathcal{H} are formalized as linear combinations of a certain number k of basis functions $\varphi_1, \dots, \varphi_k : \mathbb{R}^d \rightarrow \mathbb{R}$ that span a linear subspace, at most k -dimensional [50]. Thus, such schemes take on the form

$$(6) \quad \sum_{i=1}^k c_i \varphi_i(\cdot),$$

where the coefficients c_1, \dots, c_k are determined in such a way to minimize the distance (measured using the norm of the space \mathcal{H}) between the corresponding suboptimal strategies and the optimal (unknown) one. For example, this is the case with algebraic and trigonometric polynomials in the space of continuous functions on compact sets, orthogonal polynomials in Lebesgue spaces [54], etc. As (6) is a linear combination of k elements from a set of fixed-basis functions, it is called a *fixed-basis approximation scheme*. Although fixed-basis approximation has many convenient properties (see, e.g., [50]), often its applications are limited by the *curse of dimensionality* [7], i.e., a very fast (e.g., exponential) growth, as a function of the number d of variables (in our case, the dimensions d_i , $i = 1, \dots, n$, of the information vectors y_i available to the DMs), of the number k of basis functions needed to achieve a desired accuracy of approximation.

An alternative approximation scheme consists of using linear combinations of basis functions $\psi(\cdot, w_1), \dots, \psi(\cdot, w_k)$ obtained from a “mother function” $\psi(\cdot, w)$ by varying a vector w of adjustable parameters, i.e.,

$$(7) \quad \sum_{i=1}^k c_i \psi(\cdot, w_i),$$

where the vectors w_1, \dots, w_k are optimized together with the coefficients c_1, \dots, c_k . In general, the presence of the “inner” parameter vectors w_1, \dots, w_k “destroys” lin-

earity. So (7) is a nonlinear approximation scheme, which belongs to the family of *variable-basis approximation schemes* [22, 27, 30]. With suitable choices of the “mother function” ψ , (7) models a variety of approximating families widely used in applications, such as free-node splines, trigonometric polynomials with free frequencies and phases, radial-basis-function networks with adjustable centers and widths, and feedforward neural networks [27]. Its use in functional optimization (also called “infinite-dimensional programming”) was formalized in [58] and studied in [?, 14, 16, 28, 29, 57, 58]. Advantages of certain variable-basis approximation schemes of the form (7) over classical linear ones of the form (6) were investigated, e.g., in [5, 13, 22, 27, 30] for function approximation and in [14, 28, 58] for functional optimization.

In the following, we shall derive upper bounds on the distance between the value of the team, i.e., the quantity $\sup_{s_1, \dots, s_n} v(s_1, \dots, s_n)$, and the suboptimal values when solutions are searched for within a subset of strategies expressed via certain variable-basis approximation schemes. Then, we shall estimate the minimum number k of basis functions required to guarantee a desired accuracy in approximating the value of the team (i.e., to guarantee that the difference between the value of the team and the expected value of the team utility when suitable variable-basis strategies are used is below a desired threshold).

As a first step, the following theorem allows one to reduce Problem TO to a function approximation problem.

THEOREM 4.1. *Let $u(x, y_1, \dots, y_n, a_1, \dots, a_n)$ be Lipschitz with constant L with respect to (a_1, \dots, a_n) and suppose that Problem TO has a solution $(s_1^\circ, \dots, s_n^\circ)$. Then, for every n -tuple (s_1, \dots, s_n) of strategies one has*

$$v(s_1^\circ, \dots, s_n^\circ) - v(s_1, \dots, s_n) \leq L \sum_{i=1}^n \sqrt{\mathbb{E}_{y_i} \{ \|s_i^\circ(y_i) - s_i(y_i)\|^2 \}}.$$

According to Theorem 4.1, in order to guarantee a satisfactory approximation of the value of the team (i.e., the quantity $\sup_{s_1, \dots, s_n} v(s_1, \dots, s_n)$) it is sufficient to get a satisfactory approximation of an optimal n -tuple $(s_1^\circ, \dots, s_n^\circ)$ of strategies.

Conversely, the following theorem shows that under suitable conditions, any “sufficiently good” suboptimal solution to Problem TO is close to an optimal strategy $(s_1^\circ, \dots, s_n^\circ)$.

THEOREM 4.2. *Let Assumptions A1–A3 hold with $m \geq 3$, let $(s_1^\circ, \dots, s_n^\circ)$ be an optimal n -tuple of strategies, and assume that for some $\bar{\tau} > 0$ the team utility function $u(x, y_1, \dots, y_n, a_1, \dots, a_n)$ is strongly concave with constant $\bar{\tau}$ with respect to (a_1, \dots, a_n) . Then for every $\varepsilon > 0$ and every n -tuple (s_1, \dots, s_n) of continuous strategies one has*

$$v(s_1^\circ, \dots, s_n^\circ) - v(s_1, \dots, s_n) \leq \varepsilon \implies \sum_{i=1}^n \mathbb{E}_{y_i} \{ \|s_i^\circ(y_i) - s_i(y_i)\|^2 \} \leq \frac{\varepsilon}{\bar{\tau}}.$$

In the remainder of this section, we consider the approximation of the optimal strategies by the variable-basis scheme (7) with three kinds of basis functions: cosines with variable centers and phases, sigmoids, and Gaussians with variable centers and widths. For the sake of notational simplicity and without loss of generality, we suppose that the sets A_i are “multidimensional boxes,” as stated in the next assumption.

Assumption A4. The sets A_i , $i = 1, \dots, n$, take on the forms $A_i = \prod_{j=1}^{l_i} [a_{i,j}^l, a_{i,j}^u]$ with $a_{i,j}^l < a_{i,j}^u$, $i = 1, \dots, n$, $j = 1, \dots, l_i$.

For $i = 1, \dots, n$ and $j = 1, \dots, l_i$, we denote by $s_{i,j}$ the j th component of s_i and by $\text{Prj}_{A_{i,j}}$ the projection operator onto $A_{i,j} \triangleq [a_{i,j}^l, a_{i,j}^u]$. By Assumption A4, every admissible strategy s_i takes its values in $A_i \triangleq \prod_{j=1}^{l_i} A_{i,j}$.

We denote by

$$(8) \quad G_i(\cos, d_i) \triangleq \left\{ g_i : Y_i \rightarrow \mathbb{R} \mid g_i(y_i) = \prod_{r=1}^{d_i} \cos(\omega_{i,r} y_{i,r} + \theta_{i,r}), \right. \\ \left. \omega_{i,r} = \frac{2\pi h}{y_{i,r}^u - y_{i,r}^l}, h \in \mathbb{N}, \theta_{i,r} \in [0, 2\pi) \right\}$$

the set of cosine basis functions and by

$$(9) \quad S_i^{(k)}(\cos, d_i) \triangleq \left\{ \tilde{s}_i^{(k)} : Y_i \rightarrow A_i \mid \tilde{s}_{i,j}^{(k)}(y_i) = \text{Prj}_{A_{i,j}} \left(\sum_{q=1}^k c_{i,j,q} g_{i,j,q}(y_i) \right), \right. \\ \left. c_{i,j,q} \in \mathbb{R}, g_{i,j,q} \in G_i(\cos, d_i), j = 1, \dots, l_i \right\}$$

the corresponding approximating set.

By $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ we denote a *sigmoid*, i.e., a bounded and measurable function satisfying $\lim_{t \rightarrow -\infty} \sigma(t) = 0$ and $\lim_{t \rightarrow +\infty} \sigma(t) = 1$ (see, e.g., [11]). The basis set corresponding to sigmoids and the associated approximating set are denoted by

$$G_i(\sigma, d_i) \triangleq \left\{ g_i : Y_i \rightarrow \mathbb{R} \mid g_i(y_i) = \sigma(\langle w_i, y_i \rangle + b_i), w_i \in \mathbb{R}^{d_i}, b_i \in \mathbb{R} \right\}$$

and

$$(10) \quad S_i^{(k)}(\sigma, d_i) \triangleq \left\{ \tilde{s}_i^{(k)} : Y_i \rightarrow A_i \mid \tilde{s}_{i,j}^{(k)}(y_i) = \text{Prj}_{A_{i,j}} \left(\sum_{q=1}^k c_{i,j,q} g_{i,j,q}(y_i) \right), \right. \\ \left. c_{i,j,q} \in \mathbb{R}, g_{i,j,q} \in G_i(\sigma, d_i), j = 1, \dots, l_i \right\},$$

respectively.

For the approximation with Gaussian computational units, we denote by

$$G_i(\text{Gauss}, d_i) \triangleq \left\{ g_i : Y_i \rightarrow \mathbb{R} \mid g_i(y_i) = e^{-\frac{\|y_i - t_i\|^2}{b_i}}, t_i \in \mathbb{R}^{d_i}, b_i > 0 \right\}$$

the basis set and by

$$(11) \quad S_i^{(k)}(\text{Gauss}, d_i) \triangleq \left\{ \tilde{s}_i^{(k)} : Y_i \rightarrow A_i \mid \tilde{s}_{i,j}^{(k)}(y_i) = \text{Prj}_{A_{i,j}} \left(\sum_{q=1}^k c_{i,j,q} g_{i,j,q}(y_i) \right), \right. \\ \left. c_{i,j,q} \in \mathbb{R}, g_{i,j,q} \in G_i(\text{Gauss}, d_i), j = 1, \dots, l_i \right\}$$

the corresponding approximating set.

THEOREM 4.3. *Let Assumptions A1–A4 hold and*

(i) $m > \frac{\max_i \{d_i\}}{2} + 2$

or

(ii) m odd and $m > \max_i \{d_i\} + 1$.

Then there exists a positive constant C such that for every positive integer k there is an n -tuple of strategies $(\tilde{s}_1^{(k)}, \dots, \tilde{s}_n^{(k)})$ such that

$$v(s_1^\circ, \dots, s_n^\circ) - v(\tilde{s}_1^{(k)}, \dots, \tilde{s}_n^{(k)}) \leq C k^{-1/2},$$

where $\tilde{s}_i^{(k)} \in S_i^{(k)}(\cos, d_i)$ or $\tilde{s}_i^{(k)} \in S_i^{(k)}(\sigma, d_i)$ in the case (i) and $\tilde{s}_i^{(k)} \in S_i^{(k)}(\text{Gauss}, d_i)$ in the case (ii), $i = 1, \dots, n$.

According to Theorem 4.3, using a k -term variable-basis approximation of the optimal strategies, the difference between the value of the team and its suboptimal value is bounded from above by a term proportional to $k^{-1/2}$. Thus, to guarantee an approximation accuracy $\varepsilon > 0$ it is sufficient to use

$$k \geq C^2 \varepsilon^{-2}$$

basis functions. Hence, the minimum required number of basis functions grows at most quadratically with the inverse of the desired accuracy ε , thus avoiding the curse of dimensionality.

5. Application example: Optimal production in a multidivisional firm.

In this section, we shall illustrate our results on two instances of Problem OPMF described in section 2. More specifically, we shall approximate the optimal strategies $s_1^\circ(y_1)$ and $s_2^\circ(y_2)$ of a two-divisional firm that produces two different types of goods. In the first example, called “Instance A,” each division has at its disposal merely one price forecast of the type of goods it produces. In “Instance B,” instead, the divisions know three price forecasts of the respective products. In both cases, we report the results of numerical simulations and make comparisons with the situation in which each division has a complete knowledge of the price forecasts of both types of goods (i.e., when the decision strategies are “centralized”).

If the constraints on the decisions a_1, a_2 , the state of the world $x \triangleq (\xi, \zeta)$, and the information y_1, y_2 are removed and the joint probability density function $\rho(x, y_1, y_2)$ is Gaussian, then closed-form solutions to Problem OPMF can be derived by using classical results from team decision theory and the optimal strategies are linear in the information [34]. However, a Gaussian probability density function may be unrealistic [19]; in particular, in Problem OPMF it implies that there exists a positive probability of negative prices. When the Gaussian assumption does not hold, closed-form solutions to Problem OPMF are not available [19], even with a quadratic utility, and suboptimal solutions have to be searched for. To this end, the knowledge of smoothness properties of the (unknown) optimal solutions can be fruitfully exploited.

The following proposition guarantees for Problem OPMF the existence of optimal strategies that have Lipschitz partial derivatives up to the order $m - 2$ and estimates the accuracies of suboptimal strategies expressed as linear combinations of sinusoidal, sigmoidal, or Gaussian variable-basis functions. The proposition follows by Theorems 3.2 and 4.3.

PROPOSITION 5.1. *If Assumptions A1–A4 hold, then Problem OPMF has a solution made up of strategies with partial derivatives that are Lipschitz up to the order $m - 2$. Moreover, if*

(i) $m > \frac{\max_i \{d_i\}}{2} + 2$

or

(ii) m odd and $m > \max_i \{d_i\} + 1,$

then there exists a positive constant C such that for every positive integer k there is

a pair $(\tilde{s}_1^{(k)}, \tilde{s}_2^{(k)})$ of strategies such that

$$v(s_1^\circ, s_2^\circ) - v(\tilde{s}_1^{(k)}, \tilde{s}_2^{(k)}) \leq C k^{-1/2},$$

where $\tilde{s}_i^{(k)} \in S_i^{(k)}(\cos, d_i)$ or $\tilde{s}_i^{(k)} \in S_i^{(k)}(\sigma, d_i)$ in the case (i) and $\tilde{s}_i^{(k)} \in S_i^{(k)}(\text{Gauss}, d_i)$ in the case (ii), $i = 1, 2$.

It is worth remarking that Proposition 5.1 still holds if the quadratic utility function (4) is replaced by a function of class C^m for which Assumption A2 holds.

Let us now describe two instances of Problem OPMF and the simulation results obtained for each of them.

Instance A. The two divisions produce two types of goods in quantities $a_1 \in A_1 \triangleq [0, 12]$ and $a_2 \in A_2 \triangleq [0, 12]$, respectively. The products are sold at prices ξ and ζ , which are independent and uniformly distributed in the interval $[2, 10]$. The price forecasts y_1 and y_2 are independently generated in the intervals $Y_1 = Y_2 \triangleq [2, 10]$, according to two truncated Gaussian conditional probability density functions (with respect to ξ and ζ , respectively), with conditional means ξ and ζ and conditional variances ξ^2 and ζ^2 , respectively (computed before truncation). The coefficients of the utility function $U(\xi, \zeta, s_1(y_1), s_2(y_2))$ (see (4)) are $c_{11} = c_{22} \triangleq 1$ and $c_{12} \triangleq 0.15$.

PROPOSITION 5.2. *Instance A of Problem OPMF satisfies Assumptions A1–A4.*

The simulations were performed by constraining the strategies s_1 and s_2 to take on the form of variable-basis functions (see (7)) with sinusoidal, sigmoidal (specifically, the hyperbolic tangent was used), and Gaussian bases. The expectation operator in the firm’s expected net profit was approximated by using an empirical mean computed over a number L of realizations of the random variables. More specifically, let us denote by ξ^l, ζ^l, y_1^l , and y_2^l the l th realizations ($l = 1, \dots, L$) of the variables ξ, ζ, y_1 , and y_2 , respectively. We emphasize the dependencies of the parametrized strategies of the form (7) on vectors of adjustable parameters by writing

$$\tilde{s}_1^{(k)}(y_1^l, \omega_1) \triangleq \sum_{i=1}^k c_{1,i} \psi(y_1^l, w_{1,i}),$$

where $\omega_1 \triangleq (c_{1,1}, \dots, c_{1,k}, w_{1,1}, \dots, w_{1,k})$, and

$$\tilde{s}_2^{(k)}(y_2^l, \omega_2) \triangleq \sum_{i=1}^k c_{2,i} \psi(y_2^l, w_{2,i}),$$

where $\omega_2 \triangleq (c_{2,1}, \dots, c_{2,k}, w_{2,1}, \dots, w_{2,k})$ and the function ψ may be a sigmoid, a Gaussian, or a sinusoid. Each parameter vector $\omega_i, i = 1, 2$, contains k coefficients of the linear combinations and k vectors of “inner” parameters of the basis functions.

Once the number k of basis functions, their type, and the number L of realizations are fixed, we search for the parameter vectors ω_1^* and ω_2^* solving

$$(12) \quad \max_{\omega_1, \omega_2} v^{\text{emp}}(\omega_1, \omega_2),$$

where the superscript “emp” means “empirical” and

$$(13) \quad v^{\text{emp}}(\omega_1, \omega_2) \triangleq \frac{1}{L} \left[\sum_{l=1}^L \xi^l \tilde{s}_1^{(k)}(y_1^l, \omega_1) + \zeta^l \tilde{s}_2^{(k)}(y_2^l, \omega_2) - \frac{1}{2} c_{11} \tilde{s}_1^{(k)}(y_1^l, \omega_1)^2 - c_{12} \tilde{s}_1^{(k)}(y_1^l, \omega_1) \tilde{s}_2^{(k)}(y_2^l, \omega_2) - \frac{1}{2} \tilde{s}_2^{(k)}(y_2^l, \omega_2)^2 \right].$$

TABLE 1
Simulation results for Instance A of Problem OPMF.

Basis functions	Expected net profit				Simulation time (s)			
	$k = 5$	$k = 10$	$k = 15$	$k = 20$	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Sinusoidal	33.945	34.402	34.785	35.151	76.3	322.5	945.5	$1.58 \cdot 10^3$
Sigmoidal	33.949	34.604	34.818	34.991	110.5	500.4	748.4	$1.27 \cdot 10^3$
Gaussian	34.085	34.104	34.187	34.259	753.0	$2.17 \cdot 10^3$	$2.95 \cdot 10^3$	$3.00 \cdot 10^3$

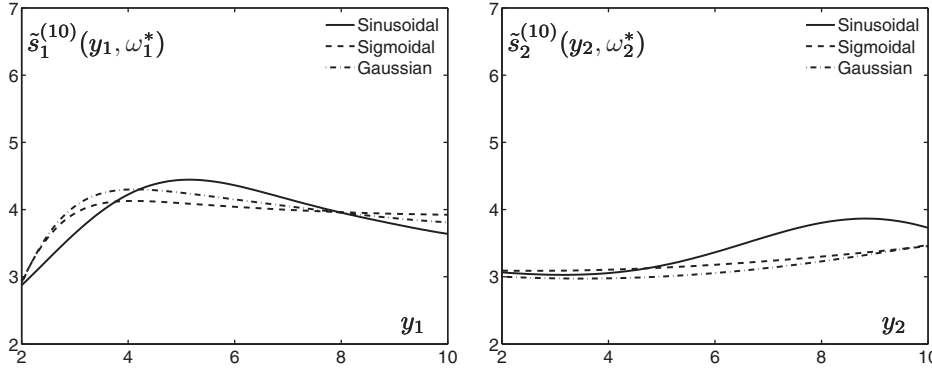


FIG. 1. Plots of the approximate optimal strategies $\tilde{s}_1^{(k)}$ and $\tilde{s}_2^{(k)}$ with $k = 10$ basis functions obtained for Instance A of Problem OPMF.

The maximization (12) entails a mathematical programming problem, to which we have applied the sequential quadratic programming algorithm [39]. Because, in general, problem (12) is nonconvex, the algorithm may be trapped in local maxima. In order to mitigate this risk, we adopted a multistart technique, which consists of solving (12) for several different initial values of the parameter vectors ω_1 and ω_2 and choosing the vectors ω_1^* and ω_2^* corresponding to the largest value of v^{emp} at the end of the optimization. In all the simulations, the empirical mean was computed over a number $L = 100$ of realizations of the random variables. The code for the simulations was written in MATLAB and the optimizations were performed using the routine *fmincon* of the MATLAB Optimization Toolbox (version 5.1 included in MATLAB 7.11). All the simulations were run on a PC with the Windows XP operating system, a 1.8 GHz Intel Core2 Duo CPU, and 2 GB RAM.

Table 1 presents the values of the firm’s expected net profit v^{emp} corresponding to the three types of variable-basis approximators with different numbers k of basis functions, together with the times (in seconds) needed to perform the simulations. Figure 1 shows the plots of the suboptimal strategies $\tilde{s}_1^{(k)}(y_1, \omega_1^*)$ and $\tilde{s}_2^{(k)}(y_2, \omega_2^*)$ with $k = 10$ basis functions.

The best performances in terms of expected net profit are obtained via sinusoidal variable-basis functions. As expected, a larger number k of basis functions provides a higher value of the profit, i.e., a higher approximation accuracy is achieved by using a larger number of basis functions. However, such an increase of performance is smaller when using Gaussian basis functions with respect to the sigmoidal and sinusoidal cases. Concerning the simulation times, the Gaussian basis functions require the largest computational effort; the simulation times of the sinusoidal and sigmoidal basis functions are smaller and quite similar.

TABLE 2
Simulation results in the presence of centralization for Instance A of Problem OPMF.

Basis functions	Expected net profit				Simulation time (s)			
	$k = 5$	$k = 10$	$k = 15$	$k = 20$	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Sinusoidal	33.988	34.791	36.150	36.748	86.5	273.5	$1.02 \cdot 10^3$	$2.41 \cdot 10^3$
Sigmoidal	34.005	34.677	34.980	35.595	155.8	$1.57 \cdot 10^3$	$2.00 \cdot 10^3$	$4.66 \cdot 10^3$
Gaussian	34.108	34.468	34.846	34.914	$2.44 \cdot 10^3$	$1.14 \cdot 10^4$	$1.67 \cdot 10^4$	$1.88 \cdot 10^4$

Let us now compare these results with the case in which each of the two divisions knows the price forecasts of both types of products. We shall refer to this case with the term “centralized,” as the decisions are taken with the knowledge of the whole information available on the two types of products. Equivalently, one can think that there exists a unique DM with information (y_1, y_2) and vector-valued strategies, to which one can apply a “centralized version” of Proposition 5.1. In this situation, each strategy depends on both forecasts y_1 and y_2 . We denote the two centralized strategies by $s_1^{\text{centr}}(y_1, y_2)$ and $s_2^{\text{centr}}(y_1, y_2)$, respectively.

Like in the previous case, the code for the simulations was written in MATLAB by using the routine *fmincon* of the MATLAB Optimization Toolbox (version 5.1 included in MATLAB 7.11). All the simulations were run on a PC with the Windows XP operating system, a 1.8 GHz Intel Core2 Duo CPU, and 2 GB RAM.

Table 2 summarizes the results obtained in the centralized case, using a suitably modified version of (13) for the empirical mean. The centralized suboptimal strategies provide better results than those obtained in the decentralized case: the values of the firm’s expected net profit reported in Table 2 are larger than the corresponding values reported in Table 1. Indeed, having at one’s disposal more information allows one to devise more effective strategies. However, the computational effort is larger than that of the decentralized case, as each strategy s_1^{centr} and s_2^{centr} is a function of two variables instead of merely one. Thus, the number k of basis functions being the same, the number of parameters to be optimized in the centralized case is larger than the corresponding number in the decentralized context. Finally, note that the gap between the centralized and decentralized cases increases when the number k of basis functions increases.

Instance B. As in Instance A, the firm is made up of two divisions that produce two types of goods in quantities $a_1 \in A_1 \triangleq [0, 12]$ and $a_2 \in A_2 \triangleq [0, 12]$, respectively, sold at prices ξ and ζ independently and uniformly distributed in the interval $[2, 10]$. Now, however, each division has at its disposal three forecasts of the prices of the good it produces: $y_{1,1}$, $y_{1,2}$, and $y_{1,3}$ for the first type and $y_{2,1}$, $y_{2,2}$, and $y_{2,3}$ for the second type. Letting $y_1 \triangleq (y_{1,1}, y_{1,2}, y_{1,3})$ and $y_2 \triangleq (y_{2,1}, y_{2,2}, y_{2,3})$, the strategies can be expressed as functions $s_1(y_1)$ and $s_2(y_2)$, respectively. The price forecasts y_1 and y_2 are independently generated in the three-dimensional cube $Y_1 = Y_2 \triangleq [2, 10]^3$. The forecasts $y_{1,1}$, $y_{1,2}$, and $y_{1,3}$ are obtained according to truncated Gaussian conditional probability density functions with respect to ξ , with conditional means ξ and conditional variances ξ^2 , $2\xi^2$, and $3\xi^2$, respectively (computed before truncation). Similarly, the forecasts $y_{2,1}$, $y_{2,2}$, and $y_{2,3}$ are generated via truncated Gaussian conditional probability density functions with respect to ζ , with conditional means ζ and conditional variances ζ^2 , $2\zeta^2$, and $3\zeta^2$, respectively (computed before truncation). Like in Instance A, the coefficients of the team utility function $U(\xi, \zeta, s_1(y_1), s_2(y_2))$ are $c_{11} = c_{22} \triangleq 1$ and $c_{12} \triangleq 0.15$.

TABLE 3
Simulation results for Instance B of Problem OPMF.

Basis functions	Expected net profit				Simulation time (s)			
	$k = 5$	$k = 10$	$k = 15$	$k = 20$	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Sinusoidal	34.349	35.634	36.711	36.892	122.3	840.6	$2.60 \cdot 10^3$	$5.06 \cdot 10^3$
Sigmoidal	33.800	34.974	36.069	36.037	237.4	$1.12 \cdot 10^3$	$1.55 \cdot 10^3$	$2.21 \cdot 10^3$
Gaussian	35.635	36.018	36.043	36.066	$1.54 \cdot 10^3$	$3.14 \cdot 10^3$	$3.62 \cdot 10^3$	$4.03 \cdot 10^3$

TABLE 4
Simulation results in the presence of centralization for Instance B of Problem OPMF.

Basis functions	Expected net profit				Simulation time (s)			
	$k = 5$	$k = 10$	$k = 15$	$k = 20$	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Sinusoidal	35.071	36.863	36.910	36.915	450.0	$3.10 \cdot 10^3$	$1.16 \cdot 10^4$	$2.21 \cdot 10^4$
Sigmoidal	34.035	35.094	36.138	36.323	$1.02 \cdot 10^3$	$2.95 \cdot 10^3$	$8.32 \cdot 10^3$	$1.19 \cdot 10^4$
Gaussian	35.769	36.104	36.203	36.366	$9.79 \cdot 10^3$	$1.62 \cdot 10^4$	$1.87 \cdot 10^4$	$2.09 \cdot 10^4$

PROPOSITION 5.3. Instance B of Problem OPMF satisfies Assumptions A1–A4.

As done in Instance A, the numerical simulations were performed by searching for strategies s_1 and s_2 expressed as variable-basis functions with sinusoidal, sigmoidal (specifically, the hyperbolic tangent was used), and Gaussian bases. The expectation in the objective function was approximated via an empirical mean (a suitably modified version of (13)) computed over $L = 100$ realizations of the random variables. The programming language and the computer platform used for the simulations are the same as in Instance A.

Table 3 shows the values of the firm’s expected net profit obtained in correspondence of the three types of approximation schemes with different numbers k of variable-basis functions, together with the times (in seconds) needed to perform the simulations.

For the values of the expected net profit, also for Instance B the best suboptimal solutions are obtained using sinusoidal variable-basis functions. Likewise in the case of Instance A, as expected, a larger number k of basis functions provides a higher value of the expected net profit; however, such an increase in performance is smaller when using Gaussian basis functions with respect to the sigmoidal and sinusoidal cases. Concerning the simulation times, the Gaussian basis functions require the largest computational effort. The simulation times of the sinusoidal and sigmoidal basis functions are smaller and quite similar. By comparing Tables 1 and 3 we note that, as expected, having at disposal more than one forecast on the price of goods provides better results with respect to having a unique forecast.

Following the modus operandi of Instance A, we compared the results of Table 3 with the case in which each of the two divisions knows the price forecasts of both types of products. We refer again to this case as “centralized,” since the decisions are taken with the knowledge of the whole information available on the two types of products. In this situation, both strategies depend on both vectors y_1 and y_2 , i.e., we have $s_1^{\text{entr}}(y_1, y_2)$ and $s_2^{\text{entr}}(y_1, y_2)$. Table 4 summarizes the results obtained in such a centralized case. The programming language and the computer platform used to perform the simulations are the same as in Instance A.

By comparing Tables 3 and 4, we note that the values of the firm’s expected net profit in the centralized case are larger than the corresponding values of the decentralized case. As in Instance A, having at one’s disposal a larger amount of information allows one to devise more effective strategies and the gap between the centralized

and decentralized cases increases if the number k of basis functions increases. The computational effort needed to perform the approximation is larger for the centralized case, as each strategy s_1^{centr} and s_2^{centr} is a function of six variables instead of the three variables on which each of the decentralized strategies s_1 and s_2 depends. Indeed, the number k of basis functions being the same, the number of parameters to be optimized in the centralized case is much larger than the corresponding number in the decentralized case, and thus the optimization problem is more difficult to solve. Finally note that, as expected, the increase of the computational time due to centralization is larger in the centralized situation of Instance B (each strategy depends on six variables) than in the centralized case of Instance A (each strategy depends on two variables).

6. Discussion. We have investigated team optimization problems with statistical information structure.

In the first part of the paper, we proved smoothness properties of the optimal strategies for a family of team optimization problems. This represents our first contribution.

Then, we exploited such properties to prove that the optimal strategies are in the closures of the convex hulls of certain sets (see the proof of Theorem 4.3) and we applied nonlinear approximation techniques in such a way to derive accurate suboptimal solutions. This is our second contribution. The literature shows that typically these nonlinear approximation tools (such as the Maurey–Jones–Barron theorem; see again the proof of Theorem 4.3) are exploited either assuming that the functions to be approximated belong to the closures of the convex hulls of some sets or restricting the search of approximators to such closures. Instead, we have proved that the (unknown) optimal strategies do belong to them. To the best of our knowledge, the proposed approach to the approximate solution of team optimization problems has not been previously investigated.

Finally, we illustrated the results on a team optimization problem that models production planning in a firm where various divisions collaborate toward the maximization of the firm’s expected net profit, on the basis of stochastic information given by forecasts of the selling prices.

6.1. The curse of dimensionality. The term “curse of dimensionality,” coined by Bellman [7], is used in the optimization literature in different contexts. Such contexts share the feature of unmanageable growth, with respect to a problem’s “dimension,” of the resources (typically, computational time and/or memory requirements) required to solve the problem itself. In this paper, the curse of dimensionality refers to an exponential growth, as a function of the number d of variables, of the number k of basis functions needed to achieve a desired accuracy of suboptimal solutions; d plays the role of “dimension” of Problem TO. Our results provide a way to cope, at least partially, with such a curse of dimensionality. More specifically, let the degree of smoothness m of the team utility and of the probability density functions depend linearly on the maximum dimension $\max_i \{d_i\}$ of the information vectors y_i , $i = 1, \dots, n$. According to Theorem 4.3, if suitable variable-basis strategies with k Gaussian, sinusoidal, or sigmoidal computational units are used, then the difference between the value of the team and the expected value of the utility is bounded from above by a quantity of order $k^{-1/2}$. So, to guarantee an approximation accuracy ε it is sufficient to use a number k of basis functions that grows at most quadratically with $1/\varepsilon$. This may be interpreted as an instance of the so-called *blessing of smoothness* [42], which compensates the curse of dimensionality.

In related studies, [43] considered three curses of dimensionality: the *curses of dimensionality in the state space, in the outcome space, and in the action space* (see [43, section 1.2]). They prevent the efficient use of the classical *dynamic programming* (DP) algorithm [7] for the solution of dynamic optimization problems with large dimensions of the state space, and/or the outcome space, and/or the action space. In some cases, such curses can be mitigated via the *approximate dynamic programming* algorithm described in [43], which exploits stochastic approximation methods to find approximate solutions to the Bellman optimality equations, on which DP is based. For some classes of dynamic optimization problems, the above-mentioned blessing of smoothness can be exploited to mitigate the *curse of dimensionality in optimal-policy-function approximation* [16].

6.2. Trade-off between decentralization and smoothness. It is worth comparing the degree of smoothness of the team utility function required to apply Theorem 4.3 with the degree of smoothness required to apply the same theorem in a centralized context, i.e., when there is only one DM with information vector $(y_1, \dots, y_n) \in \prod_{i=1}^n Y_i \subset \mathbb{R}^{\sum_{i=1}^n d_i}$. Obviously, the value of the “one-member team” is larger than or equal to the value of the “decentralized team” but the centralized version has at least two drawbacks: the cost of making the whole information available to a single DM and the larger degree m of smoothness required to apply Theorem 4.3. Indeed, in the centralized case the degree m of smoothness has to grow linearly with respect to $\sum_{i=1}^n d_i$, whereas in the decentralized case with n DMs the linear growth is required merely with respect to $\max\{d_i, i = 1, \dots, n\}$.

6.3. Application of quasi-Monte Carlo methods. Another interesting consequence of our results is the possibility for applying quasi-Monte Carlo methods [37] and related ones, such as the Korobov method (see [56] and [26, Chapter 6]), for the computations of the multivariable integral

$$v(\tilde{s}_1^{(k)}, \dots, \tilde{s}_n^{(k)}) = \mathbb{E}_{x, y_1, \dots, y_n} \left\{ u(x, \{y_i\}_{i=1}^n, \{\tilde{s}_i^{(k)}(y_i)\}_{i=1}^n) \right\},$$

where $\tilde{s}_1^{(k)}, \dots, \tilde{s}_n^{(k)}$ are approximations of the respective optimal strategies. Estimates of the accuracies of such computations can be obtained via the Koksma–Hlawka inequality [37, p. 20], which requires that the integrands have finite variation in the sense of Hardy and Krause [37, p. 19]. Considering, e.g., the case of an integrand f defined on the r -dimensional unit-cube $[0, 1]^r$, the formula (2.5) in [37, p. 19], which is typically used to prove that f has a finite variation in the sense of Hardy and Krause, requires that $f \in \mathcal{C}^r([0, 1]^r)$, i.e., the degree of smoothness has to be at least equal to the number of variables. When $m \geq \sum_{i=0}^n d_i + 2$ in Assumption A1, Theorem 3.2 provides such a degree of smoothness of the optimal strategies.

6.4. On the use of a greedy approximation algorithm. In deriving some of our results (in particular, Theorem 4.3), we have exploited the Maurey–Jones–Barron theorem [5, Lemma 1, p. 934] (see also [21, 41]). It allows one to deal with the case of a utility function $u(x, y_1, \dots, y_n, a_1, \dots, a_n)$ that is separately strongly concave with constant τ with respect to each decision variable a_1, \dots, a_n . When the utility is strongly concave with respect to the whole decision vector (a_1, \dots, a_n) , variable-basis suboptimal strategies with accuracies of the same order $k^{-1/2}$ can be obtained by exploiting, instead of the Maurey–Jones–Barron theorem, the greedy algorithm developed in [55] to maximize strongly-concave functionals over the convex hull of a set of basis functions. At each iteration, the algorithm proposed therein selects

a suitable basis function and solves a one-dimensional mathematical programming problem. The application of such algorithm to Problem TO with strongly concave utility function is made possible by the structural properties of the optimal strategies (see the proof of Theorem 4.3).

Let us consider, e.g., the proof of Theorem 4.3(i), where we have shown that each function $s_{i,j}^\circ$ belongs to the closure of the convex hull of the set $G_{i,j}(\cos, d_i)$. For a utility function $u(x, y_1, \dots, y_n, a_1, \dots, a_n)$ that is strongly concave with respect to the decision vector (a_1, \dots, a_n) , the existence of a strategy $\tilde{s}_{i,j}^{(k)}$ such that

$$\|s_{i,j}^\circ - \tilde{s}_{i,j}^{(k)}\|_{\mathcal{H}_i}^2 = \mathbb{E}_{y_i} \{|s_{i,j}^\circ(y_i) - \tilde{s}_{i,j}^{(k)}(y_i)|^2\} \leq \frac{C_{i,j}}{k}$$

follows by [55, Theorem IV.2]¹ and the properties of the modulus of concavity of a functional stated, e.g., in [28, Proposition 4.2(iii), (iv)] (the latter is applied to the objective functional in Problem TO). The advantage over the proof based on the Maurey–Jones–Barron theorem is that the approach developed in [55] also provides an algorithm (see [55, Algorithm II.1]) to find such a function $\tilde{s}_{i,j}^{(k)}$. The disadvantage is that to apply the results from [55] the utility function $u(x, y_1, \dots, y_n, a_1, \dots, a_n)$ has to be strongly concave with respect to the whole decision vector (a_1, \dots, a_n) , instead of merely separately strongly concave with respect to each decision variable.

As the numerical results that we presented in section 5 are not the core of the paper but are intended to demonstrate the way in which the theoretical results can be exploited and applied to concrete situations, in the simulations we have not implemented [55, Algorithm II.1]. Instead, we have solved a nonlinear least squares problem via nonlinear optimization (sequential quadratic programming) combined with a multistart technique, which applies to the more general case of a utility function that is separately strongly concave.

6.5. Application to network team optimization problems. Such problems arise in optimization, management, and control of *traffic networks*. Such networks include, e.g., computer networks extending in large geographical areas, store-and-forward packet-switching telecommunication networks, large-scale freeway systems, reservoir networks in water-management systems, and queueing networks in manufacturing systems. They can be modeled as graphs in which a set of nodes (with storing capabilities) are connected through a set of links (where traffic delays and transport costs may be incurred) that cannot be loaded with traffic above their capacities. In this context, the team utility function u can be written as the sum of a finite number of individual utility functions u_i , each one associated with a single DM (e.g., a telecommunications router) or a shared resource in the network (e.g., a communication link). In addition, each u_i depends only on a subset of the DMs [48]. The DMs are the nodes of a graph, and there is an edge between two DMs iff both appear in the same individual utility function. Traffic flows can be described by continuous variables, even if the “objects” exchanged among the nodes are discrete in nature (e.g., data packets, messages, cars, workpieces). This is justified whenever the number of objects is so large as to require macroscopic modeling. In store-and-forward packet-switching telecommunication networks, for instance, the DMs are the routers

¹Note that [55, Theorem IV.2] refers to an optimization problem set on the convex hull of a set of functions. However, inspection of its proof shows that for a continuous objective functional [55, Theorem IV.2] can be applied to a problem formulated in the closure of the convex hull of such a set.

acting as members of a team (they aim at maximizing a common objective related, e.g., to the congestions of the links). Each router has at its disposal some private information (e.g., the total lengths of its incoming packet queues), on the basis of which it decides how to split the incoming traffic flows into its output links.

The importance of deriving suboptimal solutions to network team optimization problems originates from the fact that closed-form solutions can be derived only in particular cases (typically, under LQG hypotheses and, in the dynamic case, partially nested information; see the introduction). The particular structure of a network team optimization allows for various simplifications in our model and results.

- As the strategy of each DM is influenced only by those of its neighbors in the network, Assumption A3 is easier to impose.
- An extension of Theorem 3.3 to $n \geq 2$ DMs can be formulated in terms of interaction terms $\beta_{i,j}$ (defined in a similar way as $\beta_{1,2}$ in Theorem 3.3), where each (i, j) is a pair of different DMs in the team. For a network team optimization problem, most $\beta_{i,j}$ are expected to be equal to 0 (since the interaction of each DM is limited to its neighbors in the graph); therefore such an extension takes on a simplified form.
- Since the utility function can be written as the sum of individual utility functions, the integral $\mathbb{E}_{x,y_1,\dots,y_n} \{u(x, \{y_i\}_{i=1}^n, \{s_i(y_i)\}_{i=1}^n)\} \triangleq v(s_1, \dots, s_n)$ (see (2)) can be decomposed into the sum of a finite number of integrals, each typically dependent on less than $\sum_{i=0}^n d_i$ variables. So, in quasi-Monte Carlo methods the minimum degree m of smoothness required by [37, p. 19, formula (2.5)] to prove for each integrand the finiteness of its variation in the sense of Hardy and Krause is smaller than $\sum_{i=0}^n d_i + 2$. (Compare with the general case discussed above.)

As to specific applications to network team optimization, our smoothness results may be applied, e.g., to stochastic versions of the congestion, routing, and bandwidth allocation problems considered in [32], which are stated in terms of smooth and concave individual utility functions.

6.6. On the Lipschitz constants of the optimal strategies. Inspection of the proof of Theorem 3.2 allows one to estimate the Lipschitz constants of the optimal strategies. By applying formulas (3) and (27) for $y''_i - y'_i$ in the direction of the gradient of $s^j_{i,h}$ and taking the limit as j goes to $+\infty$, some computations provide the following upper bound on the Lipschitz constant of s_i° for every integer $n \geq 2$ and $i = 1, \dots, n$:

$$(14) \quad \frac{\sqrt{d_i} \sqrt{l_i}}{2\tau} \times \sup_{y_i \in Y_i, q=1, \dots, d_i, r=1, \dots, l_i} \left| \frac{\int_{X \times \{Y_t\}_{t \neq i}} \rho \frac{\partial u}{\partial a_{i,r}} dx \{dy_t\}_{t \neq i} \int_{X \times \{Y_t\}_{t \neq i}} \frac{\partial \rho}{\partial y_{i,q}} dx \{dy_t\}_{t \neq i}}{\left(\int_{X \times \{Y_t\}_{t \neq i}} \rho dx \{dy_t\}_{t \neq i} \right)^2} - \frac{\int_{X \times \{Y_t\}_{t \neq i}} \rho dx \{dy_t\}_{t \neq i} \int_{X \times \{Y_t\}_{t \neq i}} \left(\frac{\partial \rho}{\partial y_{i,q}} \frac{\partial u}{\partial a_{i,r}} + \rho \frac{\partial^2 u}{\partial y_{i,q} \partial a_{i,r}} \right) dx \{dy_t\}_{t \neq i}}{\left(\int_{X \times \{Y_t\}_{t \neq i}} \rho dx \{dy_t\}_{t \neq i} \right)^2} \right|.$$

Once the joint probability density function ρ is chosen, this bound can be exploited to keep under control the Lipschitz constant of the optimal strategies, at the expense of some computations.

As an example, let us consider the case of a uniform joint probability density function ρ . Such a choice is quite meaningful: for instance, for a scalar random variable

it represents a situation of maximum uncertainty, in the sense that it maximizes the differential entropy [10] among all joint probability density functions on a compact interval. In this case, the upper bound (14) takes on the form

$$(15) \quad \frac{\sqrt{d_i} \sqrt{l_i}}{2\tau} \sup \left| \frac{\partial^2 u(x, y_1, \dots, y_n, a_1, \dots, a_n)}{\partial y_{i,q} \partial a_{i,r}} \right|,$$

where the supremum is with respect to $(x, y_1, \dots, y_n, a_1, \dots, a_n) \in X \times \prod_{i=1}^n Y_i \times \prod_{i=1}^n A_i$, $q = 1, \dots, d_i$, and $r = 1, \dots, l_i$. So, the dependence of the Lipschitz constant of the optimal strategies on the number n of DMs can be controlled via the dependence on n of the constant τ of separate concavity of the utility function u and the largest absolute values of its second-order derivatives $\frac{\partial^2 u(x, y_1, \dots, y_n, a_1, \dots, a_n)}{\partial y_{i,q} \partial a_{i,r}}$. Note that the upper bound (15) shows the dependence on the dimensions d_i and l_i . The Lipschitz constant does not blow up with d_i and l_i . Indeed, its rate of growth is quite slow: it is merely proportional to the product $\sqrt{d_i} \sqrt{l_i}$ of the square roots of the “dimensions” d_i and l_i of the problem. It is worth remarking that moderately large values of n and small values of d_i and l_i are of practical interest, as they correspond to situations in which one has many DMs, each with a simple structure (i.e., small dimensions of the decision and information vectors).

A family of problems and associated utility functions u , which can be modeled in the form of Problem TO and for which a good control of the Lipschitz constants can be obtained, is represented by the network team optimization problems discussed in section 6.5. They share the following characteristics: (i) the strategy of each DM is influenced only by those of its neighbors in the network, (ii) the utility function can be written as the sum of individual utility functions, and (iii) as the interaction of each DM is limited to its neighbors in the graph, each utility function depends only on a small number of DMs. Let us investigate the consequences of such features on the upper bound (15) on the Lipschitz constant and, in particular, its dependence on the number n of DMs. As the utility function is the sum of individual utility functions, the constant τ of separate strong concavity can be considered to be independent of n . Moreover, as each individual utility function depends only on a small number of DMs (the neighboring ones in the graph), the term $\frac{\partial^2 u(x, y_1, \dots, y_n, a_1, \dots, a_n)}{\partial y_{i,q} \partial a_{i,r}}$ can be bounded from above independently of n . In such a way, one can keep the Lipschitz constants of the optimal strategies under control.

6.7. Extensions to other n -person games. Problem TO is a particular case of n -person games (also called *pure coordination games* [49], a particular case of *potential games* [45]), in which the players share the same objective functional. Our smoothness results can be extended to games in which different players may have different objectives. In particular, the technique used in Step 1 of the proof of Theorem 3.2 may be applied to prove analogous smoothness properties for n -tuples of strategies representing a Nash equilibrium in an infinite-dimensional stochastic n -person game, like those studied in [31] and [35]. For instance, [31] provides sufficient conditions for the existence and uniqueness of a Nash equilibrium but it does not address the smoothness of the strategies. By our approach, one can investigate a sufficiently high degree of smoothness of such strategies and search for suboptimal ones implemented by variable-basis approximation schemes with k computational units, which represent an ε -Nash equilibrium [6, section 4.2] for $\varepsilon = \mathcal{O}(1/k^2)$ (thus without incurring the curse of the dimensionality). Such smoothness results may be of interest also in the context of the so-called *algorithmic game theory* [38].

Finally, we mention the following interpretation of our results in terms of a game: the larger the degree of smoothness of the optimal strategies, the smaller the relevance of each component of the information vector available to each player in finding an optimal strategy. More specifically, a small variation of such a component implies a small variation of the optimal decision and, keeping the other factors unchanged, the dependence decreases by increasing the degree of smoothness. Roughly speaking, this property allows one to efficiently approximate the optimal strategies with a small number of terms in suitable variable-basis approximation schemes.

7. Proofs. Recall that a subset \mathcal{F} of the space $\mathcal{C}(\Omega)$ of continuous functions on $\Omega \subseteq \mathbb{R}^d$ is *equicontinuous* at $z \in \Omega$ iff for every $\varepsilon > 0$ there exists a neighborhood U of z such that for every $w \in U$ and every $f \in \mathcal{F}$ one has $|f(z) - f(w)| \leq \varepsilon$. The set \mathcal{F} is *equicontinuous* iff it is equicontinuous at every $z \in \Omega$. The *Ascoli–Arzelá Theorem* [1, Theorem 1.33, p. 11] states that for a compact set $\Omega \subset \mathbb{R}^d$, a set $\mathcal{F} \subset \mathcal{C}(\Omega)$ is compact in $\mathcal{C}(\Omega)$ iff it is closed, bounded, and equicontinuous.

Proof of Lemma 3.1. The proof proceeds as in Step 1 of the proof of [17, Theorem 1]; for completeness, we report it here. We give the proof for the case of $n = 2$ DMs, then we mention the changes required for the extension to $n > 2$.

Proof for $n = 2$. Consider a sequence $\{s_1^j, s_2^j\}$ of pairs of strategies, indexed by $j \in \mathbb{N}_+$, such that

$$\lim_{j \rightarrow \infty} v(s_1^j, s_2^j) = \sup_{s_1 \in \mathcal{M}(Y_1, A_1), s_2 \in \mathcal{M}(Y_2, A_2)} v(s_1, s_2).$$

(Such a sequence exists by the definition of supremum.) From this sequence, we generate the sequence $\{\hat{s}_1^j, \hat{s}_2^j\}$ defined for every $y_1 \in Y_1$ and every $y_2 \in Y_2$ as

$$(16) \quad \hat{s}_1^j(y_1) \triangleq \operatorname{argmax}_{a_1 \in A_1} \mathbb{E}_{x, y_2 | y_1} \{u(x, y_1, y_2, a_1, s_2^j(y_2))\},$$

$$(17) \quad \hat{s}_2^j(y_2) \triangleq \operatorname{argmax}_{a_2 \in A_2} \mathbb{E}_{x, y_1 | y_2} \{u(x, y_1, y_2, \hat{s}_1^j(y_1), a_2)\}.$$

The proof is structured as follows. First, we show that for every $j \in \mathbb{N}_+$, \hat{s}_1^j and \hat{s}_2^j are well-defined (i.e., the maxima in (16) and (17) exist and are uniquely achieved) and continuous, so it makes sense to evaluate $v(\hat{s}_1^j, \hat{s}_2^j)$. Thus, by (2), (16), (17), and the possibility of interchanging maximization and integration [46, Theorem 14.60], we get $v(\hat{s}_1^j, \hat{s}_2^j) \geq v(s_1^j, s_2^j)$. Hence,

$$\lim_{j \rightarrow \infty} v(\hat{s}_1^j, \hat{s}_2^j) = \sup_{s_1 \in \mathcal{M}(Y_1, A_1), s_2 \in \mathcal{M}(Y_2, A_2)} v(s_1, s_2).$$

We detail the proof for \hat{s}_1^j ; the same arguments hold for \hat{s}_2^j .

Let us show that for every $j \in \mathbb{N}_+$ the function \hat{s}_1^j is well-defined and continuous. Let

$$M_1^j(y_1, a_1) \triangleq \mathbb{E}_{x, y_2 | y_1} \{u(x, y_1, y_2, a_1, s_2^j(y_2))\},$$

so by definition

$$(18) \quad \hat{s}_1^j(y_1) = \operatorname{argmax}_{a_1 \in A_1} M_1^j(y_1, a_1).$$

As the probability density function $\rho(x, y_1, y_2)$ is of class \mathcal{C}^m and strictly positive on $X \times Y_1 \times Y_2$, the conditional density $\rho(x, y_2 | y_1) = \frac{\rho(x, y_1, y_2)}{\int_{X \times Y_2} \rho(x, y_1, y_2) dx dy_2}$ is of class

\mathcal{C}^m on $X \times Y_1 \times Y_2$. Let Ω be the open set containing $X \times Y_1 \times Y_2 \times A_1 \times A_2$ on which the team utility function u is of class \mathcal{C}^m , and for $i = 1, 2$, let $\tilde{A}_i \supset A_i$ be compact and such that $X \times Y_1 \times Y_2 \times \tilde{A}_1 \times \tilde{A}_2 \subset \Omega$. Since $\rho(x, y_2 | y_1)$ and u are of class \mathcal{C}^m on the compact sets $X \times Y_1 \times Y_2$ and $X \times Y_1 \times Y_2 \times \tilde{A}_1 \times \tilde{A}_2$, respectively, M_1^j is of class \mathcal{C}^m on the compact set $Y_1 \times \tilde{A}_1$. (It is an integral dependent on parameters.) By [17, Lemma 1], for every $y_1 \in Y_1$ the function $M_1^j(y_1, \cdot)$ is Lipschitz and strongly concave with constant τ on the open set $\text{int } \tilde{A}_1 \supset A_1$, where $\text{int } \tilde{A}_1$ denotes the topological interior of \tilde{A}_1 .

By the above-proved continuity and strong concavity properties of M_1^j with respect to a_1 , for every $y_1 \in Y_1$ the maximum in (18) exists and is unique, so the function $\hat{s}_1^j(\cdot)$ is well-defined. Moreover, by the necessary and sufficient optimality condition stated in [8, Theorem 3.2, p. 138], 0 is a supergradient of $M_1^j(y_1, \cdot)$ for every $y_1 \in Y_1$. Take $y'_1, y''_1 \in Y_1$. By the definition of \hat{s}_1^j , exploiting the strong concavity with constant τ of $M_1^j(y'_1, \cdot)$ and taking the supergradient 0 of $M_1^j(y'_1, \cdot)$ at $\hat{s}_1^j(y'_1)$ we get

$$(19) \quad M_1^j(y'_1, \hat{s}_1^j(y''_1)) - M_1^j(y'_1, \hat{s}_1^j(y'_1)) \leq -\tau \|\hat{s}_1^j(y''_1) - \hat{s}_1^j(y'_1)\|^2.$$

Similarly, we obtain

$$(20) \quad M_1^j(y''_1, \hat{s}_1^j(y'_1)) - M_1^j(y''_1, \hat{s}_1^j(y''_1)) \leq -\tau \|\hat{s}_1^j(y'_1) - \hat{s}_1^j(y''_1)\|^2.$$

By summing (19) and (20) we have

$$(21) \quad M_1^j(y'_1, \hat{s}_1^j(y''_1)) - M_1^j(y'_1, \hat{s}_1^j(y'_1)) + M_1^j(y''_1, \hat{s}_1^j(y'_1)) - M_1^j(y''_1, \hat{s}_1^j(y''_1)) \\ \leq -2\tau \|\hat{s}_1^j(y''_1) - \hat{s}_1^j(y'_1)\|^2$$

and by changing the sign to both sides of (21) we have

$$(22) \quad - (M_1^j(y'_1, \hat{s}_1^j(y''_1)) - M_1^j(y'_1, \hat{s}_1^j(y'_1)) + M_1^j(y''_1, \hat{s}_1^j(y'_1)) - M_1^j(y''_1, \hat{s}_1^j(y''_1))) \\ \geq 2\tau \|\hat{s}_1^j(y''_1) - \hat{s}_1^j(y'_1)\|^2.$$

Together, (21) and (22) give

$$(23) \quad |M_1^j(y'_1, \hat{s}_1^j(y''_1)) - M_1^j(y'_1, \hat{s}_1^j(y'_1)) + M_1^j(y''_1, \hat{s}_1^j(y'_1)) - M_1^j(y''_1, \hat{s}_1^j(y''_1))| \\ \geq 2\tau \|\hat{s}_1^j(y''_1) - \hat{s}_1^j(y'_1)\|^2.$$

Let Λ^j be the Lipschitz constant of the function $M_1^j \in \mathcal{C}^m(Y_1 \times \tilde{A}_1)$. Then

$$(24) \quad |M_1^j(y'_1, \hat{s}_1^j(y''_1)) - M_1^j(y'_1, \hat{s}_1^j(y'_1)) + M_1^j(y''_1, \hat{s}_1^j(y'_1)) - M_1^j(y''_1, \hat{s}_1^j(y''_1))| \\ \leq |M_1^j(y'_1, \hat{s}_1^j(y''_1)) - M_1^j(y'_1, \hat{s}_1^j(y'_1))| \\ + |M_1^j(y''_1, \hat{s}_1^j(y'_1)) - M_1^j(y''_1, \hat{s}_1^j(y''_1))| \leq 2\Lambda^j \|y''_1 - y'_1\|.$$

By (23) and (24), we obtain

$$2\Lambda^j \|y''_1 - y'_1\| \geq 2\tau \|\hat{s}_1^j(y''_1) - \hat{s}_1^j(y'_1)\|^2,$$

i.e.,

$$(25) \quad \|\hat{s}_1^j(y''_1) - \hat{s}_1^j(y'_1)\| \leq \sqrt{\frac{\Lambda^j}{\tau}} \sqrt{\|y''_1 - y'_1\|},$$

which proves the Hölder continuity of \hat{s}_1^j , hence its continuity. The continuity of \hat{s}_2^j can be derived in the same way.

Extension to $n \geq 2$. One defines the n -tuple $\hat{s}_1^j, \dots, \hat{s}_n^j$ of strategies

$$\begin{aligned} \hat{s}_1^j(y_1) &\triangleq \operatorname{argmax}_{a_1 \in A_1} \mathbb{E}_{x, \{y_i\}_{i \neq 1} | y_1} \{u(x, \{y_i\}_{i=1}^n, a_1, \{s_i^j(y_i)\}_{i=2}^n)\}, \\ \hat{s}_2^j(y_2) &\triangleq \operatorname{argmax}_{a_2 \in A_2} \mathbb{E}_{x, \{y_i\}_{i \neq 2} | y_2} \{u(x, \{y_i\}_{i=1}^n, \hat{s}_1^j(y_1), a_2, \{s_i^j(y_i)\}_{i=3}^n)\}, \\ &\vdots \\ \hat{s}_n^j(y_n) &\triangleq \operatorname{argmax}_{a_n \in A_n} \mathbb{E}_{x, \{y_i\}_{i \neq n} | y_n} \{u(x, \{y_i\}_{i=1}^n, \{\hat{s}_i^j(y_i)\}_{i=1}^{n-1}, a_n)\} \end{aligned}$$

and applies the same arguments as in the case $n = 2$. \square

Proof of Theorem 3.2. Also in this case, we detail the proof for $n = 2$ DMs. The changes required for the extension to $n \geq 2$ are the same as in the proof of Lemma 3.1.

Consider the sequence $\{\hat{s}_1^j, \hat{s}_2^j\}$ of pairs of strategies, indexed by $j \in \mathbb{N}_+$, defined in the proof of Lemma 3.1 and such that

$$\lim_{j \rightarrow \infty} v(\hat{s}_1^j, \hat{s}_2^j) = \sup_{s_1 \in \mathcal{M}(Y_1, A_1), s_2 \in \mathcal{M}(Y_2, A_2)} v(s_1, s_2).$$

Step 1. Let us prove that \hat{s}_1^j and \hat{s}_2^j are of class \mathcal{C}^{m-1} with upper bounds on the absolute values of the partial derivatives (up to the order $m - 1$) of their components. We show also that such bounds are independent on j and y_1 for \hat{s}_1^j and on j and y_2 for \hat{s}_2^j . We make the proof for \hat{s}_1^j ; the same arguments hold for \hat{s}_2^j .

Step 1.a. First, we prove that \hat{s}_1^j is of class \mathcal{C}^1 and that its Lipschitz constant is independent of j . As Y_1 is convex, it is sufficient to show that the restriction of \hat{s}_1^j to each line joining every two points y_1' and y_1'' is Lipschitz with a constant that depends neither on j nor on the line. Likewise in the proof of Lemma 3.1, let

$$M_1^j(y_1, a_1) \triangleq \mathbb{E}_{x, y_2 | y_1} \{u(x, y_1, y_2, a_1, s_2^j(y_2))\},$$

so by definition $\hat{s}_1^j(y_1) = \operatorname{argmax}_{a_1 \in A_1} M_1^j(y_1, a_1)$. Consider the function $\hat{s}_1^j(y_1(t))$, where $y_1(t) \triangleq y_1' + t(y_1'' - y_1')$ and $0 \leq t \leq 1$. By Assumption A3 and the fact that the maximum point in (18) exists and is unique, for every $0 \leq t \leq 1$ we get $\hat{s}_1^j(y_1(t)) \in \operatorname{int} A_1 = \operatorname{int} \prod_{j=1}^{l_1} [a_{1,j}^l, a_{1,j}^u]$. Thus

$$(26) \quad \left. \frac{\partial M_1^j(y_1, a_1)}{\partial a_{1,h}} \right|_{y_1(t)=y_1'+t(y_1''-y_1'), a_1(t)=\hat{s}_1^j(y_1(t))} = 0, \quad h = 1, \dots, l_1.$$

As $M_1^j(y_1, \cdot)$ is of class \mathcal{C}^m and is strongly concave with constant τ on $\operatorname{int} \tilde{A}_i$, by (3) we have

$$\sup_{a_1 \in \operatorname{int} \tilde{A}_i} \lambda_{\max}(\nabla_{2,2}^2 M_1^j(y_1, a_1)) \leq -2\tau < 0,$$

where $\nabla_{2,2}^2$ denotes the Hessian with respect to the second (vector-valued) variable.

Then we can apply the vectorial form of the implicit function theorem to (26) in such a way to study the local differentiability of the vector-valued function $\hat{s}_1^j(y_1(t))$. Indeed, taking the total derivative with respect to t of both sides of (26) and exploiting

the fact that M_1^j is of class C^m with $m \geq 2$, for every $h = 1, \dots, l_1$ we get

$$(27) \quad \sum_{r=1}^{l_1} \sum_{q=1}^{d_1} \left(\frac{\partial^2 M_1^j}{\partial a_{1,r} \partial a_{1,h}} \frac{\partial a_{1,r}}{\partial y_{1,q}} \frac{\partial y_{1,q}}{\partial t} + \frac{\partial^2 M_1^j}{\partial y_{1,q} \partial a_{1,h}} \frac{\partial y_{1,q}}{\partial t} \right) \Big|_{y_1(t)=y'_1+t(y''_1-y'_1), a_1(t)=\hat{s}_1^j(y_1(t))} = 0.$$

Denoting by $(\frac{\partial^2 M_1^j}{\partial a_{1,r} \partial a_{1,h}})^{-1}$ the elements of the negative-definite inverse of the matrix with elements $\frac{\partial^2 M_1^j}{\partial a_{1,r} \partial a_{1,h}}$, using (27), and renaming the indices, for every $h = 1, \dots, l_1$ we get

$$(28) \quad \begin{aligned} \frac{da_{1,h}(t)}{dt} &= \frac{d\hat{s}_{1,h}^j(y_1(t))}{dt} \\ &= \sum_{q=1}^{d_1} \frac{\partial a_{1,h}}{\partial y_{1,q}} \frac{\partial y_{1,q}}{\partial t} \Big|_{y_1(t)=y'_1+t(y''_1-y'_1), a_1(t)=\hat{s}_1^j(y_1(t))} \\ &= - \sum_{r=1}^{l_1} \sum_{q=1}^{d_1} \left(\frac{\partial^2 M_1^j}{\partial a_{1,r} \partial a_{1,h}} \right)^{-1} \frac{\partial^2 M_1^j}{\partial y_{1,q} \partial a_{1,r}} \frac{\partial y_{1,q}}{\partial t} \Big|_{y_1(t)=y'_1+t(y''_1-y'_1), a_1(t)=\hat{s}_1^j(y_1(t))} \\ &= - \sum_{r=1}^{l_1} \sum_{q=1}^{d_1} \left(\frac{\partial^2 M_1^j}{\partial a_{1,r} \partial a_{1,h}} \right)^{-1} \frac{\partial^2 M_1^j}{\partial y_{1,q} \partial a_{1,r}} \Big|_{y_1(t)=y'_1+t(y''_1-y'_1), a_1(t)=\hat{s}_1^j(y_1(t))} (y''_{1,q} - y'_{1,q}), \end{aligned}$$

so $a_1(t)$ and $\hat{s}_{1,h}^j(y_1)$ are locally differentiable. As this holds for every $y_1 \in Y_1$, $\hat{s}_{1,h}^j(y_1)$ is of class C^1 on the whole Y_1 .

Since

$$\sup_{a_1 \in \text{int } \bar{A}_i} |\lambda_{\min}((\nabla_{2,2}^2 M_1^j(y_1, a_1))^{-1})| \leq \frac{1}{2\tau}$$

and $\|y''_{1,q} - y'_{1,q}\| \leq \text{diameter}(Y_1)$, for every q and r we search for an upper bound on $|\frac{\partial^2 M_1^j}{\partial y_{1,q} \partial a_{1,r}}|$ in (28), independent of y_1 and j . By definition, we have

$$M_1^j(y_1, a_1) = \frac{\int_{X \times Y_2} \rho(x, y_1, y_2) u(x, y_1, y_2, a_1, s_2^j(y_2)) dx dy_2}{\int_{X \times Y_2} \rho(x, y_1, y_2) dx dy_2}.$$

Simple calculations allow one to express $\frac{\partial^2 M_1^j}{\partial y_{1,q} \partial a_{1,r}}$ as a ratio whose numerator, for $i = 0, 1, 2$ and $a + b = i$, is a polynomial in

$$(29) \quad \int_{X \times Y_2} \frac{\partial^i [\rho(x, y_1, y_2) u(x, y_1, y_2, a_1, s_2^j(y_2))]}{\partial y_{1,q}^b \partial a_{1,r}^a} dx dy_2$$

and

$$(30) \quad \int_{X \times Y_2} \frac{\partial^i \rho(x, y_1, y_2)}{\partial y_{1,q}^b \partial a_{1,r}^a} dx dy_2,$$

whereas its denominator is $(\int_{X \times Y_2} \rho(x, y_1, y_2) dx dy_2)^3 \geq \delta$, where δ is a positive constant (hence independent of y_1), whose existence and independence of y_1 are guaranteed by $\rho(x, y_1, y_2) > 0$ and the continuity of $\rho(x, y_1, y_2)$ on the compact set $X \times Y_1 \times Y_2$.

Note that the change of order between expectation and up-to-second-order partial derivatives is justified by the fact that $\rho(x, y_1, y_2)$ and $u(x, y_1, y_2, a_1, a_2)$ are of class \mathcal{C}^m on compact sets with $m \geq 2$. Then, an upper bound on $|\frac{\partial^2 M_1^j}{\partial y_{1,q} \partial a_{1,r}}|$ can be expressed in terms of the quantities

$$(31) \quad \sup_{y_1 \in Y_1} \int_{X \times Y_2} \sup_{a_2 \in A_2} \left| \frac{\partial^i [\rho(x, y_1, y_2) u(x, y_1, y_2, a_1, a_2)]}{\partial y_{1,q}^b \partial a_{1,r}^a} \right| dx dy_2$$

and

$$(32) \quad \sup_{y_1 \in Y_1} \int_{X \times Y_2} \sup_{a_2 \in A_2} \left| \frac{\partial^i \rho(x, y_1, y_2)}{\partial y_{1,q}^b \partial a_{1,r}^a} \right| dx dy_2$$

related to (29) and (30), respectively, where measurability of the integrands follows by [47, Property (c), p. 38]. This bound does not depend on y_1 . Moreover, it does not depend on the particular choice of $s_2^j(y_2)$, and therefore it is also independent of j .

Summing up, for every $h = 1, \dots, l_1$ we have on $|\frac{d \hat{s}_1^j(y_1(t))}{dt}|$ an upper bound independent of y_1 and j , where $\hat{y}_1(t) \triangleq y_1' + t(y_1'' - y_1')$. Hence, \hat{s}_1^j is Lipschitz with a constant independent of j .

Step 1.b. As M_1^j is of class \mathcal{C}^m , by taking higher-order partial derivatives of both sides of (26) we conclude that $\hat{s}_1^j(y_1)$ is locally of class \mathcal{C}^{m-1} . As this holds for every $y_1 \in Y_1$, it follows that $\hat{s}_1^j(y_1)$ is of class \mathcal{C}^{m-1} on the whole Y_1 . Since M_1^j has upper bounds on the sizes of its partial derivatives up to the order m that are independent of y_1, a_1 , and j , then for every $h = 1, \dots, l_1$ and every multi-index (i_1, \dots, i_{d_1}) such that $i_1 + \dots + i_{d_1} = m - 1$, for every $y_1 \in Y_1$ there exists on $|\frac{\partial^{m-1} \hat{s}_{1,h}^j}{\partial y_{1,1}^{i_1} \dots \partial y_{1,d_1}^{i_{d_1}}}|$ a finite upper bound that is independent of y_1 and j .

Step 2. By Step 1.b, for every $h = 1, \dots, l_1$ and every multi-index (i_1, \dots, i_{d_1}) such that $i_1 + \dots + i_{d_1} = m - 2$, the elements of the sequence $\{\frac{\partial^{m-2} \hat{s}_{1,h}^j}{\partial y_{1,1}^{i_1} \dots \partial y_{1,d_1}^{i_{d_1}}}\}$ of functions are equibounded and have the same upper bound on their Lipschitz constants, so they are equicontinuous on the compact set Y_1 . Hence, by the Ascoli–Arzelà theorem there exists a subsequence of $\{\frac{\partial^{m-2} \hat{s}_{1,h}^j}{\partial y_{1,1}^{i_1} \dots \partial y_{1,d_1}^{i_{d_1}}}\}$ that converges uniformly to a function defined on Y_1 . Since this function is the pointwise limit of a sequences of equi-Lipschitz functions, it is Lipschitz with the same bound on its Lipschitz constant.

Step 3. By integrating $m - 2$ times, we conclude that there exists a subsequence of $\{\hat{s}_1^j\}$ that converges uniformly to a strategy $s_1^\circ \in \mathcal{C}^{m-2}(Y_1, A_1)$ with Lipschitz $(m - 2)$ -order partial derivatives. Similarly, we can prove that there exists a subsequence of $\{\hat{s}_2^j\}$ that converges uniformly to $s_2^\circ \in \mathcal{C}^{m-2}(Y_2, A_2)$ with partial derivatives that are Lipschitz up to the order $m - 2$.

Finally, by the continuity of the functional $v(s_1, s_2)$ on $\mathcal{C}(Y_1, A_1) \times \mathcal{C}(Y_2, A_2)$ with the respective sup-norms, we obtain $v(s_1^\circ, s_2^\circ) = \lim_{j \rightarrow \infty} v(\hat{s}_1^j, \hat{s}_2^j) = \sup_{s_1, s_2} v(s_1, s_2)$. \square

Proof of Theorem 3.3. Inspection of the proof of Lemma 3.1 shows that there exists a (possibly nonlinear) operator $\mathcal{T} : \mathcal{C}(Y_1, A_1) \times \mathcal{C}(Y_2, A_2) \rightarrow \mathcal{C}(Y_1, A_1) \times \mathcal{C}(Y_2, A_2)$ such that

$$\begin{aligned}\mathcal{T}_1(s_1, s_2) &= \operatorname{argmax}_{s_1 \in \mathcal{C}(Y_1, A_1)} v(s_1, s_2), \\ \mathcal{T}_2(s_1, s_2) &= \operatorname{argmax}_{s_2 \in \mathcal{C}(Y_2, A_2)} v(\mathcal{T}_1(s_1, s_2), s_2).\end{aligned}$$

Moreover, it also shows that

$$\begin{aligned}\mathcal{T}_1(s_1, s_2)(y_1) &= \operatorname{argmax}_{a_1} \mathbb{E}_{x, y_2 | y_1} \{u(x, y_1, y_2, a_1, s_2(y_2))\} \quad \forall y_1 \in Y_1, \\ \mathcal{T}_2(s_1, s_2)(y_2) &= \operatorname{argmax}_{a_2} \mathbb{E}_{x, y_1 | y_2} \{u(x, y_1, y_2, \mathcal{T}_1(s_1, s_2)(y_1), a_2)\} \quad \forall y_2 \in Y_2.\end{aligned}$$

Suppose by contradiction that there exist another optimal pair $(s_1^{\circ'}, s_2^{\circ'}) \in \mathcal{C}(Y_1, A_1) \times \mathcal{C}(Y_2, A_2)$ of strategies. Then $(s_1^{\circ'}, s_2^{\circ'}) = \mathcal{T}(s_1^{\circ'}, s_2^{\circ'})$ is a necessary condition for its optimality. (Otherwise there would exist a strictly better pair of strategies by an application of the theorem on interchange of maximization and integration [46, Theorem 14.60].) Moreover, by the proof of Lemma 3.1 and the compactness of Y_1 and Y_2 , for every $(s_1, s_2) \in \mathcal{C}(Y_1, A_1) \times \mathcal{C}(Y_2, A_2)$ the strategies $\mathcal{T}_1(s_1, s_2)$ and $\mathcal{T}_2(s_1, s_2)$ belong to the interiors (with respect to the associated sup-norms) of $\mathcal{C}(Y_1, A_1)$ and $\mathcal{C}(Y_2, A_2)$ as subsets of $\mathcal{C}(Y_1, \mathbb{R}^{l_1})$ and $\mathcal{C}(Y_2, \mathbb{R}^{l_2})$, respectively, so Problem TO is reduced to an unconstrained infinite-dimensional optimization problem on $\mathcal{C}(Y_2, \mathbb{R}^{l_2}) \times \mathcal{C}(Y_2, \mathbb{R}^{l_2})$.

Now, we reduce Problem TO to an unconstrained infinite-dimensional game-theory problem on a Hilbert space, to which one can apply the techniques developed in [31] to investigate the stability of Nash equilibria. This can be done since every pair of optimal strategies for Problem TO represents a Nash equilibrium for a two-player game, for which the individual utilities are the same and equal to $v(s_1, s_2)$. To this end, we first endow $\mathcal{C}(Y_1, \mathbb{R}^{l_1})$ and $\mathcal{C}(Y_2, \mathbb{R}^{l_2})$ with the norms $\sqrt{\mathbb{E}_{y_1} \{\|s_1(y_1)\|^2\}}$ and $\sqrt{\mathbb{E}_{y_2} \{\|s_2(y_2)\|^2\}}$, respectively; then we extend by density such spaces to the Lebesgue spaces $\mathcal{L}_2(Y_1, \rho_{y_1}, \mathbb{R}^{l_1})$ and $\mathcal{L}_2(Y_2, \rho_{y_2}, \mathbb{R}^{l_2})$, respectively, where ρ_{y_1} and ρ_{y_2} are the marginal densities of y_1 and y_2 . Note that thanks to the choice of a team utility function that is quadratic with respect to a_1 and a_2 , by removing the constraints $a_1 \in A_1$ and $a_2 \in A_2$ the definition of the functional $v(s_1, s_2)$ can be extended on the space $\mathcal{L}_2(Y_1, \rho_{y_1}, \mathbb{R}^{l_1}) \times \mathcal{L}_2(Y_2, \rho_{y_2}, \mathbb{R}^{l_2})$. Moreover, definition (33) of the operator \mathcal{T} can be extended on $\mathcal{L}_2(Y_1, \rho_{y_1}, \mathbb{R}^{l_1}) \times \mathcal{L}_2(Y_2, \rho_{y_2}, \mathbb{R}^{l_2})$ by setting

$$(33) \quad \begin{aligned}\mathcal{T}_1(s_1, s_2) &\triangleq \operatorname{argmax}_{s_1 \in \mathcal{L}_2(Y_1, \rho_{y_1}, \mathbb{R}^{l_1})} v(s_1, s_2), \\ \mathcal{T}_2(s_1, s_2) &\triangleq \operatorname{argmax}_{s_2 \in \mathcal{L}_2(Y_2, \rho_{y_2}, \mathbb{R}^{l_2})} v(\mathcal{T}_1(s_1, s_2), s_2).\end{aligned}$$

Indeed, nonemptiness of the sets argmax in (33) follows by the direct method of the calculus of variations [12] by standard coercivity and upper-semicontinuity arguments. The fact that they are singletons follows by strict concavity arguments.

Proceeding as in [23, Chapter XVII, section 3], we get the following expressions for the Fréchet differentials of the integral functional v up to the second order (we

denote by h_i and h'_i the independent increments of s_i , $i = 1, 2$):

$$\begin{aligned} D_{s_1} v(s_1, s_2)\{h_1\} &= \mathbb{E}_{x, y_1, y_2} \{h_1(y_1)^T \nabla_4 u(x, y_1, y_2, s_1(y_1), s_2(y_2))\} \\ &\triangleq \mathbb{E}_{y_1} \{h_1(y_1)^T (\nabla_{s_1} v(s_1, s_2))(y_1)\}, \end{aligned}$$

$$\begin{aligned} D_{s_2} v(s_1, s_2)\{h_2\} &= \mathbb{E}_{x, y_1, y_2} \{h_2(y_2)^T \nabla_5 u(x, y_1, y_2, s_1(y_1), s_2(y_2))\} \\ &\triangleq \mathbb{E}_{y_2} \{h_2(y_2)^T (\nabla_{s_2} v(s_1, s_2))(y_2)\}, \end{aligned}$$

$$\begin{aligned} D_{s_1, s_1}^2 v(s_1, s_2)\{h_1, h'_1\} &= \mathbb{E}_{x, y_1, y_2} \{h_1(y_1)^T \nabla_{4,4}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2)) h'_1(y_1)\} \\ &\triangleq \mathbb{E}_{y_1} \{h_1(y_1)^T (\nabla_{s_1, s_1}^2 v(s_1, s_2) h'_1)(y_1)\}, \end{aligned}$$

$$(34) \quad \begin{aligned} D_{s_1, s_2}^2 v(s_1, s_2)\{h_1, h_2\} &= \mathbb{E}_{x, y_1, y_2} \{h_1(y_1)^T \nabla_{4,5}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2)) h_2(y_2)\} \\ &\triangleq \mathbb{E}_{y_1} \{h_1(y_1)^T (\nabla_{s_1, s_2}^2 v(s_1, s_2) h_2)(y_1)\}, \end{aligned}$$

$$(35) \quad \begin{aligned} D_{s_2, s_1}^2 v(s_1, s_2)\{h_2, h_1\} &= \mathbb{E}_{x, y_1, y_2} \{h_2(y_2)^T \nabla_{5,4}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2)) h_1(y_1)\} \\ &\triangleq \mathbb{E}_{y_2} \{h_2(y_2)^T (\nabla_{s_2, s_1}^2 v(s_1, s_2) h_1)(y_2)\}, \end{aligned}$$

$$\begin{aligned} D_{s_2, s_2}^2 v(s_1, s_2)\{h_2, h'_2\} &= \mathbb{E}_{x, y_1, y_2} \{h_2(y_2)^T \nabla_{5,5}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2)) h'_2(y_2)\} \\ &\triangleq \mathbb{E}_{y_2} \{h_2(y_2)^T (\nabla_{s_2, s_2}^2 v(s_1, s_2) h'_2)(y_2)\}, \end{aligned}$$

where, for every $j, k = 1, 2$, the symbols D_{s_j} and D_{s_j, s_k} denote Frechét derivatives, $\nabla_{s_j} v(s_1, s_2)$ is a function in $\mathcal{L}_2(Y_j, \rho_j, \mathbb{R}^{l_j})$, and $\nabla_{s_j, s_k} v(s_1, s_2)$ is a bounded linear operator from $\mathcal{L}_2(Y_k, \rho_k, \mathbb{R}^{l_k})$ to $\mathcal{L}_2(Y_j, \rho_j, \mathbb{R}^{l_j})$. In particular, it follows by Assumption A2 that

$$\mathbb{E}_{x, y_1, y_2} \{h_1(y_1)^T \nabla_{4,4}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2)) h_1(y_1)\} \leq -2\tau \mathbb{E}_{y_1} \{\|h_1(y_1)\|^2\}$$

and

$$\mathbb{E}_{x, y_1, y_2} \{h_2(y_2)^T \nabla_{5,5}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2)) h_2(y_2)\} \leq -2\tau \mathbb{E}_{y_2} \{\|h_2(y_2)\|^2\}.$$

Then for every $j = 1, 2$, by the expression of such an operator $\nabla_{s_j, s_j} v(s_1, s_2)$ (or by an application of the Lax–Milgram theorem [2, p. 69]) it follows that $(\nabla_{s_j, s_j}^2 v(s_1, s_2))^{-1}$ exists, it is a bounded operator, and $\|(\nabla_{s_j, s_j}^2 v(s_1, s_2))^{-1}\| \leq \frac{1}{2\tau}$. By exploiting (34) and (35), as the second-order terms $\nabla_{4,5}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2))$ and $\nabla_{5,4}^2 u(x, y_1, y_2, s_1(y_1), s_2(y_2))$ do not depend on $s_1(y_1)$ and $s_2(y_2)$ we get

$$(36) \quad \begin{aligned} \|\nabla_{s_1, s_2}^2 v(s_1, s_2)\| &= \|\nabla_{s_2, s_1}^2 v(s_1, s_2)\| \leq \beta_{1,2} \\ &\triangleq \sqrt{d_1} \sqrt{d_2} \max_{(x, y_1, y_2) \in X \times Y_1 \times Y_2} \max_{q=1, \dots, l_1, r=1, \dots, l_2} \left| \frac{\partial^2}{\partial a_{1,q} \partial a_{2,r}} u(x, y_1, y_2, a_1, a_2) \right|. \end{aligned}$$

Then, by applying [31, Theorem 1, formula (1)], it follows that \mathcal{T} is a contraction operator on $\mathcal{L}_2(Y_1, \rho_1, \mathbb{R}^{l_1}) \times \mathcal{L}_2(Y_2, \rho_2, \mathbb{R}^{l_2})$ with a contraction constant bounded from above by $\frac{\beta_{1,2}^2}{4\tau^2} < 1$. Hence \mathcal{T} has a unique fixed point $(s_1^{\circ'}, s_2^{\circ'})$, which necessarily

coincides with (s_1°, s_2°) . (It is also a fixed point of \mathcal{T} and belongs to $\mathcal{C}^{m-2}(Y_1, A_1) \times \mathcal{C}^{m-2}(Y_2, A_2)$.) \square

Proof of Theorem 3.4. Suppose by contradiction that there exists another optimal n -tuple $(s_1', \dots, s_n') \in \mathcal{C}(Y_1, A_1) \times \dots \times \mathcal{C}(Y_n, A_n)$ of strategies. For $\lambda \in (0, 1)$, let $(s_1', \dots, s_n') \triangleq \lambda(s_1^\circ, \dots, s_n^\circ) + (1 - \lambda)(s_1', \dots, s_n')$. (Note that this is an admissible n -tuple of strategies, as $s_i' : Y_i \rightarrow A_i$.) By the strict concavity of the team utility function u with respect to (a_1, \dots, a_n) and the fact that the n -tuples $(s_1'(y_1), \dots, s_n'(y_n))$ and $(s_1^\circ(y_1), \dots, s_n^\circ(y_n))$ differ on a set of (strictly) positive measure, we have

$$\begin{aligned} & \mathbb{E}_{x, y_1, \dots, y_n} \{u(x, \{y_i\}_{i=1}^n, \{s_i'(y_i)\}_{i=1}^n)\} \\ & > \lambda \mathbb{E}_{x, y_1, \dots, y_n} \{u(x, \{y_i\}_{i=1}^n, \{s_i^\circ(y_i)\}_{i=1}^n)\} \\ & \quad + (1 - \lambda) \mathbb{E}_{x, y_1, \dots, y_n} \{u(x, \{y_i\}_{i=1}^n, \{s_i'(y_i)\}_{i=1}^n)\} \\ & = \sup \left\{ v(s_1, \dots, s_n) \mid s_i \in \mathcal{C}(Y_i, A_i), i = 1, \dots, n \right\}, \end{aligned}$$

which contradicts the optimality of $(s_1^\circ, \dots, s_n^\circ)$. Thus, $(s_1^\circ, \dots, s_n^\circ)$ is the unique optimal n -tuple of strategies in $\mathcal{C}(Y_1, A_1) \times \dots \times \mathcal{C}(Y_n, A_n)$ (and also in $\mathcal{C}^{m-2}(Y_1, A_1) \times \dots \times \mathcal{C}^{m-2}(Y_n, A_n)$). \square

Proof of Theorem 4.1. Let us consider $n = 2$ DMs. The general case can be proved similarly. For an integrable real-valued random variable z and a measurable concave function f , the Jensen inequality [47, Theorem 3.3, p. 62] gives

$$(37) \quad f(\mathbb{E}_z\{z\}) \geq \mathbb{E}_z\{f(z)\}.$$

By the existence of an optimal solution, the Lipschitz continuity of the team utility function u , and (37) we get

$$\begin{aligned} v(s_1^\circ, s_2^\circ) - v(s_1, s_2) &= \mathbb{E}_{x, y_1, y_2} \{u(x, y_1, y_2, s_1^\circ(y_1), s_2^\circ(y_2)) - u(x, y_1, y_2, s_1(y_1), s_2(y_2))\} \\ &\leq L \cdot \mathbb{E}_{x, y_1, y_2} \left\{ \sqrt{\|s_1^\circ(y_1) - s_1(y_1)\|^2 + \|s_2^\circ(y_2) - s_2(y_2)\|^2} \right\} \\ &\leq L \cdot \mathbb{E}_{x, y_1, y_2} \left\{ \sqrt{\|s_1^\circ(y_1) - s_1(y_1)\|^2} + \sqrt{\|s_2^\circ(y_2) - s_2(y_2)\|^2} \right\} \\ &\leq L \cdot \left([\mathbb{E}_{y_1} \{\|s_1^\circ(y_1) - s_1(y_1)\|^2\}]^{1/2} + [\mathbb{E}_{y_2} \{\|s_2^\circ(y_2) - s_2(y_2)\|^2\}]^{1/2} \right). \quad \square \end{aligned}$$

Proof of Theorem 4.2. Proceeding as in the proof of Theorem 3.3, Problem TO is reduced to an unconstrained infinite-dimensional optimization problem on $\mathcal{C}(Y_1, A_1) \times \dots \times \mathcal{C}(Y_n, A_n)$, where each space $\mathcal{C}(Y_i, \mathbb{R}^{l_i}) \supset \mathcal{C}(Y_i, A_i)$ is endowed with the respective sup-norm, $i = 1, \dots, n$. By the necessary first-order optimality condition, the first-order Fréchet derivatives of the integral functional v computed at $(s_1^\circ, \dots, s_n^\circ)$ are equal to 0. So, by exploiting the concavity of the functional v (which follows by the assumptions on u) and the Taylor theorem with Lagrange's remainder, we get

$$(38) \quad v(s_1, \dots, s_n) - v(s_1^\circ, \dots, s_n^\circ) = \frac{1}{2} D^2 v(\chi) \{h, h\},$$

where $h = (s_1 - s_1^\circ, \dots, s_n - s_n^\circ)$, (s_1, \dots, s_n) is any other n -tuple of admissible continuous strategies, and $\chi \triangleq (\chi_1, \dots, \chi_n)$ belongs to the segment between (s_1, \dots, s_n) and the optimal solution $(s_1^\circ, \dots, s_n^\circ)$.

As by assumption $m \geq 3$, proceeding as in [23, Chapter XVII, section 3], the second-order Fréchet differential $D^2v(\chi)\{h, h\}$ has the expression

$$\mathbb{E}_{x, y_1, \dots, y_n} \{ (h_1(y_1), \dots, h_n(y_n))^T \nabla_{(n+2, \dots, 2n+1), (n+2, \dots, 2n+1)}^2 u(x, y_1, \dots, y_n, \chi_1(y_1), \dots, \chi_n(y_n))(h_1(y_1), \dots, h_n(y_n)) \},$$

which by (3) is bounded from above by

$$-2\bar{\tau} \mathbb{E}_{x, y_1, \dots, y_n} \{ \| (h_1(y_1), \dots, h_n(y_n)) \|^2 \} = -2\bar{\tau} \sum_{i=1}^n \mathbb{E}_{y_i} \{ \| s_i^\circ(y_i) - s_i(y_i) \|^2 \}.$$

The statement follows by this last bound and (38). \square

For $\Omega \subseteq \mathbb{R}^d$ and $1 \leq p < \infty$, we denote the corresponding Lebesgue space and norm by $\mathcal{L}_p(\Omega)$ and $\| \cdot \|_{\mathcal{L}_p(\Omega)}$, respectively. For Ω open, a positive integer m and $1 \leq p < \infty$, we denote by $\mathcal{W}^{m,p}(\Omega)$ the Sobolev space of functions whose weak partial derivatives up to the order m are in $\mathcal{L}_p(\Omega)$. By $\mathcal{C}^\infty(\Omega)$ we denote the space of functions on Ω that are continuous together with their partial derivatives up to every order; $\mathcal{C}_0(\Omega)$ and $\mathcal{C}_0^\infty(\Omega)$ are the spaces of functions in $\mathcal{C}(\Omega)$ and $\mathcal{C}^\infty(\Omega)$, respectively, with compact supports in Ω [1, p. 2]. We denote by $\mathcal{W}_0^{m,p}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in the Sobolev space $\mathcal{W}^{m,p}(\Omega)$ (see [1, p. 59]). For $\alpha > 0$, the Bessel potential of order α on \mathbb{R}^d , denoted by $\mathcal{B}_\alpha(x)$, is the inverse Fourier transform of the function $\hat{\mathcal{B}}_\alpha(\omega) \triangleq (2\pi)^{-\frac{d}{2}}(1 + \|\omega\|^2)^{-\alpha/2}$. When $\alpha > d$, the function \mathcal{B}_α is continuous [51, p. 132]. For $\alpha > 0$ and $1 \leq p < \infty$, $\mathcal{B}_p^\alpha(\mathbb{R}^d)$ is the Bessel potential space, whose elements are functions u such that $u = f * \mathcal{B}_\alpha$, where $f \in \mathcal{L}_p(\mathbb{R}^d)$ [51, p. 134].

For completeness, we report the statement of [15, Corollary 5.2], as we shall exploit it in the proof of Theorem 4.3(ii).

THEOREM 7.1 (see [15, Corollary 5.2]). *Let d be a positive integer and $\alpha > d$. For every $f \in \mathcal{B}_1^\alpha(\mathbb{R}^d)$ and every positive integer k , there exist $t_1, \dots, t_k \in \mathbb{R}^d$, $b_1, \dots, b_k > 0$, $c_1, \dots, c_k \in \{-1, +1\}$, and an absolute positive constant C such that*

$$\sup_{x \in \mathbb{R}^d} \left| f(x) - \frac{K(\alpha, d)}{k} \sum_{i=1}^k c_i e^{-\frac{\|x-t_i\|^2}{b_i}} \right| \leq C K(\alpha, d) \sqrt{\frac{d+3}{k}},$$

where $K(\alpha, d) \triangleq 2^{-d/2} \frac{\Gamma(\alpha/2 - d/2)}{\Gamma(\alpha/2)} \|\lambda\|_{\mathcal{L}_1(\mathbb{R}^d)}$ and $\Gamma(z) \triangleq \int_0^\infty s^{z-1} e^{-s} ds$ is the Gamma function.

Proof of Theorem 4.3. (i), first case: $\tilde{s}_i^{(k)} \in S_i^{(k)}(\cos, d_i)$. By Theorem 3.2, there exist optimal strategies $s_i^\circ : Y_i \rightarrow A_i$ such that for $j = 1, \dots, l_i$, $s_{i,j}^\circ$ is of class \mathcal{C}^{m-2} and has partial derivatives that are Lipschitz up to the order $m-2$, so $s_{i,j}^\circ \in \mathcal{W}^{m-1, \infty}(\text{int}(Y_i))$.

Since Y_i is bounded, $s_{i,j}^\circ \in \mathcal{W}^{m-1, \infty}(\text{int}(Y_i))$ implies $s_{i,j}^\circ \in \mathcal{W}^{m-1, p}(\text{int}(Y_i))$ for every $1 \leq p < \infty$. As every Y_i is a bounded convex set, by the Sobolev extension theorem (see [51, Theorem 5, p. 181] and [51, Example 2, p. 189]), $s_{i,j}^\circ$ can then be extended to a function $s_{i,j}^{\circ, \text{ext}, p} \in \mathcal{W}^{m-1, p}(\mathbb{R}^{d_i})$.

Let $Y_i' \triangleq (y_{i,1}^l, y_{i,1}^u) \times \dots \times (y_{i,d_i}^l, y_{i,d_i}^u) \supset Y_i$ and consider a function $\psi_i \in \mathcal{C}_0^\infty(Y_i')$ such that $\psi_i(y_i) = 1$ for every $y_i \in Y_i$. By an application of [1, Theorem 3.22, p. 68], $s_{i,j}^{\circ, p} \triangleq s_{i,j}^{\circ, \text{ext}, p} \cdot \psi_i \in \mathcal{W}_0^{m-1, p}(Y_i')$ if $1 \leq p < \infty$. By the Sobolev embedding theorem [1, Theorem 4.12, p. 85], if $d_i < p < \infty$, then $\mathcal{W}_0^{m-1, p}(Y_i') \subset \mathcal{C}^{m-2}([y_{i,1}^l, y_{i,1}^u] \times \dots \times [y_{i,d_i}^l, y_{i,d_i}^u])$.

From now on, we assume $d_i < p < \infty$. Let $\{\hat{S}_{i,j,r_1,\dots,r_{d_i}}\}$ be the set of the coefficients of the Fourier series expansion of $s_{i,j}^{\circ,\text{ext},p}$ on Y'_i . Note that by construction, the periodic extension of $s_{i,j}^{\circ,p}$ on \mathbb{R}^{d_i} is of class $\mathcal{C}^{m-2}(\mathbb{R}^{d_i})$. We apply a particular case of an extension to multiple Fourier series [18, p. 647] of the Bernstein theorem in Fourier analysis, according to which the condition $m - 2 > \frac{d_i}{2}$ implies

$$K_{i,j} \triangleq \sum_{r_1,\dots,r_{d_i}=0}^{\infty} |\hat{S}_{i,j,r_1,\dots,r_{d_i}}| < \infty.$$

Thus, $s_{i,j}^{\circ,p}$ belongs to the closure (with respect to the $\mathcal{L}_2(Y'_i)$ -norm) of the convex hull of the set

$$G'_{i,j}(\cos, d_i) \triangleq \left\{ g_i : Y'_i \rightarrow \mathbb{R} \mid g_i(y_i) = b \prod_{r=1}^{d_i} \cos(\omega_{i,r} y_{i,r} + \theta_{i,r}), \right. \\ \left. |b| \leq K_{i,j}, \omega_{i,r} = \frac{2\pi h}{y_{i,r}^u - y_{i,r}^l}, h \in \mathbb{N}, \theta_{i,r} \in [0, 2\pi) \right\}.$$

Let \mathcal{H}_i be the Hilbert space of functions $f_i : Y_i \rightarrow \mathbb{R}$ such that $\mathbb{E}_{y_i}\{|f_i(y_i)|^2\} < \infty$, where the expected value is evaluated on Y_i with the marginal probability density function ρ_{y_i} induced by ρ . As the probability density function ρ is bounded, $s_{i,j}^{\circ}$ belongs to the closure (with respect to the norm of \mathcal{H}_i) of the convex hull of the set

$$G_{i,j}(\cos, d_i) \triangleq \left\{ g_i : Y_i \rightarrow \mathbb{R} \mid g_i(y_i) = b \prod_{r=1}^{d_i} \cos(\omega_{i,r} y_{i,r} + \theta_{i,r}), \right. \\ \left. |b| \leq K_{i,j}, \omega_{i,r} = \frac{2\pi h}{y_{i,r}^u - y_{i,r}^l}, h \in \mathbb{N}, \theta_{i,r} \in [0, 2\pi) \right\}. \tag{39}$$

Moreover, for every $g_i \in G_{i,j}(\cos, d_i) \subset G_i(\cos, d_i)$ we have

$$\|g_i\|_{\mathcal{H}_i} = \sqrt{\mathbb{E}_{y_i}\{|g_i(y_i)|^2\}} \leq K_{i,j}.$$

Hence, by the Maurey–Jones–Barron theorem [5, Lemma 1, p. 934] (see also [21, 41]), for every positive integer $k \geq 1$ and every $C_{i,j} > K_{i,j}^2 - \|s_{i,j}^{\circ}\|_{\mathcal{H}_i}^2$ there exists a function $\tilde{s}_{i,j}^{(k)}$ in the convex hull of k elements of $G_{i,j}(\cos, d_i)$ such that

$$\|s_{i,j}^{\circ} - \tilde{s}_{i,j}^{(k)}\|_{\mathcal{H}_i}^2 = \mathbb{E}_{y_i} \left\{ \left| s_{i,j}^{\circ}(y_i) - \tilde{s}_{i,j}^{(k)}(y_i) \right|^2 \right\} \leq \frac{C_{i,j}}{k}. \tag{40}$$

We conclude the proof by taking projections on $A_{i,j}$ and applying (40) and Theorem 4.1.

(i), *second case*: $\tilde{s}_i^{(k)} \in S_i^{(k)}(\sigma, d_i)$. As in the proof of the first case, by choosing $p = 2$ the function $s_{i,j}^{\circ}$ can be extended to $s_{i,j}^{\circ,\text{ext},2} \in \mathcal{W}^{m-1,2}(\mathbb{R}^{d_i})$. Let $\hat{s}_{i,j}^{\circ,\text{ext},2}$ be its Fourier transform. By similar arguments as in [5, Example 15, p. 941] (based on the Cauchy–Schwarz inequality), the condition $m - 1 > d_i/2 + 1$ implies $\int_{\mathbb{R}^{d_i}} |\hat{s}_{i,j}^{\circ,\text{ext},2}(\omega)| \|\omega\|_2 d\omega < +\infty$. So one can apply [5, Theorem 1] to $s_{i,j}^{\circ}$ on Y_i , thus obtaining for sigmoidal computational units the same approximation error bound as in (40), in general with

different constants $C_{i,j}$. We conclude by taking projections onto $A_{i,j}$ and applying such a bound and Theorem 4.1.

(ii) As in the proof of item (i), by choosing $p = 1$ the function $s_{i,j}^{\circ}$ can be extended to $s_{i,j}^{\circ,\text{ext},1} \in \mathcal{W}^{m-1,1}(\mathbb{R}^{d_i})$. As $m - 1$ is even, we can exploit the inclusion of Sobolev spaces into Bessel potential spaces stated in [51, Remark 6.6 (b), p. 160]. Then, there exists $\lambda_{i,j} \in \mathcal{L}_1(\mathbb{R}^{d_i})$ such that $s_{i,j}^{\circ,\text{ext},1} = \mathcal{B}_{m-1} * \lambda_{i,j}$. Since $m - 1 > d_i$ we can apply [15, Corollary 5.2], reported above as Theorem 7.1. This, together with the fact that the \mathcal{L}_2 -norm of a function defined on the bounded domain Y_i can be bounded from above by a constant times its sup-norm, provides for the case of Gaussian computational units the same approximation error bound as in (40), in general with different constants $C_{i,j}$. As before, we conclude the proof by taking projections onto $A_{i,j}$ and applying such an error bound and Theorem 4.1. \square

Proof of Proposition 5.1. The statement follows by Theorems 3.2 and 4.3. \square

Proof of Proposition 5.2. The joint probability density function $\rho(\xi, \zeta, y_1, y_2)$ can be written as

$$\begin{aligned} \rho(\xi, \zeta, y_1, y_2) &= \rho(\xi, y_1)\rho(\zeta, y_2) = \rho(y_1|\xi)\rho(\xi)\rho(y_2|\zeta)\rho(\zeta) \\ &= \frac{(2\pi\xi^2)^{-1/2} e^{-(y_1-\xi)^2/(2\xi^2)}}{(2\pi\xi^2)^{-1/2} \int_2^{10} e^{-(y_1-\xi)^2/(2\xi^2)} dy_1} \cdot \frac{1}{8} \\ &\quad \cdot \frac{(2\pi\zeta^2)^{-1/2} e^{-(y_2-\zeta)^2/(2\zeta^2)}}{(2\pi\zeta^2)^{-1/2} \int_2^{10} e^{-(y_2-\zeta)^2/(2\zeta^2)} dy_2} \cdot \frac{1}{8}. \end{aligned}$$

This function is positive on $[\xi_{\min}, \xi_{\max}] \times [\zeta_{\min}, \zeta_{\max}] \times Y_1 \times Y_2$ and for every $m \geq 2$ enjoys the properties required in Assumption A1.

As the team utility function u has only linear and quadratic terms, Assumption A2 is satisfied by taking $\tau \triangleq \frac{1}{2} \min\{c_{11}, c_{22}\} = \frac{1}{2} > 0$.

Assumption A3 can be verified as follows (recall that $\text{int } A_1$ denotes the topological interior of A_1). For every $y_1 \in Y_1$ and every admissible strategy s_2 , the function

$$M_1(y_1, a_1) \triangleq \mathbb{E}_{x,y_2|y_1} \left\{ \xi a_1 + \zeta s_2(y_2) - \frac{1}{2} c_{11} a_1^2 - c_{12} a_1 s_2(y_2) - \frac{1}{2} c_{22} s_2(y_2)^2 \right\}$$

is strictly concave and differentiable, and $\text{argmax}_{a_1 \in A_1} M_1(y_1, a_1) \in \text{int } A_1$ iff for some $a_1^* \in \text{int } A_1$ one has

$$\frac{\partial}{\partial a_1} M_1(y_1, a_1) \Big|_{a_1=a_1^*} = \mathbb{E}_{\xi|y_1} \{ \xi \} - c_{11} a_1^* - c_{12} \mathbb{E}_{y_2|y_1} \{ s_2(y_2) \} = 0.$$

Note that $a_1^* \triangleq \frac{\mathbb{E}_{\xi|y_1} \{ \xi \} - c_{12} \mathbb{E}_{y_2|y_1} \{ s_2(y_2) \}}{c_{11}} \in \text{int } A_1$, as

$$2 - 0.15 \cdot 10 = 0.5 \leq \frac{\mathbb{E}_{\xi|y_1} \{ \xi \} - c_{12} \mathbb{E}_{y_2|y_1} \{ s_2(y_2) \}}{c_{11}} \leq 10 - 0.15 \cdot 2 = 9.7$$

and $[0.5, 9.7] \subset \text{int } A_1$. Similar arguments hold for every admissible strategy of the other DM, so we conclude that Assumption A3 is satisfied.

Finally, Assumption A4 holds since the sets A_1 and A_2 are closed and bounded nonempty intervals. \square

Proof of Proposition 5.3. The proof exploits the same arguments as those in the proof of Proposition 5.2. \square

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