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Supplementary Material for "Interacting Reinforced Stochastic Processes: Statistical Inference based on the Weighted Empirical Means"

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In this document we collect some technical results and computations necessary for the proof of (Aletti, Crimaldi and Ghiglietti, 2019, Lemma 5.1) and we give the entries of the matrices S_γ^{11}, S_γ^{12} and S_γ^{22} introduced in (Aletti, Crimaldi and Ghiglietti, 2019, Theorem 5.1). Therefore, the notation and the assumptions used here are the same as those used in that paper.

A. Technical details for the proof of (Aletti, Crimaldi and Ghiglietti, 2019, Lemma 5.1)

In all the sequel, given $(z_n)_n, (z'_n)_n$ two sequences of complex numbers, the notation $z_n = O(z'_n)$ means $|z_n| \leq C|z'_n|$ for a suitable constant $C > 0$ and n large enough. Moreover, if $z'_n \neq 0$, the notation $z_n \sim z z'_n$ with $z \in \mathbb{C} \setminus \{0\}$ means $\lim_n z_n/z'_n = z$ and, finally, the notation $z_n = o(z'_n)$ means $\lim_n z_n/z'_n = 0$.

Given $1/2 < \delta \leq 1$, $x = a_x + i\, b_x \in \mathbb{C}$ with $a_x > 0$ and an integer $m_0 \geq 2$ such that $a_x m^{-\delta} < 1$ for all $m \geq m_0$, let us set

$$p_n^\delta(x) := \prod_{m=m_0}^{n} \left(1 - \frac{x}{m^\delta}\right) \quad \text{for } n \geq m_0\,. \tag{A.1}$$

A.1. Some technical results

We first recall the following result, which has been proved in Aletti, Crimaldi and Ghiglietti (2017).

Lemma A.1. *(Aletti, Crimaldi and Ghiglietti, 2017, Lemma A.4) We have*

$$|p_n^\delta(x)| \;=\; \begin{cases} O\left(\exp\left(-a_x \frac{n^{1-\delta}}{1-\delta}\right)\right) & \textit{for } 1/2 < \delta < 1 \\ O\left(n^{-a_x}\right) & \textit{for } \delta = 1 \end{cases} \tag{A.2}$$

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and

$$|p_n^\delta(x)^{-1}| \;=\; \begin{cases} O\left(\exp\left(a_x \frac{n^{1-\delta}}{1-\delta}\right)\right) & \text{for } 1/2<\delta<1 \\ O\left(n^{a_x}\right) & \text{for } \delta=1\,. \end{cases} \tag{A.3}$$

Therefore , if we set

$$F_{k+1,n}^\delta(x) := \frac{p_n^\delta(x)}{p_k^\delta(x)} \qquad \text{for } m_0\leq k\leq n\,, \tag{A.4}$$

we have

$$|F_{k+1,n}^\delta(x)| \;=\; \begin{cases} O\left(\exp\left(\frac{a_x}{1-\delta}(k^{1-\delta}-n^{1-\delta})\right)\right) & \text{for } 1/2<\delta<1 \\ O\left(\left(\frac{k}{n}\right)^{a_x}\right) & \text{for } \delta=1. \end{cases} \tag{A.5}$$

Now, we prove two other results.

Lemma A.2. *Given $\beta>1$ and $e>0$, we have*

$$\sum_{k=m_0}^{n}\frac{1}{k^\beta}|F_{k+1,n}^\delta(x)|^e \;=\; \begin{cases} O\left(n^{-(\beta-\delta)}\right) & \text{if } 1/2<\delta<1, \\ O\left(n^{-ea_x}\right) & \text{if } \delta=1 \text{ and } ea_x<\beta-1, \\ O\left(n^{-(\beta-1)}\ln(n)\right) & \text{if } \delta=1 \text{ and } ea_x=\beta-1, \\ O\left(n^{-(\beta-1)}\right) & \text{if } \delta=1 \text{ and } ea_x>\beta-1. \end{cases} \tag{A.6}$$

Proof. The desired relations immediately follows from (A.5) using the well-known relation

$$\sum_{k=1}^{n}\frac{1}{k^{1-a}} \;=\; \begin{cases} O(1) \quad \text{for } a<0, \\ \ln(n)+d+O(n^{-1})=\ln(n)+O(1) \quad \text{for } a=0, \\ a^{-1}\,n^a+O(1) \quad \text{for } 0<a\leq 1, \\ a^{-1}\,n^a+O(n^{a-1}) \quad \text{for } a>1, \end{cases} \tag{A.7}$$

where d is the Euler-Mascheroni constant, and the relation

$$\begin{aligned}\sum_{k=1}^{n}\frac{\exp(ak^b/b)}{k^\beta} &= O\left(\int_1^n\frac{\exp(at^b/b)}{t^\beta}\,dt\right) = O\left(\left[\frac{\exp(at^b/b)}{at^{b+\beta-1}}\right]_1^n + \frac{(b+\beta-1)}{a}\int_1^n\frac{\exp(at^b/b)}{t^{\beta+b}}\,dt\right) \\ &= O\left(\frac{\exp(an^b/b)}{n^{b+\beta-1}}\right) \qquad \text{for } a>0,\ b>0,\ \beta>1. \end{aligned} \tag{A.8}$$

Indeed, for the case $\delta=1$, it is enough to apply (A.7) with $a=ea_x-(\beta-1)$; while, for the case $1/2<\delta<1$, it is enough to apply (A.8) with $a=ea_x$, $b=1-\delta$ and β. □

The following lemma extends (Aletti, Crimaldi and Ghiglietti, 2017, Lemma A.5).

Lemma A.3. *Given $1/2<\delta_1\leq\delta_2\leq 1$, $\beta>\delta_1$ and $x_1,x_2\in\mathbb{C}$ with $\mathcal{R}e(x_1)>0$, $\mathcal{R}e(x_2)>0$, let $m_0\geq 2$ be an integer such that $\max\{\mathcal{R}e(x_1),\mathcal{R}e(x_2)\}m^{-\delta_1}<1$ for all $m\geq m_0$. Then we have*

$$\lim_n n^{\beta-\delta_1}\sum_{k=m_0}^{n}k^{-\beta}F_{k+1,n}^{\delta_1}(x_1)F_{k+1,n}^{\delta_2}(x_2) \;=\; \begin{cases} \frac{1}{x_1+x_2} & \text{if } 1/2<\delta_1=\delta_2<1, \\ \frac{1}{x_1+x_2-\beta+1} & \text{if } \delta_1=\delta_2=1 \text{ and } \mathcal{R}e(x_1+x_2)>\beta-1, \\ \frac{1}{x_1} & \text{if } 1/2<\delta_1<\delta_2\leq 1. \end{cases} \tag{A.9}$$

Proof. Let us start with observing that, in each considered case, relation (A.2) implies

$$\lim_n n^{\beta-\delta_1}|p_n^{\delta_1}(x_1)|\,|p_n^{\delta_2}(x_2)| = 0. \tag{A.10}$$

Indeed, in particular, when $\delta_1 = \delta_2 = 1$ we have the additional condition $\mathcal{R}e(x_1 + x_2) > \beta - 1$.

Now, fix $k \geq 2$ and let us set $\eta := \beta - \delta_1$ and $\ell_n^\delta(x) := 1/p_n^\delta(x)$ and define the following quantity

$$\begin{aligned} D_k \quad &:= \frac{1}{k^\eta}\ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2) - \frac{1}{(k-1)^\eta}\ell_{k-1}^{\delta_1}(x_1)\ell_{k-1}^{\delta_2}(x_2) \\ &= \left(\frac{1}{k^\eta} - \frac{1}{(k-1)^\eta}\right)\ell_{k-1}^{\delta_1}(x_1)\ell_{k-1}^{\delta_2}(x_2) + \frac{1}{k^\eta}\left(\ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2) - \ell_{k-1}^{\delta_1}(x_1)\ell_{k-1}^{\delta_2}(x_2)\right) \\ &= \ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)\left[\left(\frac{1}{k^\eta} - \frac{1}{(k-1)^\eta}\right)\frac{\ell_{k-1}^{\delta_1}(x_1)\ell_{k-1}^{\delta_2}(x_2)}{\ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)} + \frac{1}{k^\eta}\left(1 - \frac{\ell_{k-1}^{\delta_1}(x_1)\ell_{k-1}^{\delta_2}(x_2)}{\ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)}\right)\right]. \end{aligned}$$

Then, we observe the following:

$$\left(\frac{1}{k^\eta} - \frac{1}{(k-1)^\eta}\right) = -\frac{\eta}{k^{1+\eta}} + O\left(\frac{1}{k^{2+\eta}}\right) \quad \text{for } k \to +\infty \tag{A.11}$$

and

$$\frac{\ell_{k-1}^{\delta_1}(x_1)\ell_{k-1}^{\delta_2}(x_2)}{\ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)} = \left(1 - \frac{x_1}{k^{\delta_1}}\right)\left(1 - \frac{x_2}{k^{\delta_2}}\right) = 1 + \frac{x_1x_2}{k^{(\delta_1+\delta_2)}} - \frac{x_1}{k^{\delta_1}} - \frac{x_2}{k^{\delta_2}}. \tag{A.12}$$

Now, by using (A.11) and (A.12) in the above expression of D_k, we have for $k \to +\infty$

$$\begin{aligned} D_k &= \ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)\left[\left(-\frac{\eta}{k^{\eta+1}} + O(1/k^{\eta+2})\right)\left(1 + \frac{x_1x_2}{k^{(\delta_1+\delta_2)}} - \frac{x_1}{k^{\delta_1}} - \frac{x_2}{k^{\delta_2}}\right) + \frac{1}{k^\eta}\left(-\frac{x_1x_2}{k^{(\delta_1+\delta_2)}} + \frac{x_1}{k^{\delta_1}} + \frac{x_2}{k^{\delta_2}}\right)\right] \\ &= \begin{cases} \ell_k^{\delta}(x_1)\ell_k^{\delta}(x_2)\left[\frac{x_1+x_2}{k^{\eta+\delta}} - \frac{\eta}{k^{\eta+1}} + o(1/k^{\eta+\delta})\right] & \text{if } \delta_1 = \delta_2 = \delta, \\ \ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)\left[\frac{x_1}{k^{\eta+\delta_1}} + o(1/k^{\eta+\delta_1})\right] & \text{if } \delta_1 < \delta_2 \end{cases} \\ &= \begin{cases} \ell_k^{\delta}(x_1)\ell_k^{\delta}(x_2)\frac{x_1+x_2}{k^{\eta+\delta}} + o(\ell_k^{\delta}(x_1)\ell_k^{\delta}(x_2)/k^{\eta+\delta}) & \text{if } \delta_1 = \delta_2 = \delta < 1, \\ \ell_k^{1}(x_1)\ell_k^{1}(x_2)\frac{x_1+x_2-\eta}{k^{\eta+1}} + o(\ell_k^{1}(x_1)\ell_k^{1}(x_2)/k^{\eta+1}) & \text{if } \delta_1 = \delta_2 = 1 \text{ and } \mathcal{R}e(x_1+x_2) > \eta, \\ \ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)\frac{x_1}{k^{\eta+\delta_1}} + o(\ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2)/k^{\eta+\delta_1}) & \text{if } \delta_1 < \delta_2. \end{cases} \end{aligned}$$

that is

$$D_k \sim \begin{cases} \frac{x_1+x_2}{k^{\eta+\delta}}\,\ell_k^{\delta}(x_1)\ell_k^{\delta}(x_2) & \text{if } 1/2 < \delta_1 = \delta_2 = \delta < 1, \\ \frac{x_1+x_2-\eta}{k^{\eta+1}}\,\ell_k^{1}(x_1)\ell_k^{1}(x_2) & \text{if } \delta_1 = \delta_2 = 1 \text{ and } \mathcal{R}e(x_1+x_2) > \eta, \\ \frac{x_1}{k^{\eta+\delta_1}}\,\ell_k^{\delta_1}(x_1)\ell_k^{\delta_2}(x_2) & \text{if } 1/2 < \delta_1 < \delta_2 \leq 1. \end{cases} \tag{A.13}$$

Now, following the same arguments used in the proof of (Aletti, Crimaldi and Ghiglietti, 2017, Lemma A.5), in order to conclude, we apply (Aletti, Crimaldi and Ghiglietti, 2017, Corollary A.2) with

$$z_n = D_n, \quad v_n = n^\eta\, p_n^{\delta_1}(x_1)p_n^{\delta_2}(x_2), \quad w_n = \frac{\ell_{n,1}\ell_{n,2}}{n^{\eta+\delta_1}D_n}, \quad w = \begin{cases} \frac{1}{x_1+x_2} & \text{if } 1/2 < \delta_1 = \delta_2 = \delta < 1 \\ \frac{1}{x_1+x_2-\eta} & \text{if } \delta_1 = \delta_2 = 1 \text{ and } \mathcal{R}e(x_1+x_2) > \eta \\ \frac{1}{x_1} & \text{if } 1/2 < \delta_1 < \delta_2 \leq 1. \end{cases}$$

Indeed, $\lim_n v_n = 0$ by (A.10), $\lim_n w_n = w \neq 0$ by (A.13),

$$\begin{aligned} \lim_n v_n \sum_{k=m_0}^{n} z_k &= \lim_n n^\eta p_n^{\delta_1}(x_1)p_n^{\delta_2}(x_2)\sum_{k=m_0}^{n} D_k \\ &= \lim_n n^\eta p_n^{\delta_1}(x_1)p_2^{\delta_2}(x_2)\left(\frac{\ell_n^{\delta_1}(x_1)\ell_n^{\delta_2}(x_2)}{n^\eta} - \frac{\ell_{m_0-1}^{\delta_1}(x_1)\ell_{m_0-1}^{\delta_2}(x_2)}{(m_0-1)^\eta}\right) = 1 \end{aligned}$$

by (A.10) and $z'_n = z_n w_n = r_n^2\ell_{n,1}\ell_{n,2}$. □

A.2. Analytic expression of $A^j_{k+1,n-1}$ with $j \geq 2$

Let us recall the definition of the following quantities for $j \geq 2$:

$$A^j_{k+1,n-1} = \prod_{m=k+1}^{n-1} (I - D_{Q,j,m}), \qquad \text{where} \qquad D_{Q,j,n} = \begin{pmatrix} \widehat{r}_{n-1}(1-\lambda_j) & 0 \\ -\lambda_j h_n(\lambda_j) & \widehat{q}_{n,n} \end{pmatrix}$$

with the function h_n defined in (Aletti, Crimaldi and Ghiglietti, 2019, Equation (5.10)). The aim of this section is to compute the product above and so finding the following useful expression of $A^j_{k+1,n-1}$:

$$A^j_{k+1,n-1} = \begin{pmatrix} F^{\gamma}_{k+1,n-1}(c\alpha_j) & 0 \\ \lambda_j G_{k+1,n-1}(c\alpha_j, q) & F^{\nu}_{k+1,n-1}(q) \end{pmatrix},$$

where

$$G_{k+1,n-1}(c\alpha_j, q) \;=\; \sum_{l=k+1}^{n-1} F^{\gamma}_{l+1,n-1}(c\alpha_j) h_l(\lambda_j) F^{\nu}_{k+1,l-1}(q).$$

It is straightforward to see that $[A^j_{k+1,n-1}]_{21} = 0$,

$$[A^j_{k+1,n-1}]_{11} \;=\; \prod_{m=k+1}^{n-1} (1 - \widehat{r}_{m-1}(1-\lambda_j)) = F^{\gamma}_{k+1,n-1}(c\alpha_j),$$

$$[A^j_{k+1,n-1}]_{22} \;=\; \prod_{m=k+1}^{n-1} (1 - \widehat{q}_{m,m}) = F^{\nu}_{k+1,n-1}(q),$$

while it is not immediate to determine $[A^j_{k+1,n-1}]_{12}$. To this end, let us set $x_{n-1} := [A^j_{k+1,n-1}]_{21}$ and observe that, since $A^j_{k+1,n-1} = A^j_{k+1,n-2}(I - D_{Q,j,n-1})$ and $x_{k+1} = \lambda_j h_{k+1}(\lambda_j)$, we have that

$$\begin{aligned}
x_{n-1} &= x_{n-2}(1 - \widehat{r}_{n-1}(1-\lambda_j)) + [A^j_{k+1,n-2}]_{22} \lambda_j h_{n-1}(\lambda_j) \\
&= x_{n-2} F^{\gamma}_{n-1,n-1}(c\alpha_j) + F^{\nu}_{k+1,n-2}(q) \lambda_j h_{n-1}(\lambda_j) \\
&= x_{n-3} F^{\gamma}_{n-2,n-1}(c\alpha_j) + F^{\nu}_{k+1,n-3}(q) \lambda_j h_{n-2}(\lambda_j) F^{\gamma}_{n-1,n-1}(c\alpha_j) + F^{\nu}_{k+1,n-2}(q) \lambda_j h_{n-1}(\lambda_j) \\
&= \;\ldots \\
&= x_{k+1} F^{\gamma}_{k+2,n-1}(c\alpha_j) + \sum_{l=k+2}^{n-2} F^{\nu}_{k+1,l-1}(q) \lambda_j h_l(\lambda_j) F^{\gamma}_{l+1,n-1}(c\alpha_j) + F^{\nu}_{k+1,n-2}(q) \lambda_j h_{n-1}(\lambda_j) \\
&= \sum_{l=k+1}^{n-1} F^{\nu}_{k+1,l-1}(q) \lambda_j h_l(\lambda_j) F^{\gamma}_{l+1,n-1}(c\alpha_j) \\
&= \lambda_j G_{k+1,n-1}(c\alpha_j, q).
\end{aligned}$$

A.3. Asymptotic behavior of $G_{k+1,n-1}(x,q)$

Let us recall the definition

$$G_{k+1,n-1}(x,q) \;:=\; \sum_{l=k+1}^{n-1} F^{\gamma}_{l+1,n-1}(x) h_l(1 - c^{-1}x) F^{\nu}_{k+1,l-1}(q).$$

Here we prove the following result:

Lemma A.4. *When $\nu = \gamma$, we have for $x \in \mathbb{C} \setminus \{0\}$*

$$G_{k+1,n-1}(x,q) = \begin{cases} \frac{q}{x-q}\left(F^{\gamma}_{k+1,n-1}(q) - F^{\gamma}_{k+1,n-1}(x)\right) & \text{if } x \neq q, \\ \frac{q}{1-\gamma}F^{\gamma}_{k+1,n-1}(q)\left[(n-1)^{1-\gamma} - (k+1)^{1-\gamma}\right] + O(F^{\gamma}_{k+1,n-1}(q)) & \text{if } x = q \text{ and } 1/2 < \gamma < 1, \\ qF^{\gamma}_{k+1,n-1}(q)\ln\left(\frac{n-1}{k+1}\right) + O(k^{-1}F^{\gamma}_{k+1,n-1}(q)) & \text{if } x = q \text{ and } \gamma = 1. \end{cases} \tag{A.14}$$

When $\nu \neq \gamma$, we have for $x \in \mathbb{C} \setminus \{0\}$

$$G_{k+1,n-1}(x,q) = C(x,q)\left(\frac{F^{\nu}_{k+1,n-1}(q)}{(n-1)^{\mu}} - \frac{F^{\gamma}_{k+1,n-1}(x)}{k^{\mu}}\right) + O\left(\frac{|F^{\nu}_{k+1,n-1}(q)|}{n^{2\mu}} + \frac{|F^{\gamma}_{k+1,n-1}(x)|}{k^{2\mu}}\right), \tag{A.15}$$

where $\mu := |\gamma - \nu|$ and

$$C(x,q) := \begin{cases} -\frac{x}{q} & \text{if } \nu < \gamma, \\ \frac{q}{x} & \text{if } \gamma < \nu. \end{cases}$$

Proof. Recalling the definition (A.4), we can write

$$\begin{aligned} G_{k+1,n-1}(x,q) &= \sum_{l=k+1}^{n-1} F^{\gamma}_{l+1,n-1}(x)h_l(1-c^{-1}x)F^{\nu}_{k+1,l-1}(q) = \sum_{l=k+1}^{n-1}\frac{p^{\gamma}_{n-1}(x)}{p^{\gamma}_l(x)}h_l(1-c^{-1}x)\frac{p^{\nu}_{l-1}(q)}{p^{\nu}_k(q)} \\ &= \frac{p^{\gamma}_{n-1}(x)}{p^{\nu}_k(q)}\sum_{l=k+1}^{n-1}\frac{h_l(1-c^{-1}x)}{(1-\widehat{q}_{l,l})}X_l, \qquad \text{where} \qquad X_l := \frac{p^{\nu}_l(q)}{p^{\gamma}_l(x)}. \end{aligned} \tag{A.16}$$

Moreover, recalling the definition of the function h_l (see (Aletti, Crimaldi and Ghiglietti, 2019, Equation (5.10))), we have for $x \neq 0$

$$h_l(1-c^{-1}x) = \begin{cases} \widehat{r}_{l-1}c^{-1}x = xl^{-\gamma} & \text{if } \nu < \gamma, \\ \widehat{q}_{l,l} = ql^{-\nu} & \text{if } \nu \geq \gamma. \end{cases}$$

Let us start with the case $\nu = \gamma$. In this case, we have

$$\begin{aligned} \Delta X_l := X_l - X_{l-1} &= \left(1 - \frac{X_{l-1}}{X_l}\right)X_l = \left(\frac{x-q}{q}\frac{q}{l^{\gamma}(1-ql^{-\gamma})}\right)X_l \\ &= \frac{x-q}{q}\frac{\widehat{q}_{l,l}}{1-\widehat{q}_{l,l}}X_l = \frac{x-q}{q}\frac{h_l(1-c^{-1}x)}{1-\widehat{q}_{l,l}}X_l. \end{aligned}$$

It follows that

$$\frac{x-q}{q}\sum_{l=k+1}^{n-1}\frac{h_l(1-c^{-1}x)}{1-\widehat{q}_{l,l}}X_l = X_{n-1} - X_k.$$

Since

$$\begin{aligned} \frac{p^{\gamma}_{n-1}(x)}{p^{\nu}_k(q)}X_{n-1} &= \frac{p^{\gamma}_{n-1}(x)}{p^{\nu}_k(q)}\frac{p^{\nu}_{n-1}(q)}{p^{\gamma}_{n-1}(x)} = F^{\nu}_{k+1,n-1}(q) \text{ and} \\ \frac{p^{\gamma}_{n-1}(x)}{p^{\nu}_k(q)}X_k &= \frac{p^{\gamma}_{n-1}(x)}{p^{\nu}_k(q)}\frac{p^{\nu}_k(q)}{p^{\gamma}_k(x)} = F^{\gamma}_{k+1,n-1}(x), \end{aligned} \tag{A.17}$$

we find by (A.16)

$$\frac{x-q}{q}G_{k+1,n-1}(x,q) = \left(F^{\gamma}_{k+1,n-1}(q) - F^{\gamma}_{k+1,n-1}(x)\right)$$

and so for $x \neq q$ we get

$$G_{k+1,n-1}(x,q) = \frac{q}{x-q}\left(F^{\gamma}_{k+1,n-1}(q) - F^{\gamma}_{k+1,n-1}(x)\right).$$

When $\nu = \gamma$ and $x = q$, we have $X_l = 1$ and so we obtain (by (A.16) together with (A.7))

$$\begin{aligned} G_{k+1,n-1}(x,q) &= qF^{\gamma}_{k+1,n-1}(q)\sum_{l=k+1}^{n-1}\frac{1}{l^{\gamma}(1-ql^{-\gamma})} = qF^{\gamma}_{k+1,n-1}(q)\sum_{l=k+1}^{n-1}\frac{1}{l^{\gamma}} + O\left(\sum_{l\geq k+1} l^{-2\gamma}\right) \\ &= qF^{\gamma}_{k+1,n-1}(q)\begin{cases} \frac{(n-1)^{1-\gamma}}{1-\gamma} - \frac{(k+1)^{1-\gamma}}{1-\gamma} + O(1) + O(k^{-(2\gamma-1)}) & \text{if } 1/2<\gamma<1 \\ \ln(n-1) - \ln(k+1) + O(n^{-1}) + O(k^{-1}) & \text{if } \gamma = 1, \end{cases} \end{aligned}$$

which implies the two different asymptotic behavior in (A.14) according to the value of γ.

Now, let us consider the case $\nu \neq \gamma$ and introduce the sequence $\{y_l; l \geq 1\}$ defined as $y_l := l^{-\mu}$, with $\mu = |\gamma - \nu|$. Then, we have

$$\begin{aligned} y_l X_l - y_{l-1}X_{l-1} = \Delta y_l X_l + y_{l-1}\Delta X_l &= \left(\frac{1}{l^{\mu}} - \frac{1}{(l-1)^{\mu}}\right)X_l + \left(\frac{1}{l^{\mu}} + O\left(\frac{1}{l^{1+\mu}}\right)\right)\Delta X_l \\ &= \left(\frac{-\mu}{l^{1+\mu}} + O\left(\frac{1}{l^{2+\mu}}\right)\right)X_l + \left(\frac{1}{l^{\mu}} + O\left(\frac{1}{l^{1+\mu}}\right)\right)\Delta X_l\,, \end{aligned}$$

where

$$\Delta X_l := X_l - X_{l-1} = \left(1 - \frac{X_{l-1}}{X_l}\right)X_l = R_l X_l$$

with

$$R_l := \left(1 - \frac{X_{l-1}}{X_l}\right) = \frac{xl^{-\gamma} - ql^{-\nu}}{1-ql^{-\nu}} = \frac{\widehat{r}_{l-1}c^{-1}x - \widehat{q}_{l,l}}{1-\widehat{q}_{l,l}} = O\left(\frac{1}{l^{\min\{\gamma,\nu\}}}\right).$$

Taking into account that $\mu + \min\{\gamma,\nu\} < 1 + \mu$ for $\nu \neq \gamma$, we obtain that

$$y_l X_l - y_{l-1}X_{l-1} = \left[\frac{R_l}{l^{\mu}} + O\left(\frac{1}{l^{1+\mu}}\right)\right]X_l = K(x,q)\frac{h_l(1-c^{-1}x)}{(1-\widehat{q}_{l,l})}X_l + Q_l X_l,$$

where

$$K(x,q) \;:=\; \left(-\frac{q}{x}\right)\mathbb{1}_{\{\nu<\gamma\}} + \left(\frac{x}{q}\right)\mathbb{1}_{\{\nu>\gamma\}} = C(x,q)^{-1}$$

and

$$Q_l := \begin{cases} \frac{xl^{-(2\gamma-\nu)}}{1-\widehat{q}_{l,l}} + O(l^{-(1+\mu)}) & \text{if } \nu<\gamma, \\ -\frac{ql^{-(2\nu-\gamma)}}{1-\widehat{q}_{l,l}} + O(l^{-(1+\mu)}) & \text{if } \nu>\gamma. \end{cases}$$

Note that $Q_l \sim \kappa l^{-(2\mu+\min\{\gamma,\nu\})}$ with a suitable $\kappa \neq 0$. The above expression implies that

$$\frac{X_{n-1}}{(n-1)^{\mu}} - \frac{X_k}{k^{\mu}} = \sum_{l=k+1}^{n-1}(y_l X_l - y_{l-1}X_{l-1}) = K(x,q)\sum_{l=k+1}^{n-1}\frac{h_l(1-c^{-1}x)}{(1-\widehat{q}_{l,l})}X_l + \sum_{l=k+1}^{n-1}Q_l X_l. \tag{A.18}$$

With similar computations, setting

$$R^*_l := 1 - \frac{|X_{l-1}|}{|X_l|} = \frac{|1-ql^{-\nu}| - |1-xl^{-\gamma}|}{|1-ql^{-\nu}|}$$

and taking into account that $R_l^* l^{-2\mu} \sim \kappa' l^{-(2\mu+\min\{\gamma,\nu\})}$ with a suitable $\kappa' \neq 0$ and $\min\{\gamma,\nu\} < 1$ for $\nu \neq \gamma$, we find

$$\frac{|X_l|}{l^{2\mu}} - \frac{|X_{l-1}|}{(l-1)^{2\mu}} = \left[\frac{R_l^*}{l^{2\mu}} + O\left(\frac{1}{l^{1+2\mu}}\right)\right]|X_l| = Q_l^*|X_l|.$$

Then, since $Q_l \sim \kappa'' Q_l^*$ with a suitable $\kappa'' \neq 0$,

$$\sum_{l=k+1}^{n-1} Q_l X_l = O\left(\sum_{l=k+1}^{n-1} Q_l^*|X_l|\right) = O\left(\frac{|X_{n-1}|}{(n-1)^{2\mu}} - \frac{|X_k|}{k^{2\mu}}\right).$$

Finally, by (A.16), (A.17), (A.18) and the last above relations, we obtain for $x \neq 0$

$$G_{k+1,n-1}(x,q) = C(x,q)\left(\frac{F^{\nu}_{k+1,n-1}(q)}{(n-1)^{\mu}} - \frac{F^{\gamma}_{k+1,n-1}(x)}{k^{\mu}}\right) + O\left(\frac{|F^{\nu}_{k+1,n-1}(q)|}{n^{2\mu}} + \frac{|F^{\gamma}_{k+1,n-1}(x)|}{k^{2\mu}}\right).$$

□

A.4. Asymptotic behavior of $C^{J(I)}_{m_0,n-1}\sum_{j\in J}\mathbf{Y}_{j,m_0}$

Recalling (Aletti, Crimaldi and Ghiglietti, 2019, Equation (5.37)) and taking into account the fact that in all the considered cases with $1 \in J$, i.e. $(ii), (iii)$ and (v), we have $1 \notin I_1$, we get

$$\left|C^{J(I)}_{m_0,n}\right| = O(C_n^{11}) + O(C_n^{21}) + O(C_n^{22}),$$

where

$$\begin{aligned} C_n^{11} &:= \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{1\in I_j\}} |F^{\gamma}_{m_0,n-1}(c\alpha_j)|\,, \\ C_n^{21} &:= \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} |G_{m_0,n-1}(c\alpha_j,q)|\,, \\ C_n^{22} &:= \sum_{j\in J} \mathbb{1}_{\{2\in I_j\}} |F^{\nu}_{m_0,n-1}(q)|\,. \end{aligned}$$

Using (A.5) and denoting by a^* the real part of $\alpha^* := 1-\lambda^*$, it is immediate to see that

$$C_n^{11} = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{1\in I_j\}} \begin{cases} O\left(\exp\left(-ca^*\frac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2<\gamma<1 \\ O(n^{-ca^*}) & \text{if } \gamma = 1 \end{cases}$$

and

$$C_n^{22} = \sum_{j\in J} \mathbb{1}_{\{2\in I_j\}} \begin{cases} O\left(\exp\left(-q\frac{n^{1-\nu}}{1-\nu}\right)\right) & \text{if } 1/2<\nu<1 \\ O(n^{-q}) & \text{if } \nu = 1. \end{cases}$$

For the term C_n^{21}, we apply Lemma A.4 so that we get:

Case $\nu < \gamma$ We have $G_{m_0,n-1}(c\alpha_j,q) = O\big(n^{-(\gamma-\nu)}|F^{\nu}_{m_0,n-1}(q)| + |F^{\gamma}_{m_0,n-1}(c\alpha_j)|\big)$ by means of Lemma A.4 and so

$$C_n^{21} = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} O\left(n^{-(\gamma-\nu)}|F^{\nu}_{m_0,n-1}(q)| + |F^{\gamma}_{m_0,n-1}(c\alpha_j)|\right),$$

where, as above, by (A.5), we have $|F^{\nu}_{m_0,n-1}(q)| = O\left(\exp\left(-q\frac{n^{1-\nu}}{1-\nu}\right)\right)$ and

$$|F^{\gamma}_{m_0,n-1}(c\alpha_j)| = \begin{cases} O\left(\exp\left(-ca^*\frac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2<\gamma<1 \\ O(n^{-ca^*}) & \text{if } \gamma = 1. \end{cases}$$

Case $\nu > \gamma$ We have $G_{m_0,n-1}(c\alpha_j, q) = O\big(n^{-(\nu-\gamma)}|F^{\nu}_{m_0,n-1}(q)| + |F^{\gamma}_{m_0,n-1}(c\alpha_j)|\big)$ by means of Lemma A.4 and so

$$C^{21}_n = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} O\left(n^{-(\nu-\gamma)}|F^{\nu}_{m_0,n-1}(q)| + |F^{\gamma}_{m_0,n-1}(c\alpha_j)|\right),$$

where, as above, by (A.5), we have $|F^{\gamma}_{m_0,n-1}(c\alpha_j)| = O\left(\exp\left(-ca^* \frac{n^{1-\gamma}}{1-\gamma}\right)\right)$ and

$$|F^{\nu}_{m_0,n-1}(q)| = \begin{cases} O\left(\exp\left(-q\dfrac{n^{1-\nu}}{1-\nu}\right)\right) & \text{if } 1/2 < \nu < 1 \\ O(n^{-q}) & \text{if } \nu = 1. \end{cases}$$

Case $\nu = \gamma$ Assuming $q \neq c\alpha_j$ for all $j \geq 2$, by Lemma A.4, we have[1] $G_{m_0,n-1}(c\alpha_j, q) = O\big(|F^{\gamma}_{m_0,n-1}(q)| + |F^{\gamma}_{m_0,n-1}(c\alpha_j)|\big)$ and so

$$C^{21}_n = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} O\left(|F^{\gamma}_{m_0,n-1}(q)| + |F^{\gamma}_{m_0,n-1}(c\alpha_j)|\right),$$

where, as above, by (A.5), we have for $x = q$ or $x \in \{c\alpha_j : j \in J,\, j \neq 1\}$

$$|F^{\gamma}_{m_0,n-1}(x)| = \begin{cases} O\left(\exp\left(-a_x\dfrac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2 < \nu = \gamma < 1 \\ O(n^{-a_x}) & \text{if } \nu = \gamma = 1 \end{cases}$$

and so, setting $x^* := \min\{q, ca^*\}$, we can write

$$C^{21}_n = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} \begin{cases} O\left(\exp\left(-x^*\dfrac{n^{1-\gamma}}{1-\gamma}\right)\right) & \text{if } 1/2 < \nu = \gamma < 1 \\ O(n^{-x^*}) & \text{if } \nu = \gamma = 1. \end{cases}$$

Summing up, taking into account the conditions $ca^* > 1/2$ when $\gamma = 1$ and $q > 1/2$ when $\nu = 1$, we can conclude that in all the six cases (i)-(vi) we have $t_n(J(I))\left|C^{J(I)}_{m_0,n-1}\right| \to 0$ and so

$$t_n(J(I))C^{J(I)}_{m_0,n-1}\sum_{j\in J}\mathbf{Y}_{j,m_0} \xrightarrow{a.s.} \mathbf{0}.$$

A.5. Asymptotic behavior of $\sum_{k=m_0}^{n-1} \boldsymbol{\rho}^{J(I)}_{k+1,n-1}$

Recall (Aletti, Crimaldi and Ghiglietti, 2019, Equation (5.39)) and the fact that $\Delta\mathbf{R}_{Y,k+1} = (\Delta\mathbf{R}_{Z,k+1}, \Delta\mathbf{R}_{N,k+1})^{\top}$, where, by (Aletti, Crimaldi and Ghiglietti, 2019, Assumption 2.2), we have $|\Delta\mathbf{R}_{Z,k+1}| = O(k^{-2\gamma})$ and $|\Delta\mathbf{R}_{N,k+1}| = O(k^{-2\nu})$. Then, taking into account the fact that in all the considered cases with $1 \in J$, i.e. (ii), (iii) and (v), we have $1 \notin I_1$, we get

$$\left|\sum_{k=m_0}^{n-1} \boldsymbol{\rho}^{J(I)}_{k+1,n-1}\right| = O(\rho^{11}_n) + O(\rho^{21}_n) + O(\rho^{22}_n),$$

[1] If there exists $j \geq 2$ such that $q = c\alpha_j$, we have to consider the other asymptotic expression given in Lemma A.4.

where

$$\rho_n^{11} := \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{1\in I_j\}} \sum_{k=m_0}^{n-1} k^{-2\gamma} |F^{\gamma}_{k+1,n-1}(c\alpha_j)|\,,$$
$$\rho_n^{21} := \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} \sum_{k=m_0}^{n-1} k^{-2\gamma} |G_{k+1,n-1}(c\alpha_j, q)|\,,$$
$$\rho_n^{22} := \sum_{j\in J} \mathbb{1}_{\{2\in I_j\}} \sum_{k=m_0}^{n-1} (k^{-2\gamma} + k^{-2\nu}) |F^{\nu}_{k+1,n-1}(q)|.$$

Using Lemma A.2 (with $\beta = 2\gamma > 1$, $e = 1$ and $\delta = \gamma$), we get

$$\rho_n^{11} = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{1\in I_j\}} \begin{cases} O\left(n^{-\gamma}\right) & \text{if } 1/2 < \gamma < 1, \\ O\left(n^{-ca^*}\right) & \text{if } \gamma = 1 \text{ and } 1/2 < ca^* < 1, \\ O\left(n^{-1}\ln(n)\right) & \text{if } \gamma = 1 \text{ and } ca^* = 1, \\ O\left(n^{-1}\right) & \text{if } \gamma = 1 \text{ and } ca^* > 1. \end{cases}$$

For ρ_n^{22}, we observe that we have $k^{-2\gamma} = O(k^{-2\nu})$ when $\nu \leq \gamma$ and $k^{-2\nu} = O(k^{-2\gamma})$ when $\nu > \gamma$. Therefore, using Lemma A.2 (with $e = 1$ and $\delta = \nu$ and $\beta = 2\nu > 1$ if $\nu \leq \gamma$ and $\beta = 2\gamma > 1$ if $\nu > \gamma$), we obtain for the case $\nu \leq \gamma$

$$\begin{aligned} \rho_n^{22} &= \sum_{j\in J} \mathbb{1}_{\{2\in I_j\}} O\left(\sum_{k=m_0}^{n-1} k^{-2\nu} |F^{\nu}_{k+1,n-1}(q)| \right) \\ &= \sum_{j\in J} \mathbb{1}_{\{2\in I_j\}} \begin{cases} O\left(n^{-\nu}\right) & \text{if } 1/2 < \nu < 1, \\ O\left(n^{-q}\right) & \text{if } \nu = 1 \text{ and } 1/2 < q < 1, \\ O\left(n^{-1}\ln(n)\right) & \text{if } \nu = 1 \text{ and } q = 1, \\ O\left(n^{-1}\right) & \text{if } \nu = 1 \text{ and } q > 1\,, \end{cases} \end{aligned}$$

and for the case $\nu > \gamma$

$$\begin{aligned} \rho_n^{22} &= \sum_{j\in J} \mathbb{1}_{\{2\in I_j\}} O\left(\sum_{k=m_0}^{n-1} k^{-2\gamma} |F^{\nu}_{k+1,n-1}(q)| \right) \\ &= \sum_{j\in J} \mathbb{1}_{\{2\in I_j\}} \begin{cases} O\left(n^{-2\gamma+\nu}\right) & \text{if } 1/2 < \nu < 1, \\ O\left(n^{-q}\right) & \text{if } \nu = 1 \text{ and } 1/2 < q < 2\gamma - 1, \\ O\left(n^{-q}\ln(n)\right) & \text{if } \nu = 1 \text{ and } q = 2\gamma - 1 > 1/2, \\ O\left(n^{-2\gamma+1}\right) & \text{if } \nu = 1 \text{ and } q > \max\{1/2, 2\gamma - 1\}\,. \end{cases} \end{aligned} \tag{A.19}$$

For the term ρ_n^{21}, we apply Lemma A.2 and Lemma A.4 so that we get:

Case $\nu < \gamma$ We have $G_{k+1,n-1}(c\alpha_j, q) = O\big(n^{-(\gamma-\nu)}|F^{\nu}_{k+1,n-1}(q)| + k^{-(\gamma-\nu)}|F^{\gamma}_{k+1,n-1}(c\alpha_j)|\big)$ by means of Lemma A.4, and so we get

$$\rho_n^{21} = \sum_{j\in J\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} O\left(n^{-(\gamma-\nu)} \sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}} |F^{\nu}_{k+1,n-1}(q)| + \sum_{k=m_0}^{n-1} \frac{1}{k^{3\gamma-\nu}} |F^{\gamma}_{k+1,n-1}(c\alpha_j)| \right)\,,$$

where, by Lemma A.2, the first term is $O(n^{-3\gamma+2\nu})$, while for the second term we have

$$\sum_{k=m_0}^{n-1} \frac{1}{k^{3\gamma-\nu}}|F^{\gamma}_{k+1,n-1}(c\alpha_j)| = \begin{cases} O\left(n^{-2\gamma+\nu}\right) & \text{if } 1/2<\gamma<1,\\ O\left(n^{-ca^*}\right) & \text{if } \gamma=1 \text{ and } 1/2<ca^*<2-\nu,\\ O\left(n^{-2+\nu}\ln(n)\right) & \text{if } \gamma=1 \text{ and } ca^*=2-\nu,\\ O\left(n^{-2+\nu}\right) & \text{if } \gamma=1 \text{ and } ca^*>2-\nu\,. \end{cases}$$

Case $\nu>\gamma$ We have $G_{k+1,n-1}(c\alpha_j,q) = O\big(n^{-(\nu-\gamma)}|F^{\nu}_{k+1,n-1}(q)| + k^{-(\nu-\gamma)}|F^{\gamma}_{k+1,n-1}(c\alpha_j)|\big)$ by means of Lemma A.4, and so we get

$$\rho^{21}_n = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} O\left(n^{-(\nu-\gamma)}\sum_{k=m_0}^{n-1}\frac{1}{k^{2\gamma}}|F^{\nu}_{k+1,n-1}(q)| + \sum_{k=m_0}^{n-1}\frac{1}{k^{\gamma+\nu}}|F^{\gamma}_{k+1,n-1}(c\alpha_j)|\right),$$

where, by Lemma A.2, the second term is $O(n^{-\nu})$, while the sum in the first term has the asymptotic behavior given in (A.19).

Case $\nu=\gamma$ Assuming $q\neq c\alpha_j$ for all $j\geq 2$, by Lemma A.4, we have[2] $G_{k+1,n-1}(c\alpha_j,q) = O\big(|F^{\gamma}_{k+1,n-1}(q)| + |F^{\gamma}_{k+1,n-1}(c\alpha_j)|\big)$, and so we get

$$\rho^{21}_n = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} O\left(\sum_{k=m_0}^{n-1}\frac{1}{k^{2\gamma}}|F^{\gamma}_{k+1,n-1}(q)| + \sum_{k=m_0}^{n-1}\frac{1}{k^{2\gamma}}|F^{\gamma}_{k+1,n-1}(c\alpha_j)|\right),$$

where, by Lemma A.2, we have for $x=q$ or $x\in\{c\alpha_j : j\in J,\, j\neq 1\}$

$$\sum_{k=m_0}^{n-1} \frac{1}{k^{2\gamma}}|F^{\gamma}_{k+1,n-1}(x)| = \begin{cases} O\left(n^{-\gamma}\right) & \text{if } 1/2<\nu=\gamma<1,\\ O\left(n^{-a_x}\right) & \text{if } \nu=\gamma=1 \text{ and } 1/2<a_x<1,\\ O\left(n^{-1}\ln(n)\right) & \text{if } \nu=\gamma=1 \text{ and } a_x=1,\\ O\left(n^{-1}\right) & \text{if } \nu=\gamma=1 \text{ and } a_x>1 \end{cases}$$

and so, setting $x^* := \min\{q, ca^*\}$, we can write

$$\rho^{21}_n = \sum_{j\in J,\, j\neq 1} \mathbb{1}_{\{2\in I_j\}} \begin{cases} O\left(n^{-\gamma}\right) & \text{if } 1/2<\nu=\gamma<1,\\ O\left(n^{-x^*}\right) & \text{if } \nu=\gamma=1 \text{ and } 1/2<x^*<1,\\ O\left(n^{-1}\ln(n)\right) & \text{if } \nu=\gamma=1 \text{ and } x^*=1,\\ O\left(n^{-1}\right) & \text{if } \nu=\gamma=1 \text{ and } x^*>1\,. \end{cases}$$

Summing up, taking into account the conditions $ca^*>1/2$ when $\gamma=1$ and $q>1/2$ when $\nu=1$, from the asymptotic behavior given above we easily obtain that in all the cases (i)-(v) we have $t_n(J(I))\left|\sum_{k=m_0}^{n-1}\boldsymbol{\rho}^{J(I)}_{n,k}\right|\to 0$ a.s. and so

$$t_n(J(I))\sum_{k=m_0}^{n-1}\boldsymbol{\rho}^{J(I)}_{n,k} \xrightarrow{a.s.} \mathbf{0}\,. \tag{A.20}$$

In the case (vi), the evaluation of the asymptotic behavior given in (A.19) for the term ρ^{22}_n is not enough in order to conclude that $t_n(J(I))\rho^{22}_n\to 0$. Therefore, we need a better evaluation, that we can get applying Lemma A.2 in a different way. Indeed, in the case (vi), taking $u>1$ and applying Lemma A.2 with $e=u$,

[2] If there exists $j\geq 2$ such that $q=c\alpha_j$, we have to consider the other asymptotic expression given in Lemma A.4.

$\delta = \nu$ and $\beta = 2\gamma u > 1$, we find

$$\left(t_n(J(I))\rho_n^{22}\right)^u = n^{u\nu/2} O\left(\sum_{k=m_0}^{n-1} k^{-2\gamma u} |F_{k+1,n-1}^{\nu}(q)|^u\right)$$
$$= n^{u\nu/2} \begin{cases} O\left(n^{-2\gamma u+\nu}\right) & \text{if } 1/2 < \nu < 1, \\ O\left(n^{-qu}\right) & \text{if } \nu = 1 \text{ and } 1/2 < q < 2\gamma - u^{-1}, \\ O\left(n^{-qu}\ln(n)\right) & \text{if } \nu = 1 \text{ and } q = 2\gamma - u^{-1} > 1/2, \\ O\left(n^{-2\gamma u+1}\right) & \text{if } \nu = 1 \text{ and } q > \max\{1/2, 2\gamma - u^{-1}\}. \end{cases}$$

Hence, from the above relations we get that it is possible to find $u > 1$ large enough such that $(t_n(J(I))\rho_n^{22})^u \to 0$, that trivially implies $t_n(J(I))\rho_n^{22} \to 0$. Therefore also in the case (vi), we can conclude that (A.20) holds true.

A.6. Computation of the limit $d^{j_1(i_1),j_2(i_2)}$

Recall that we have

$$d_{k,n}^{j(1)} = \begin{cases} \widehat{r}_{k-1} & \text{for } j = 1 \\ \widehat{r}_{k-1} F_{k+1,n-1}^{\gamma}(c\alpha_j) & \text{for } j \geq 2 \end{cases}$$

and

$$d_{k,n}^{j(2)} = \begin{cases} \left(\widehat{q}_{k,k} - \widehat{r}_{k-1}\right) F_{k+1,n-1}^{\nu}(q) \text{ for } j = 1 \\ \lambda_j \widehat{r}_{k-1} G_{k+1,n-1}(c\alpha_j, q) + \left(\widehat{q}_{k,k} - \widehat{r}_{k-1} g(\lambda_j)\right) F_{k+1,n-1}^{\nu}(q) \text{ for } j \geq 2\,, \end{cases}$$

where, for each $j \geq 2$, we have $g(\lambda_j) = \lambda_j$ when $\nu < \gamma$, while $g(\lambda_j) = 0$ when $\nu \geq \gamma$.

Here, for each of the six cases $(i) - (vi)$ listed in the statement of (Aletti, Crimaldi and Ghiglietti, 2019, Lemma 5.1), we compute the limit

$$d^{j_1(i_1),j_2(i_2)} \;=\; \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)}\,.$$

For all the computations, we make the assumptions stated in (Aletti, Crimaldi and Ghiglietti, 2019, Section 2) and we use Lemma A.3 and Lemma A.4.

Case (i) Take $\nu < \gamma$, $j_1, j_2 \in \{2, \ldots, N\}$ and $i_1 = i_2 = 1$. We have

$$\begin{aligned} \lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} &= \lim_n n^{\gamma} \sum_{k=m_0}^{n-1} \widehat{r}_{k-1}^{2} F_{k+1,n-1}^{\gamma}(c\alpha_{j_1}) F_{k+1,n-1}^{\gamma}(c\alpha_{j_2}) \\ &= c^2 \lim_n n^{\gamma} \sum_{k=m_0}^{n-1} k^{-2\gamma} F_{k+1,n-1}^{\gamma}(c\alpha_{j_1}) F_{k+1,n-1}^{\gamma}(c\alpha_{j_2}) \\ &= \frac{c^2}{c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}}\,. \end{aligned}$$

Case (ii) Take $\nu<\gamma$, $j_1,j_2\in\{1,\dots,N\}$ and $i_1=i_2=2$. For $j_1=j_2=1$, we have

$$\begin{aligned}\lim_n t_n(J(I))^2\sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)}d_{k,n}^{j_2(i_2)} &= \lim_n n^{\nu}\sum_{k=m_0}^{n-1}\left(\widehat{q}_{k,k}-\widehat{r}_{k-1}\right)^2 F_{k+1,n-1}^{\nu}(q)^2\\ &=\lim_n n^{\nu}\sum_{k=m_0}^{n-1}\widehat{q}_{k,k}^2 F_{k+1,n-1}^{\nu}(q)^2\\ &= q^2\lim_n n^{\nu}\sum_{k=m_0}^{n-1}k^{-2\nu}F_{k+1,n-1}^{\nu}(q)^2=\frac{q}{2}.\end{aligned}$$

(Note that the above second equality is due to the fact that some terms are $o(n^{-\nu})$ and so we can cancel them.) Similarly, for the cases $j_1\geq 2$, $j_2\geq 2$ and $j_1=1$, $j_2\geq 2$ and $j_1\geq 2$, $j_2=1$, using Lemma A.4, which allows us to replace in the computation of the desired limit the quantity $G_{k+1,n-1}(c\alpha_j,q)$ by

$$-\frac{c\alpha_j}{q}\left(\frac{F_{k+1,n-1}^{\nu}(q)}{(n-1)^{\gamma-\nu}}-\frac{F_{k+1,n-1}^{\gamma}(c\alpha_j)}{k^{\gamma-\nu}}\right),$$

and removing the terms which are $o(n^{-\nu})$, we obtain

$$\begin{aligned}\lim_n t_n(J(I))^2\sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)}d_{k,n}^{j_2(i_2)} &= \lim_n n^{\nu}\sum_{k=m_0}^{n-1}\widehat{q}_{k,k}^2 F_{k+1,n-1}^{\nu}(q)^2\\ &= q^2\lim_n n^{\nu}\sum_{k=m_0}^{n-1}k^{-2\nu}F_{k+1,n-1}^{\nu}(q)^2=\frac{q}{2}.\end{aligned}$$

Case (iii) Take $\nu=\gamma$, $j_1,j_2\in\{1,\dots,N\}$ and $i_1,i_2\in\{1,2\}$ with $i_h\neq 1$ if $j_h=1$. Recall that we are assuming $q\neq c\alpha_j$ for all $j\geq 2$[3]. Therefore, for $j_1=j_2=1$ and $i_1=i_2=2$, we have

$$\begin{aligned}\lim_n t_n(J(I))^2\sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)}d_{k,n}^{j_2(i_2)} &= \lim_n n^{\gamma}\sum_{k=m_0}^{n-1}\left(\widehat{q}_{k,k}-\widehat{r}_{k-1}\right)^2 F_{k+1,n-1}^{\gamma}(q)^2\\ &=\lim_n (q-c)^2 n^{\gamma}\sum_{k=m_0}^{n-1}\frac{1}{k^{2\gamma}}F_{k+1,n-1}^{\gamma}(q)^2=\frac{(q-c)^2}{2q-\mathbb{1}_{\{\gamma=1\}}}.\end{aligned}$$

For $j_1=1$, $j_2\geq 2$, $i_1=2$ and $i_2=1$, we have

$$\begin{aligned}&\lim_n t_n(J(I))^2\sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)}d_{k,n}^{j_2(i_2)} =\\ &\lim_n n^{\gamma}\sum_{k=m_0}^{n-1}\left(\widehat{q}_{k,k}-\widehat{r}_{k-1}\right) F_{k+1,n-1}^{\gamma}(q)\widehat{r}_{k-1}F_{k+1,n-1}^{\gamma}(c\alpha_{j_2}) =\\ &(q-c)c\lim_n n^{\gamma}\sum_{k=m_0}^{n-1}\frac{1}{k^{2\gamma}}F_{k+1,n-1}^{\gamma}(q)F_{k+1,n-1}^{\gamma}(c\alpha_{j_2})=\frac{c(q-c)}{c\alpha_{j_2}+q-\mathbb{1}_{\{\gamma=1\}}}.\end{aligned}$$

By symmetry, for $j_1\geq 2$, $j_2=1$, $i_1=1$ and $i_2=2$, we have

$$d^{j_1(i_1),j_2(i_2)}=\frac{c(q-c)}{c\alpha_{j_1}+q-\mathbb{1}_{\{\gamma=1\}}}.$$

[3] In the case $q=c\alpha_j$ for some $j\geq 2$ the computations are similar, but we have to consider the other asymptotic expression given in Lemma A.4.

For $j_1 = 1$, $j_2 \geq 2$ and $i_1 = i_2 = 2$, we observe that, by means of Lemma A.4, in the computation of the considered limit, we can replace $G_{k+1,n-1}(c\alpha_j, q)$ by

$$q(c\alpha_j - q)^{-1}\big(F^{\gamma}_{k+1,n-1}(q) - F^{\gamma}_{k+1,n-1}(c\alpha_j)\big),$$

that is we can replace $d^{j(2)}_{k,n}$, with $j \geq 2$ by

$$\widehat{q}_{k,k}\left(\frac{(c\alpha_j - c)F^{\gamma}_{k+1,n-1}(c\alpha_j) - (q-c)F^{\gamma}_{k+1,n-1}(q)}{c\alpha_j - q}\right).$$

Therefore, we have

$$\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d^{j_1(i_1)}_{k,n} d^{j_2(i_2)}_{k,n} =$$
$$\lim_n n^{\gamma} \sum_{k=m_0}^{n-1} \left(\widehat{q}_{k,k} - \widehat{r}_{k-1}\right) F^{\gamma}_{k+1,n-1}(q)\widehat{q}_{k,k}\left(\frac{(c\alpha_{j_2} - c)F^{\gamma}_{k+1,n-1}(c\alpha_{j_2}) - (q-c)F^{\gamma}_{k+1,n-1}(q)}{c\alpha_{j_2} - q}\right) =$$
$$\frac{q(q-c)(c\alpha_{j_2} - c)}{c\alpha_{j_2} - q}\lim_n n^{\gamma}\sum_{k=m_0}^{n-1}\frac{1}{k^{2\gamma}}F^{\gamma}_{k+1,n-1}(c\alpha_{j_2})F^{\gamma}_{k+1,n-1}(q) + \frac{q(q-c)^2}{c\alpha_{j_2} - q}\lim_n n^{\gamma}\sum_{k=m_0}^{n-1}\frac{1}{k^{2\gamma}}F^{\gamma}_{k+1,n-1}(q)^2 =$$
$$\frac{q(q-c)(c+q-\mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})(2q - \mathbb{1}_{\{\gamma=1\}})}.$$

By symmetry, for $j_1 \geq 2$, $j_2 = 1$ and $i_1 = i_2 = 2$, we get

$$d^{j_1(i_1),j_2(i_2)} = \frac{q(q-c)(c+q-\mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(2q - \mathbb{1}_{\{\gamma=1\}})}.$$

Similarly, for $j_1 \geq 2$, $j_2 \geq 2$, $i_1 = i_2 = 1$, we have

$$\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d^{j_1(i_1)}_{k,n} d^{j_2(i_2)}_{k,n} = \lim_n n^{\gamma}\sum_{k=m_0}^{n-1}\widehat{r}^2_{k-1}F^{\gamma}_{k+1,n-1}(c\alpha_{j_1})F^{\gamma}_{k+1,n-1}(c\alpha_{j_2})$$
$$= \frac{c^2}{c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}}.$$

For $j_1 \geq 2$, $j_2 \geq 2$, $i_1 = 1$ and $i_2 = 2$, we have

$$\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d^{j_1(i_1)}_{k,n} d^{j_2(i_2)}_{k,n} =$$
$$\lim_n n^{\gamma}\sum_{k=m_0}^{n-1}\widehat{r}_{k-1}F^{\gamma}_{k+1,n-1}(c\alpha_{j_1})\widehat{q}_{k,k}\left(\frac{(c\alpha_{j_2} - c)F^{\gamma}_{k+1,n-1}(c\alpha_{j_2}) - (q-c)F^{\gamma}_{k+1,n-1}(q)}{c\alpha_{j_2} - q}\right) =$$
$$\frac{cq(c\alpha_{j_1} + c - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + c\alpha_{j_2} - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})}.$$

By symmetry, for $j_1 \geq 2$, $j_2 \geq 2$, $i_1 = 2$ and $i_2 = 1$, we get

$$d^{j_1(i_1),j_2(i_2)} = \frac{cq(c\alpha_{j_2} + c - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + c\alpha_{j_2} - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})}.$$

Finally, for $j_1 \geq 2$, $j_2 \geq 2$ and $i_1 = i_2 = 2$, we have

$$\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} =$$

$$\lim_n n^\gamma \sum_{k=m_0}^{n-1} \widehat{q}_{k,k}^2 \left(\frac{(c\alpha_{j_1} - c)F_{k+1,n-1}^\gamma(c\alpha_{j_1}) - (q-c)F_{k+1,n-1}^\gamma(q)}{c\alpha_{j_1} - q} \right) \left(\frac{(c\alpha_{j_2} - c)F_{k+1,n-1}^\gamma(c\alpha_{j_2}) - (q-c)F_{k+1,n-1}^\gamma(q)}{c\alpha_{j_2} - q} \right) =$$

$$q^2 \frac{c^3(\alpha_{j_1} + \alpha_{j_2}) + 2c^2 q(\alpha_{j_1}\alpha_{j_2} + 1) - \mathbb{1}_{\{\gamma=1\}} c^2(\alpha_{j_1}\alpha_{j_2} + \alpha_{j_1} + \alpha_{j_2} + 2)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})}$$
$$+ q^2 \frac{c(q - \mathbb{1}_{\{\gamma=1\}})^2(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}(2c + q - 1)(q-1)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1} + \alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})}.$$

Case (iv) Take $\gamma < \nu$, $j_1, j_2 \in \{2, \ldots, N\}$ and $i_1 = i_2 = 1$. We have

$$\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} = \lim_n n^\gamma \sum_{k=m_0}^{n-1} \widehat{r}_{k-1}^2 F_{k+1,n-1}^\gamma(c\alpha_{j_1}) F_{k+1,n-1}^\gamma(c\alpha_{j_2})$$
$$= \lim_n n^\gamma \sum_{k=m_0}^{n-1} k^{-2\gamma} F_{k+1,n-1}^\gamma(c\alpha_{j_1}) F_{k+1,n-1}^\gamma(c\alpha_{j_2}) = \frac{c}{\alpha_{j_1} + \alpha_{j_2}}.$$

The difference with the computations in the case $\nu < \gamma$ concerns only the fact that here it is not possible that $\gamma = 1$ since $\gamma < \nu \leq 1$.

Case (v) Take $\gamma < \nu$, $j_1, j_2 = 1$ and $i_1 = i_2 = 2$. We have

$$\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} = \lim_n n^{2\gamma-\nu} \sum_{k=m_0}^{n-1} \left(\widehat{q}_{k,k} - \widehat{r}_{k-1}\right)^2 F_{k+1,n-1}^\nu(q)^2$$
$$= \lim_n n^{2\gamma-\nu} \sum_{k=m_0}^{n-1} \widehat{r}_{k-1}^2 F_{k+1,n-1}^\nu(q)^2 = \frac{c^2}{2q - \mathbb{1}_{\{\nu=1\}}(2\gamma - 1)}.$$

(Note that the above second equality is due to the fact that some terms are $o(n^{-(2\gamma-\nu)})$ and so we can cancel them.)

Case (vi) Take $\gamma < \nu$, $j_1, j_2 \in \{2, \ldots, N\}$ and $i_1 = i_2 = 2$. Using Lemma A.4, which allows us to replace in the computation of the desired limit the quantity $G_{k+1,n-1}(c\alpha_j, q)$ by

$$\frac{q}{c\alpha_j} \left(\frac{F_{k+1,n-1}^\nu(q)}{(n-1)^{\nu-\gamma}} - \frac{F_{k+1,n-1}^\gamma(c\alpha_j)}{k^{\nu-\gamma}} \right),$$

and removing the terms which are $o(n^{-\nu})$, we have

$$\lim_n t_n(J(I))^2 \sum_{k=m_0}^{n-1} d_{k,n}^{j_1(i_1)} d_{k,n}^{j_2(i_2)} =$$

$$\lim_n n^{\nu} \sum_{k=m_0}^{n-1} \left(\lambda_{j_1}\widehat{r}_{k-1}G_{k+1,n-1}(c\alpha_{j_1},q) + \widehat{q}_{k,k}F^{\nu}_{k+1,n-1}(q)\right)\left(\lambda_{j_2}\widehat{r}_{k-1}G_{k+1,n-1}(c\alpha_{j_2},q) + \widehat{q}_{k,k}F^{\nu}_{k+1,n-1}(q)\right) =$$

$$\frac{\lambda_{j_1}\lambda_{j_2}}{\alpha_{j_1}\alpha_{j_2}} q^2 \lim_n n^{2\gamma-\nu} \sum_{k=m_0}^{n-1} k^{-2\gamma} F^{\nu}_{k+1,n-1}(q)^2 + \left(\frac{\lambda_{j_1}}{\alpha_{j_1}} + \frac{\lambda_{j_2}}{\alpha_{j_2}}\right) q^2 \lim_n n^{\gamma} \sum_{k=m_0}^{n-1} k^{-(\gamma+\nu)} F^{\nu}_{k+1,n-1}(q)^2$$

$$+ q^2 \lim_n n^{\nu} \sum_{k=m_0}^{n-1} k^{-2\nu} F^{\nu}_{k+1,n-1}(q)^2 =$$

$$\left(\frac{\lambda_{j_1}\lambda_{j_2}}{\alpha_{j_1}\alpha_{j_2}}\right) \frac{q^2}{2q - \mathbb{1}_{\{\nu=1\}}(2\gamma-1)} + \left(\frac{\lambda_{j_1}}{\alpha_{j_1}} + \frac{\lambda_{j_2}}{\alpha_{j_2}}\right) \frac{q^2}{2q - \mathbb{1}_{\{\nu=1\}}\gamma} + \frac{q^2}{2q - \mathbb{1}_{\{\nu=1\}}}.$$

B. Entries of the matrices S_γ^{11}, S_γ^{12} and S_γ^{22} in (Aletti, Crimaldi and Ghiglietti, 2019, Theorem 5.1)

Starting from (Aletti, Crimaldi and Ghiglietti, 2019, Equation (5.27)), using the values $d^{j_1(i_1),j_2(i_2)}$ computed above and taking $2 \le j_1, j_2 \le N$, we get

$$[S_\gamma^{11}]_{11} = [S^{11}]_{j_1 1} = [S^{11}]_{1 j_2} := 0, \qquad [S_\gamma^{11}]_{j_1 j_2} := \frac{c^2}{c(\alpha_{j_1}+\alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2},$$

$$[S_\gamma^{12}]_{11} = [S^{12}]_{1 j_2} := 0, \qquad [S_\gamma^{12}]_{j_1 1} := \frac{c(q-c)}{c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}}} \mathbf{v}_{j_1}^\top \mathbf{v}_1,$$

$$[S_\gamma^{12}]_{j_1 j_2} := \frac{cq(c\alpha_{j_1} + c - \mathbb{1}_{\{\gamma=1\}})}{(c\alpha_{j_1} + c\alpha_{j_2} - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2},$$

$$[S_\gamma^{22}]_{11} := \frac{(q-c)^2}{2q - \mathbb{1}_{\{\gamma=1\}}} \|\mathbf{v}_1\|^2, \qquad [S_\gamma^{22}]_{j1} = [S_\gamma^{22}]_{1j} := \frac{q(q-c)(c+q-\mathbb{1}_{\{\gamma=1\}})}{(c\alpha_j + q - \mathbb{1}_{\{\gamma=1\}})(2q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_j^\top \mathbf{v}_1,$$

$$[S_\gamma^{22}]_{j_1 j_2} := q^2 \frac{c^3(\alpha_{j_1}+\alpha_{j_2}) + 2c^2 q(\alpha_{j_1}\alpha_{j_2}+1) - \mathbb{1}_{\{\gamma=1\}} c^2(\alpha_{j_1}\alpha_{j_2} + \alpha_{j_1} + \alpha_{j_2} + 2)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1}+\alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}$$

$$+ q^2 \frac{c(q - \mathbb{1}_{\{\gamma=1\}})^2(\alpha_{j_1}+\alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}}(2c+q-1)(q-1)}{(2q - \mathbb{1}_{\{\gamma=1\}})(c(\alpha_{j_1}+\alpha_{j_2}) - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_1} + q - \mathbb{1}_{\{\gamma=1\}})(c\alpha_{j_2} + q - \mathbb{1}_{\{\gamma=1\}})} \mathbf{v}_{j_1}^\top \mathbf{v}_{j_2}.$$

References

ALETTI, G., CRIMALDI, I. and GHIGLIETTI, A. (2017). Synchronization of reinforced stochastic processes with a network-based interaction. *Ann. Appl. Probab.* **27** 3787-3844.

ALETTI, G., CRIMALDI, I. and GHIGLIETTI, A. (2019). Interacting Reinforced Stochastic Processes: Statistical Inference based on the Weighted Empirical Means. *Bernoulli.* `htpps://doi.org/10.3150/19-BEJ1143`