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Supplementary Material
Networks of reinforced stochastic processes: asymptotics for the empirical means

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A. Some technical results

In all the sequel, given $(a_n)_n, (b_n)_n$ two sequences of real numbers with $b_n \geq 0$, the notation $a_n = O(b_n)$ means $|a_n| \leq C b_n$ for a suitable constant $C > 0$ and n large enough. Therefore, if we also have $a_n^{-1} = O(b_n^{-1})$, then $C' b_n \leq |a_n| \leq C b_n$ for suitable constants $C, C' > 0$ and n large enough. Moreover, given $(z_n)_n, (z'_n)_n$ two sequences of complex numbers, with $z'_n \neq 0$, the notation $z_n = o(z'_n)$ means $\lim_n z_n/z'_n = 0$.

A.1. Asymptotic results for sums of complex numbers

We start recalling an extension of the Toeplitz lemma (see Linero and Rosalsky (2013)) to complex numbers provided in Aletti, Crimaldi and Ghiglietti (2017), from which we get useful technical results employed in our proofs.

Lemma A.1. *(Aletti, Crimaldi and Ghiglietti, 2017, Lemma A.2) (Generalized Toeplitz lemma)*
Let $\{z_{n,k} : 1 \leq k \leq k_n\}$ be a triangular array of complex numbers such that

i) $\lim_n z_{n,k} = 0$ *for each fixed* k*;*
ii) $\lim_n \sum_{k=1}^{k_n} z_{n,k} = s \in \{0, 1\}$*;*
iii) $\sum_{k=1}^{k_n} |z_{n,k}| = O(1)$.

Let $(w_n)_n$ be a sequence of complex numbers with $\lim_n w_n = w \in \mathbb{C}$. Then, we have $\lim_n \sum_{k=1}^{k_n} z_{n,k} w_k = sw$.

From this lemma we can easily get the following corollary, which slightly extends the generalized version of the Kronecker lemma provided in (Aletti, Crimaldi and Ghiglietti, 2017, Corollary A.3):

Corollary A.1. *(Generalized Kronecker lemma)*
Let $\{v_{n,k} : 1 \leq k \leq n\}$ and $(z_n)_n$ be respectively a triangular array and a sequence of complex numbers such that $v_{n,k} \neq 0$ and

$$\lim_n v_{n,k} = 0, \quad \lim_n v_{n,n} \text{ exists finite}, \quad \sum_{k=1}^{n} |v_{n,k} - v_{n,k-1}| = O(1)$$

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and $\sum_n z_n$ is convergent. Then

$$\lim_n \sum_{k=1}^{n} v_{n,k} z_k = 0.$$

Proof. Without loss of generality, we can suppose $\lim_n v_{n,n} = s \in \{0,1\}$. Set $w_n = \sum_{k=n}^{+\infty} z_k$ and observe that, since $\sum_n z_n$ is convergent, we have $\lim_n w_n = w = 0$ and, moreover, we can write

$$\sum_{k=1}^{n} v_{n,k} z_k = \sum_{k=1}^{n} v_{n,k}(w_k - w_{k+1}) = \sum_{k=2}^{n} (v_{n,k} - v_{n,k-1}) w_k + v_{n,1} w_1 - v_{n,n} w_{n+1}.$$

The second and the third term obviously converge to zero. In order to prove that the first term converges to zero, it is enough to apply Lemma A.1 with $z_{n,k} = v_{n,k} - v_{n,k-1}$. □

The above corollary is useful to get the following result for complex random variables, which again slightly extends the version provided in (Aletti, Crimaldi and Ghiglietti, 2017, Lemma A.3):

Lemma A.2. *Let $\mathcal{H} = (\mathcal{H}_n)_n$ be a filtration and $(Y_n)_n$ a $\mathcal{H}$-adapted sequence of complex random variables such that $E[Y_n|\mathcal{H}_{n-1}] \to Y$ almost surely. Moreover, let $(c_n)_n$ be a sequence of strictly positive real numbers such that $\sum_n E\left[|Y_n|^2\right]/c_n^2 < +\infty$ and let $\{v_{n,k}, 1 \le k \le n\}$ be a triangular array of complex numbers such that $v_{n,k} \neq 0$ and*

$$\lim_n v_{n,k} = 0, \quad \lim_n v_{n,n} \text{ exists finite}, \quad \lim_n \sum_{k=1}^{n} \frac{v_{n,k}}{c_k} = \eta \in \mathbb{C}, \tag{A.1}$$

$$\sum_{k=1}^{n} \frac{|v_{n,k}|}{c_k} = O(1), \quad \sum_{k=1}^{n} |v_{n,k} - v_{n,k-1}| = O(1). \tag{A.2}$$

Then $\sum_{k=1}^{n} v_{n,k} Y_k / c_k \xrightarrow{a.s.} \eta Y$.

Proof. Let A be an event such that $P(A) = 1$ and $\lim_n E[Y_n|\mathcal{H}_{n-1}](\omega) = Y(\omega)$ for each $\omega \in A$. Fix $\omega \in A$ and set $w_n = E[Y_n|\mathcal{H}_{n-1}](\omega)$ and $w = Y(\omega)$. If $\eta \neq 0$, applying Lemma A.1 to $z_{n,k} = v_{n,k}/(c_k \eta)$, $s = 1$ and w_n, we obtain

$$\lim_n \sum_{k=1}^{n} v_{n,k} \frac{E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k \eta} = Y(\omega).$$

If $\eta = 0$, applying Lemma A.1 to $z_{n,k} = v_{n,k}/c_k$, $s = 0$ and w_n, we obtain

$$\lim_n \sum_{k=1}^{n} v_{n,k} \frac{E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k} = 0.$$

Therefore, for both cases, we have

$$\sum_{k=1}^{n} v_{n,k} \frac{E[Y_k|\mathcal{H}_{k-1}]}{c_k} \xrightarrow{a.s.} \eta Y.$$

Now, consider the martingale $(M_n)_n$ defined by

$$M_n = \sum_{k=1}^{n} \frac{Y_k - E[Y_k|\mathcal{H}_{k-1}]}{c_k}.$$

It is bounded in L^2 since $\sum_{k=1}^{n} \frac{E[|Y_k|^2]}{c_k^2} < +\infty$ by assumption and so it is almost surely convergent, that means

$$\sum_k \frac{Y_k(\omega) - E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k} < +\infty$$

for $\omega \in B$ with $P(B) = 1$. Therefore, fixing $\omega \in B$ and setting $z_k = \frac{Y_k(\omega) - E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k}$, by Corollary A.1, we get

$$\lim_n \sum_{k=1}^{n} v_{n,k} \frac{Y_k(\omega) - E[Y_k|\mathcal{H}_{k-1}](\omega)}{c_k} = 0$$

and so

$$\sum_{k=1}^{n} v_{n,k} \frac{Y_k - E[Y_k|\mathcal{H}_{k-1}]}{c_k} \xrightarrow{a.s.} 0.$$

In order to conclude, it is enough to observe that

$$\sum_{k=1}^{n} v_{n,k} \frac{Y_k}{c_k} = \sum_{k=1}^{n} v_{n,k} \frac{Y_k - E[Y_k|\mathcal{H}_{k-1}]}{c_k} + \sum_{k=1}^{n} v_{n,k} \frac{E[Y_k|\mathcal{H}_{k-1}]}{c_k}.$$

□

We conclude this subsection recalling the following well-known relations for $a \in \mathbb{R}$:

$$\sum_{k=1}^{n} \frac{1}{k^{1-a}} = \begin{cases} O(1) & \text{for } a < 0, \\ \ln(n) + O(1) & \text{for } a = 0, \\ a^{-1}\, n^a + O(1) & \text{for } 0 < a \leq 1, \\ a^{-1}\, n^a + O(n^{a-1}) & \text{for } a > 1. \end{cases} \tag{A.3}$$

More precisely, in the case $a = 0$, we have

$$d_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n) = d + O(n^{-1}) \tag{A.4}$$

where d denotes the Euler-Mascheroni constant.

A.2. Asymptotic results for products of complex numbers

Fix $\gamma = 1$ and $c > 0$, and consider a sequence $(r_n)_n$ of real numbers such that $0 \leq r_n < 1$ for each n and

$$nr_n - c \;=\; O\left(n^{-1}\right). \tag{A.5}$$

Obviously, we have $r_n > 0$ for n large enough and so in the sequel, without loss of generality, we will assume $0 < r_n < 1$ for all n.

Let $x = a_x + i\, b_x \in \mathbb{C}$ and $y = a_y + i\, b_y \in \mathbb{C}$ with $a_x, a_y > 0$ and $c(a_x + a_y) \geq 1$. Denote by $m_0 \geq 2$ an integer such that $\max\{a_x,\, a_y\} r_m < 1$ for all $m \geq m_0$ and set:

$$p_{m_0-1}(x) := 1, \quad p_n(x) := \prod_{m=m_0}^{n} (1 - xr_m) \text{ for } n \geq m_0 \quad \text{and} \quad F_{k+1,n}(x) := \frac{p_n(x)}{p_k(x)} \text{ for } m_0 - 1 \leq k \leq n-1.$$

We recall the following result, which has been proved in Aletti, Crimaldi and Ghiglietti (2017).

Lemma A.3. *(Aletti, Crimaldi and Ghiglietti, 2017, Lemma A.4) We have that*

$$|p_n(x)| \;=\; O\left(n^{-ca_x}\right) \qquad \text{and} \qquad |p_n^{-1}(x)| \;=\; O\left(n^{ca_x}\right).$$

Inspired by the computation done in Aletti, Crimaldi and Ghiglietti (2017); Crimaldi et al. (2019), we can prove the following other technical result:

Lemma A.4. *(i) When $c(a_x + a_y) = 1$, we have*

$$\lim_n \frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 F_{k+1,n}(x) F_{k+1,n}(y) = \begin{cases} c^2 & \text{if } b_x + b_y = 0, \\ 0 & \text{if } b_x + b_y \neq 0; \end{cases} \tag{A.6}$$

while when $c(a_x + a_y) > 1$, we have

$$\begin{aligned} \lim_n n \sum_{k=m_0}^{n-1} r_k^2 F_{k+1,n}(x) F_{k+1,n}(y) &= \frac{c^2}{c(x+y)-1}, \\ \lim_n n \sum_{k=m_0}^{n-1} r_k^2 \ln\left(\frac{n}{k}\right) F_{k+1,n}(x) F_{k+1,n}(y) &= \frac{c^2}{(c(x+y)-1)^2}, \\ \lim_n n \sum_{k=m_0}^{n-1} r_k^2 \ln^2\left(\frac{n}{k}\right) F_{k+1,n}(x) F_{k+1,n}(y) &= \frac{2c^2}{(c(x+y)-1)^3}. \end{aligned} \tag{A.7}$$

(ii) Moreover, for any $u \geq 1$, we have:
when $c(a_x + a_y) = 1$

$$\sum_{k=m_0}^{n-1} r_k^{2u} \frac{|p_n(x)|^u |p_n(y)|^u}{|p_k(x)|^u |p_k(y)|^u} = \begin{cases} O(\ln(n)/n) & \text{for } u = 1, \\ O\left(n^{-u}\right) & \text{for } u > 1; \end{cases} \tag{A.8}$$

while when $c(a_x + a_y) > 1$ and $e \in \{0, 1, 2\}$

$$\sum_{k=m_0}^{n-1} r_k^{2u} \ln^{eu}\left(\frac{n}{k}\right) \frac{|p_n(x)|^u |p_n(y)|^u}{|p_k(x)|^u |p_k(y)|^u} = \begin{cases} O(n^{-uc(a_x+a_y)} \ln^{eu}(n)) & \text{for } uc(a_x + a_y) < 2u - 1, \\ O(n^{-(2u-1)} \ln^{eu+1}(n)) & \text{for } uc(a_x + a_y) = 2u - 1, \\ O(n^{-(2u-1)}) & \text{for } uc(a_x + a_y) > 2u - 1 \end{cases} \tag{A.9}$$

(note that for $u = 1$ only the third case is possible).

Proof. (i) First of all, let us notice that the limit (A.6) and the first of the limits (A.7) have already been proved in (Aletti, Crimaldi and Ghiglietti, 2017, Eq. (A.11),(A.18)). Therefore, we can focus on the second and the third limits in (A.7). To this end, let us set

$$\mathcal{S}_{1,n} := \sum_{k=m_0}^{n-1} \frac{r_k^2}{p_k(x)p_k(y)}, \qquad \mathcal{S}_{2,n} := \sum_{k=m_0}^{n-1} \frac{r_k^2 \ln(k)}{p_k(x)p_k(y)}, \qquad \mathcal{S}_{3,n} := \sum_{k=m_0}^{n-1} \frac{r_k^2 \ln^2(k)}{p_k(x)p_k(y)},$$

so that, recalling the equality $F_{k+1,n}(x) = p_n(x)/p_k(x)$, we can write:

$$\begin{aligned} & n \sum_{k=m_0}^{n-1} r_k^2 F_{k+1,n}(x) F_{k+1,n}(y) = n p_n(x) p_n(y) \mathcal{S}_{1,n}, \\ & n \sum_{k=m_0}^{n-1} r_k^2 \ln\left(\frac{n}{k}\right) F_{k+1,n}(x) F_{k+1,n}(y) = n p_n(x) p_n(y) \left(\ln(n)\mathcal{S}_{1,n} - \mathcal{S}_{2,n}\right), \\ & n \sum_{k=m_0}^{n-1} r_k^2 \ln^2\left(\frac{n}{k}\right) F_{k+1,n}(x) F_{k+1,n}(y) = n p_n(x) p_n(y) \left(\ln^2(n)\mathcal{S}_{1,n} - 2\ln(n)\mathcal{S}_{2,n} + \mathcal{S}_{3,n}\right). \end{aligned}$$

Now, set $G_{1,k} := c^2/[k p_k(x) p_k(y)]$ and recall that, as seen in (Aletti, Crimaldi and Ghiglietti, 2017, Proof of Lemma A.5), when $c(a_x + a_y) > 1$ we have

$$\Delta G_{1,k} = (c(x+y)-1)\Delta \mathcal{S}_{1,k} + O\left(k^{-1}|\Delta \mathcal{S}_{1,k}|\right). \tag{A.10}$$

Using analogous arguments, we can set $G_{2,k} := c^2 \ln(k)/[kp_k(x)p_k(y)]$ and observe that we have:

$$\begin{aligned}\Delta G_{2,k} &= \frac{c^2}{p_k(x)p_k(y)}\left[\left(\frac{\ln(k)}{k} - \frac{\ln(k-1)}{k-1}\right)\left(1-(x+y)r_k + r_k^2xy\right) + \frac{\ln(k)}{k}\left((x+y)r_k - r_k^2xy\right)\right]\\
&= \frac{c^2}{p_k(x)p_k(y)}\left[\left(-\frac{\ln(k)}{k^2} + \frac{1}{k^2} + o(k^{-2})\right)\left(1-(x+y)r_k + r_k^2xy\right) + \frac{\ln(k)}{k}\left((x+y)r_k - r_k^2xy\right)\right]\\
&= (c(x+y)-1)\Delta\mathcal{S}_{2,k} + \Delta\mathcal{S}_{1,k} + O(k^{-1}|\Delta\mathcal{S}_{2,k}|).\end{aligned}$$

Therefore, when $c(a_x + a_y) > 1$, we obtain

$$\frac{\Delta G_{2,k}}{c(x+y)-1} - \Delta\mathcal{S}_{2,k} = \frac{\Delta\mathcal{S}_{1,k}}{c(x+y)-1} + O\left(k^{-1}\ln(k)|\Delta\mathcal{S}_{1,k}|\right). \tag{A.11}$$

The relations (A.10), (A.11) and the first limit in (A.7) imply

$$\begin{aligned}&\lim_n np_n(x)p_n(y)\Big(\ln(n)\mathcal{S}_{1,n} - \mathcal{S}_{2,n}\Big)\\
&\quad= \lim_n np_n(x)p_n(y)\Big(\frac{\ln(n)G_{1,n}}{c(x+y)-1} - \mathcal{S}_{2,n}\Big) + O\Big(\ln(n)n|p_n(x)p_n(y)|\sum_{k=m_0}^{n-1} k^{-1}|\Delta\mathcal{S}_{1,k}|\Big)\\
&\quad= \lim_n np_n(x)p_n(y)\Big(\frac{G_{2,n}}{c(x+y)-1} - \mathcal{S}_{2,n}\Big)\\
&\quad= (c(x+y)-1)^{-1}\lim_n np_n(x)p_n(y)\mathcal{S}_{1,n} + O\Big(n|p_n(x)p_n(y)|\sum_{k=m_0}^{n-1} k^{-1}\ln(k)|\Delta\mathcal{S}_{1,k}|\Big)\\
&\quad= (c(x+y)-1)^{-1}\lim_n np_n(x)p_n(y)\mathcal{S}_{1,n} = \frac{c^2}{(c(x+1)-1)^2},\end{aligned}$$

where we have used the fact that, by Lemma A.3 and relation (A.3), we have

$$O\Big(\ln(n)n|p_n(x)p_n(y)|\sum_{k=m_0}^{n-1} k^{-1}|\Delta\mathcal{S}_{1,k}|\Big) = O\Big(\frac{\ln(n)}{n^{c(a_x+a_y)-1}}\sum_{k=m_0}^{n-1}\frac{1}{k^{1-(c(a_x+a_y)-2)}}\Big) \longrightarrow 0.$$

For the last limit, we can set $G_{3,k} := c^2\ln^2(k)/[kp_k(x)p_k(y)]$ and, similarly as above, observe that we have:

$$\begin{aligned}\Delta G_{3,k} &= \frac{c^2}{p_k(x)p_k(y)}\left[\left(\frac{\ln^2(k)}{k} - \frac{\ln^2(k-1)}{k-1}\right)\left(1-(x+y)r_k + r_k^2xy\right) + \frac{\ln^2(k)}{k}\left((x+y)r_k - r_k^2xy\right)\right]\\
&= \frac{c^2}{p_k(x)p_k(y)}\times\\
&\left[\left(-\frac{\ln^2(k)}{k^2} + 2\frac{\ln(k)}{k^2} + O(k^{-3}\ln^2(k))\right)\left(1-(x+y)r_k + r_k^2xy\right) + \frac{\ln^2(k)}{k}\left((x+y)r_k - r_k^2xy\right)\right]\\
&= (c(x+y)-1)\Delta\mathcal{S}_{3,k} + 2\Delta\mathcal{S}_{2,k} + O(k^{-1}|\Delta\mathcal{S}_{3,k}|).\end{aligned}$$

Therefore, when $c(a_x + a_y) > 1$, we obtain

$$\frac{\Delta G_{3,k}}{c(x+y)-1} - \Delta\mathcal{S}_{3,k} = \frac{2\Delta\mathcal{S}_{2,k}}{c(x+y)-1} + O(k^{-1}\ln^2(k)|\Delta\mathcal{S}_{1,k}|). \tag{A.12}$$

By means of analogous computations as above, the relations (A.10), (A.11), (A.12) and the already proved second limit in (A.7) imply

$$
\begin{aligned}
&\lim_n n p_n(x)p_n(y)\left(\ln^2(n)\mathcal{S}_{1,n}-2\ln(n)\mathcal{S}_{2,n}+\mathcal{S}_{3,n}\right)\\
&=\lim_n np_n(x)p_n(y)\Big(\frac{\ln^2(n)G_{1,n}}{c(x+y)-1}-2\ln(n)\mathcal{S}_{2,n}+\mathcal{S}_{3,n}\Big)+O\Big(\ln^2(n)n|p_n(x)p_n(y)|\sum_{k=m_0}^{n-1}k^{-1}|\Delta\mathcal{S}_{1,k}|\Big)\\
&=\lim_n np_n(x)p_n(y)\Big(\frac{\ln(n)G_{2,n}}{c(x+y)-1}-2\ln(n)\mathcal{S}_{2,n}+\mathcal{S}_{3,n}\Big)\\
&=\lim_n np_n(x)p_n(y)\Big(\frac{\ln(n)G_{2,n}}{c(x+y)-1}-2\frac{\ln(n)(G_{2,n}-\mathcal{S}_{1,n})}{c(x+y)-1}+\mathcal{S}_{3,n}\Big)\\
&\quad+O\Big(\ln(n)n|p_n(x)p_n(y)|\sum_{k=m_0}^{n-1}k^{-1}\ln(k)|\Delta\mathcal{S}_{1,k}|\Big)\\
&=\lim_n np_n(x)p_n(y)\Big(\frac{2\ln(n)\mathcal{S}_{1,n}}{c(x+y)-1}-\frac{G_{3,n}}{c(x+y)-1}+\mathcal{S}_{3,n}\Big)\\
&=\frac{2}{c(x+y)-1}\lim_n np_n(x)p_n(y)\Big(\ln(n)\mathcal{S}_{1,n}-\mathcal{S}_{2,n}\Big)+O\Big(n|p_n(x)p_n(y)|\sum_{k=m_0}^{n-1}k^{-1}\ln^2(k)|\Delta\mathcal{S}_{1,k}|\Big)\\
&=\frac{2}{c(x+y)-1}\lim_n np_n(x)p_n(y)\Big(\ln(n)\mathcal{S}_{1,n}-\mathcal{S}_{2,n}\Big)=\frac{2c^2}{(c(x+1)-1)^3},
\end{aligned}
$$

where we have used the fact that, by Lemma A.3 and relation (A.3), we have

$$
O\Big(\ln^2(n)n|p_n(x)p_n(y)|\sum_{k=m_0}^{n-1}k^{-1}|\Delta\mathcal{S}_{1,k}|\Big)=O\Big(\frac{\ln^2(n)}{n^{c(a_x+a_y)-1}}\sum_{k=m_0}^{n-1}\frac{1}{k^{1-(c(a_x+a_y)-2)}}\Big)\longrightarrow 0.
$$

ii) For the second part of the proof, note that by condition (A.5) on $(r_n)_n$, relation (A.3) and Lemma A.3, when $c(a_x+a_y)=1$, we have

$$
\sum_{k=m_0}^{n-1}r_k^{2u}\frac{|p_n(x)|^u|p_n(y)|^u}{|p_k(x)|^u|p_k(y)|^u}=O(n^{-u})\sum_{k=m_0}^{n-1}O(k^{-u})=\begin{cases}O(\ln(n)/n) & \text{for } u=1,\\ O\left(n^{-u}\right) & \text{for } u>1.\end{cases}
$$

For the case $c(a_x+a_y)>1$, note that for $u\geq 1$ and $e\in\{0,1,2\}$, we have

$$
\sum_{k=m_0}^{n-1}r_k^{2u}\ln^{eu}\left(\frac{n}{k}\right)\frac{|p_n(x)|^u|p_n(y)|^u}{|p_k(x)|^u|p_k(y)|^u}=\sum_{k=m_0}^{n-1}O(k^{-2u})\ln^{eu}\left(\frac{n}{k}\right)O\Big(\left(\frac{k}{n}\right)^{uc(a_x+a_y)}\Big)=
$$

$$
n^{-2u}\sum_{k=m_0}^{n-1}\ln^{eu}\left(\frac{n}{k}\right)O\Big(\left(\frac{k}{n}\right)^{u(c(a_x+a_y)-2)}\Big).
$$

Then, for $e=0$, using relation (A.3), it is easy to see that

$$
n^{-2u}\sum_{k=m_0}^{n-1}O\left(\left(\frac{k}{n}\right)^{u(c(a_x+a_y)-2)}\right)=\begin{cases}O(n^{-uc(a_x+a_y)}) & \text{for } uc(a_x+a_y)<2u-1,\\ O(n^{-(2u-1)}\ln(n)) & \text{for } uc(a_x+a_y)=2u-1,\\ O(n^{-(2u-1)}) & \text{for } uc(a_x+a_y)>2u-1\end{cases}
$$

(note that for $u=1$ only the third case is possible).

Now we consider the cases $e=1$ and $e=2$. Note that, setting $\alpha:=2u-uc(a_x+a_y)\in\mathbb{R}$ and $\beta:=eu\geq 1$, we have that

$$
\frac{1}{n}\sum_{k=m_0}^{n-1}\ln^{\beta}\left(\frac{n}{k}\right)O\Big(\left(\frac{k}{n}\right)^{-\alpha}\Big)\ =\ O(1)\ +\ O\left(\int_{\frac{m_0-1}{n}}^{\epsilon}x^{-\alpha}\ln^{\beta}(x^{-1})dx\right),
$$

where $\epsilon \in (0,1)$ has been chosen such that $g(x) = x^{-\alpha}\ln^{\beta}(x^{-1})$ is monotone in $(0,\epsilon]$ and we recall that $(m_0 - 1) \geq 1$. Then, we have that

$$\int_{\frac{m_0-1}{n}}^{\epsilon} x^{-\alpha}\ln^{\beta}(x^{-1})dx = \begin{cases} O(n^{\alpha-1}\ln^{\beta}(n)) & \text{for } \alpha > 1 \\ O(\ln^{\beta+1}(n)) & \text{for } \alpha = 1, \\ O(1) & \text{for } \alpha < 1. \end{cases}$$

Finally, we can conclude that, for the cases $e = 1$ and $e = 2$, we have

$$n^{-2u}\sum_{k=m_0}^{n-1} \ln^{eu}\left(\frac{n}{k}\right) O\Big(\Big(\frac{k}{n}\Big)^{u(c(a_x+a_y)-2)}\Big) = \begin{cases} O(n^{-uc(a_x+a_y)}\ln^{eu}(n)) & \text{for } uc(a_x+a_y) < 2u-1, \\ O(n^{-(2u-1)}\ln^{eu+1}(n)) & \text{for } uc(a_x+a_y) = 2u-1, \\ O(n^{-(2u-1)}) & \text{for } uc(a_x+a_y) > 2u-1 \end{cases}$$

(note again that for $u = 1$ only the third case is possible). □

Remark A.1. Setting $v_{n,k}^{(e)} := (n/k)\ln^{e}(n/k)F_{k+1,n}(x)F_{k+1,n}(y)$ for any $e \in \{0,1,2\}$ and $m_0 - 1 \leq k \leq n-1$, and using the relations (A.10), (A.11), (A.12) found in the proof of Lemma A.4, for $c(a_x + a_y) > 1$ we have:

$$\begin{aligned}
|v_{n,k}^{(0)} - v_{n,k-1}^{(0)}| &= n|p_n(x)p_n(y)|O(|\Delta G_{1,k}|) = n|p_n(x)p_n(y)|O(|\Delta\mathcal{S}_{1,k}|) = O\Big(nr_k^2\frac{|p_n(x)||p_n(y)|}{|p_k(x)||p_k(y)|}\Big); \\
|v_{n,k}^{(1)} - v_{n,k-1}^{(1)}| &= n|p_n(x)p_n(y)|O(|\ln(n)\Delta G_{1,k} - \Delta G_{2,k}|) \\
&= n|p_n(x)p_n(y)|O(|\ln(n)\Delta\mathcal{S}_{1,k} - \Delta\mathcal{S}_{2,k}| + |\Delta\mathcal{S}_{1,k}|) = O\Big(nr_k^2\left(\ln\left(\frac{n}{k}\right)+1\right)\frac{|p_n(x)||p_n(y)|}{|p_k(x)||p_k(y)|}\Big); \\
|v_{n,k}^{(2)} - v_{n,k-1}^{(2)}| &= n|p_n(x)p_n(y)|O(|\ln^2(n)\Delta G_{1,k} - 2\ln(n)\Delta G_{2,k} + \Delta G_{3,k}|) \\
&= n|p_n(x)p_n(y)|O\left(|\ln^2(n)\Delta\mathcal{S}_{1,k} - 2\ln(n)\Delta\mathcal{S}_{2,k} + \Delta\mathcal{S}_{3,k}| + |\ln(n)\Delta\mathcal{S}_{1,k} - \Delta\mathcal{S}_{2,k}|\right) \\
&= O\Big(nr_k^2\Big(\ln^2\left(\frac{n}{k}\right) + \ln\left(\frac{n}{k}\right)\Big)\frac{|p_n(x)||p_n(y)|}{|p_k(x)||p_k(y)|}\Big),
\end{aligned}$$

Moreover, setting $v_{n,k} := v_{n,k}^{(0)}/\ln(n)$ for any $m_0 - 1 \leq k \leq n-1$, in the case $c(a_x + a_y) = 1$ we have: $|v_{n,k} - v_{n,k-1}| = O\left(r_k^2 k/\ln(n)\right)$ when $b_x + b_y \neq 0$ since Lemma A.3 and

$$|v_{n,k}^{(0)} - v_{n,k-1}^{(0)}| = n|p_n(x)p_n(y)|O(|\Delta G_{1,k}|) = n|p_n(x)p_n(y)|O(|\Delta\mathcal{S}_{1,k}|) = O\Big(nr_k^2\frac{|p_n(x)||p_n(y)|}{|p_k(x)||p_k(y)|}\Big);$$

while $|v_{n,k} - v_{n,k-1}| = O\left(r_k^2/\ln(n)\right)$ when $b_x + b_y = 0$ since Lemma A.3 and

$$|v_{n,k}^{(0)} - v_{n,k-1}^{(0)}| = n|p_n(x)p_n(y)|O(|\Delta G_{1,k}|) = n|p_n(x)p_n(y)|O(k^{-1}|\Delta\mathcal{S}_{1,k}|) = O\Big(r_k^2\frac{n|p_n(x)||p_n(y)|}{k|p_k(x)||p_k(y)|}\Big).$$

A.3. Technical computations for the proofs of (Aletti, Crimaldi and Ghiglietti, 2019, Theorem 4.3 and Theorem 4.4)

In this subsection we collect some technical computations necessary for the proofs of (Aletti, Crimaldi and Ghiglietti, 2019, Theorem 4.3 and Theorem 4.4). Therefore, the notation and the assumptions used here are the same as those used in these theorems.
The first technical result is the following:

Lemma A.5. *Let the matrix $A_{k+1,n}$ be defined as in (Aletti, Crimaldi and Ghiglietti, 2019, (4.22)) for $m_0-1\le k\le n-1$. Then, we have that*

$$\begin{aligned} [A^{11}_{k+1,n}]_{jj} &= F_{k+1,n}(\alpha_j),\\ [A^{33}_{k+1,n}]_{jj} &= a^{22}_{k+1,n} \;=\; F_{k+1,n}(c^{-1}),\\ [A^{31}_{k+1,n}]_{jj} &= \begin{cases} \left(\frac{1-\alpha_j}{c\alpha_j-1}\right)(F_{k+1,n}(c^{-1})-F_{k+1,n}(\alpha_j)), & \text{for } c\alpha_j\neq 1,\\ (1-c^{-1})F_{k+1,n}(c^{-1})\ln\left(\frac{n}{k}\right)+O(n^{-1}), & \text{for } c\alpha_j=1. \end{cases} \end{aligned}$$

Proof. By means of (Aletti, Crimaldi and Ghiglietti, 2019, (4.20) and (4.22)), after standard calculations, the elements in $A_{k+1,n}$ for $m_0-1\le k\le n-1$ can be written as follows: $[A^{11}_{k+1,n}]_{jj}=F_{k+1,n}(\alpha_j)$, $[A^{33}_{k+1,n}]_{jj}=a^{22}_{k+1,n}=F_{k+1,n}(c^{-1})$ and

$$[A^{31}_{k+1,n}]_{jj} \;=\; (1-\alpha_j)\frac{p_n(\alpha_j)}{p_k(c^{-1})}S^j_{k+1,n},$$

where

$$S^j_{k+1,n}:=\sum_{l=k+1}^{n}\left(\frac{r_lc^{-1}}{1-r_lc^{-1}}\right)X^j_l \qquad\text{and}\qquad X^j_l:=\frac{p_l(c^{-1})}{p_l(\alpha_j)}.$$

Setting $\Delta X^j_l:=(X^j_l-X^j_{l-1})$, notice that we have

$$\Delta X^j_l=\left(\frac{1-r_lc^{-1}}{1-r_l\alpha_j}-1\right)X^j_{l-1}=(c\alpha_j-1)\left(\frac{r_lc^{-1}}{1-r_l\alpha_j}\right)X^j_{l-1}=(c\alpha_j-1)\left(\frac{r_lc^{-1}}{1-r_lc^{-1}}\right)X^j_l.$$

Hence, in the case $c\alpha_j\neq 1$, we have that

$$(X^j_n-X^j_k)=\sum_{l=k+1}^{n}\Delta X^j_l=(c\alpha_j-1)S^j_{k+1,n},$$

which implies

$$S^j_{k+1,n}=\frac{X^j_n-X^j_k}{c\alpha_j-1}=(c\alpha_j-1)^{-1}\left(\frac{p_n(c^{-1})}{p_n(\alpha_j)}-\frac{p_k(c^{-1})}{p_k(\alpha_j)}\right).$$

Using the above expression of $S^j_{k+1,n}$ in the definition of $A^{31}_{k+1,n}$, we obtain (for $c\alpha_j\neq 1$) that

$$[A^{31}_{k+1,n}]_{jj}=\frac{1-\alpha_j}{c\alpha_j-1}\frac{p_n(\alpha_j)}{p_k(c^{-1})}\left(\frac{p_n(c^{-1})}{p_n(\alpha_j)}-\frac{p_k(c^{-1})}{p_k(\alpha_j)}\right)=\left(\frac{1-\alpha_j}{c\alpha_j-1}\right)\Big(F_{k+1,n}(c^{-1})-F_{k+1,n}(\alpha_j)\Big).$$

When $c\alpha_j=1$, observing that $X^j_l=1$ for any $l\ge 1$ and using condition (A.5) we get

$$S^j_{k+1,n}=\sum_{l=k+1}^{n}\frac{r_lc^{-1}}{1-r_lc^{-1}}=\sum_{l=k+1}^{n}\frac{1}{l-1}+\sum_{l=k+1}^{n}O\left(\frac{1}{l^2}\right)=\sum_{l=k}^{n}\frac{1}{l}-\frac{1}{n}+O\left(\sum_{l\ge k}\frac{1}{l^2}\right)=\sum_{l=k}^{n}\frac{1}{l}+O(k^{-1}),$$

where, for the last equality, we have used the fact that $k<n$ and $\sum_{l\ge k}1/l^2=O(1/k)$. Then, using (A.4) for $a=0$, we have

$$\sum_{l=k}^{n}\frac{1}{l}=\ln\left(\frac{n}{k}\right)+d_n-d_k=\ln\left(\frac{n}{k}\right)+O(n^{-1})-O(k^{-1})=\ln\left(\frac{n}{k}\right)+O(k^{-1})$$

(where the last passage follows again by the fact that $k<n$). Finally, since Lemma A.3 we have $|F_{k+1,n}(c^{-1})|=O(k/n)$, we obtain (for $c\alpha_j=1$) that

$$[A^{31}_{k+1,n}]_{jj}=(1-c^{-1})\frac{p_n(c^{-1})}{p_k(c^{-1})}\Big(\ln(n/k)+O(1/k)\Big)=(1-c^{-1})F_{k+1,n}(c^{-1})\ln\left(\frac{n}{k}\right)+O(n^{-1}).$$

□

A.3.1. Computations for the almost sure limits of the elements in (Aletti, Crimaldi and Ghiglietti, 2019, (4.27))

- *a.s.* $-\lim_n n\sum_{k=m_0}^{n-1} r_k^2[A^1_{k+1,n}B_{k+1}A^1_{k+1,n}]_{h,j}$:
  By using the first limit in (Aletti, Crimaldi and Ghiglietti, 2019, (4.29)), we have

$$n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}[A^1_{k+1,n}]_{h,h}[A^1_{k+1,n}]_{j,j} = n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}F_{k+1,n}(\alpha_h)F_{k+1,n}(\alpha_j)$$
$$\xrightarrow{a.s} \frac{c^2}{c(\alpha_h+\alpha_j)-1}(\mathbf{v}_h^\top \mathbf{v}_j)Z_\infty(1-Z_\infty).$$

- *a.s.* $-\lim_n n\sum_{k=m_0}^{n-1} r_k^2[A^3_{k+1,n}B_{k+1}A^3_{k+1,n}]_{h,j}$:
  First, note that when $c\alpha_h\neq 1$ and $c\alpha_j\neq 1$, we have that $n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j}$ has the same limit as

$$\frac{(1-c^{-1})^2}{(c\alpha_h-1)(c\alpha_j-1)}\,n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}F^2_{k+1,n}(c^{-1})$$
$$+\ \frac{(1-\alpha_h)(1-\alpha_j)}{(c\alpha_h-1)(c\alpha_j-1)}\,n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}F_{k+1,n}(\alpha_h)F_{k+1,n}(\alpha_j)$$
$$-\ \frac{(1-\alpha_h)(1-c^{-1})}{(c\alpha_h-1)(c\alpha_j-1)}\,n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}F_{k+1,n}(\alpha_h)F_{k+1,n}(c^{-1})$$
$$-\ \frac{(1-\alpha_j)(1-c^{-1})}{(c\alpha_h-1)(c\alpha_j-1)}\,n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}F_{k+1,n}(\alpha_j)F_{k+1,n}(c^{-1}).$$

  Then, when $c\alpha_h\neq 1$ and $c\alpha_j\neq 1$, using the first limit in (4.29) we obtain, after some standard calculations,

$$n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j} \xrightarrow{a.s} \frac{1+(c-1)(\alpha_h^{-1}+\alpha_j^{-1})}{c(\alpha_h+\alpha_j)-1}(\mathbf{v}_h^\top \mathbf{v}_j)Z_\infty(1-Z_\infty).$$

  When $c\alpha_h=c\alpha_j=1$, we have that $n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j}$ has the same limit as

$$(1-c^{-1})^2\,n\sum_{k=m_0}^{n-1}\ln^2(n/k)r_k^2[B_{k+1}]_{h,j}F^2_{k+1,n}(c^{-1})$$
$$+\ 2c^{-1}(1-c^{-1})\,n\sum_{k=m_0}^{n-1}\ln(n/k)r_k^2[B_{k+1}]_{h,j}F^2_{k+1,n}(c^{-1})$$
$$+\ c^{-2}\,n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}F^2_{k+1,n}(c^{-1}),$$

  from which, using the three limits in (Aletti, Crimaldi and Ghiglietti, 2019, (4.29)), we obtain

$$n\sum_{k=m_0}^{n-1} r_k^2[B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j} \xrightarrow{a.s} (1+2c(c-1))(\mathbf{v}_h^\top \mathbf{v}_j)Z_\infty(1-Z_\infty).$$

Finally, when $c\alpha_h \neq 1$ and $c\alpha_j = 1$, we have that $n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j}$ has the same limit as

$$\frac{(1-c^{-1})^2}{(c\alpha_h - 1)}\, n\sum_{k=m_0}^{n-1} \ln(n/k) r_k^2 [B_{k+1}]_{h,j} F^2_{k+1,n}(c^{-1}) \\ + \frac{c^{-1}(1-c^{-1})}{(c\alpha_h - 1)}\, n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F^2_{k+1,n}(c^{-1}) \\ - \frac{(1-\alpha_h)(1-c^{-1})}{(c\alpha_h - 1)}\, n\sum_{k=m_0}^{n-1} \ln(n/k) r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(c^{-1}) \\ - \frac{c^{-1}(1-\alpha_h)}{(c\alpha_h - 1)}\, n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(c^{-1}),$$

which implies, using the first two limits in (Aletti, Crimaldi and Ghiglietti, 2019, (4.29)), that

$$n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j} \ \xrightarrow{a.s} \ \frac{1+(c-1)(c+\alpha_h^{-1})}{c\alpha_h} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

The case $c\alpha_h = 1$ and $c\alpha_j \neq 1$ is analogous. Therefore, we can summarize the limits in all the above cases with the formula:

$$\frac{1+(c-1)(\alpha_h^{-1}+\alpha_j^{-1})}{c(\alpha_h+\alpha_j)-1} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

- $a.s. - \lim_n n\sum_{k=m_0}^{n-1} r_k^2 (a^2_{k+1,n})^2 b_{k+1}$:
  Using the first limit in (Aletti, Crimaldi and Ghiglietti, 2019, (4.29)), we have

$$n\sum_{k=m_0}^{n-1} r_k^2 (a^2_{k+1,n})^2 b_{k+1} \ = \ (c^{-1}-1)^2 n\sum_{k=m_0}^{n-1} r_k^2 b_{k+1} F^2_{k+1,n}(c^{-1}) \ \xrightarrow{a.s} \ (c-1)^2 \|\mathbf{v}_1\|^2 Z_\infty(1-Z_\infty).$$

- $a.s. - \lim_n n\sum_{k=m_0}^{n-1} r_k^2 [A^1_{k+1,n} B_{k+1} A^3_{k+1,n}]_{h,j}$:
  First, when $c\alpha_j \neq 1$ notice that $n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^1_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j}$ has the same limit as

$$\frac{1-c^{-1}}{c\alpha_j - 1} n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(c^{-1}) - \frac{1-\alpha_j}{c\alpha_j - 1} n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(\alpha_j),$$

and hence, after standard calculations, we obtain

$$n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^1_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j} \ \xrightarrow{a.s} \ \frac{\alpha_h^{-1}(c-1)+c}{c(\alpha_h+\alpha_j)-1} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

When $c\alpha_j = 1$, $n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^1_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j}$ has the same limit as

$$(1-c^{-1}) n\sum_{k=m_0}^{n-1} \ln(n/k) r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(c^{-1}) + c^{-1} n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(c^{-1}),$$

and hence

$$n\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^1_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j} \ \xrightarrow{a.s} \ \frac{\alpha_h^{-1}(c-1)+c}{c\alpha_h} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

Therefore we can summarize the limits of the above two cases with the formula

$$\frac{\alpha_h^{-1}(c-1)+c}{c(\alpha_h+\alpha_j)-1}(\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

- $a.s.-\lim_n n\sum_{k=m_0}^{n-1} r_k^2 a_{k+1,n}^2 [\mathbf{b}_{k+1}^\top A_{k+1,n}^1]_j$:
Notice that

$$n\sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j [A_{k+1,n}^1]_{jj} a_{k+1,n}^2 = (c^{-1}-1)n\sum_{k=m_0}^{n-1} r_k^2[\mathbf{b}_{k+1}]_j F_{k+1,n}(\alpha_j)F_{k+1,n}(c^{-1}),$$

which implies that

$$n\sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j [A_{k+1,n}^1]_{jj} a_{k+1,n}^2 \xrightarrow{a.s} \frac{1-c}{\alpha_j}(\mathbf{v}_1^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

- $a.s.-\lim_n n\sum_{k=m_0}^{n-1} r_k^2 a_{k+1,n}^2 [\mathbf{b}_{k+1}^\top A_{k+1,n}^3]_j$:
First, when $c\alpha_j \neq 1$, notice that
$n\sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j [A_{k+1,n}^3]_{jj} a_{k+1,n}^2$ has the same limit as

$$\frac{(1-c^{-1})(1-\alpha_j)}{c\alpha_j-1} n\sum_{k=m_0}^{n-1} r_k^2[\mathbf{b}_{k+1}]_j F_{k+1,n}(\alpha_j)F_{k+1,n}(c^{-1})$$
$$-\frac{(1-c^{-1})^2}{c\alpha_j-1} n\sum_{k=m_0}^{n-1} r_k^2[\mathbf{b}_{k+1}]_j F_{k+1,n}^2(c^{-1}),$$

which implies after some calculations

$$n\sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j [A_{k+1,n}^3]_{jj} a_{k+1,n}^2 \xrightarrow{a.s} \frac{1-c}{\alpha_j}(\mathbf{v}_1^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

When $c\alpha_j = 1$, $n\sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j [A_{k+1,n}^3]_{jj} a_{k+1,n}^2$ has the same limit as

$$-(1-c^{-1})^2 n\sum_{k=m_0}^{n-1} \ln(n/k) r_k^2[\mathbf{b}_{k+1}]_j F_{k+1,n}^2(c^{-1}) - c^{-1}(1-c^{-1})n\sum_{k=m_0}^{n-1} r_k^2[\mathbf{b}_{k+1}]_j F_{k+1,n}^2(c^{-1}),$$

from which we can obtain

$$n\sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j [A_{k+1,n}^3]_{jj} a_{k+1,n}^2 \xrightarrow{a.s} c(1-c)(\mathbf{v}_1^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

Therefore, we can summarize the limits of the above two cases with the formula

$$\frac{1-c}{\alpha_j}(\mathbf{v}_1^\top \mathbf{v}_j) Z_\infty(1-Z_\infty).$$

A.3.2. Computations for the almost sure limits of the elements in (Aletti, Crimaldi and Ghiglietti, 2019, (4.30))

- $a.s. - \lim_n \frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [A^1_{k+1,n} B_{k+1} A^1_{k+1,n}]_{h,j}$:
  By using (Aletti, Crimaldi and Ghiglietti, 2019, (4.31)), we have

$$\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^1_{k+1,n}]_{h,h}[A^1_{k+1,n}]_{j,j} = \frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h)F_{k+1,n}(\alpha_j)$$
$$\xrightarrow{a.s} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty(1-Z_\infty)\begin{cases} c^2 & \text{if } b_{\alpha_h}+b_{\alpha_j}=0,\\ 0 & \text{if } b_{\alpha_h}+b_{\alpha_j}\neq 0.\end{cases}$$

- $a.s. - \lim_n \frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [A^3_{k+1,n} B_{k+1} A^3_{k+1,n}]_{h,j}$:
  Since $c(\alpha_h+\alpha_j)=1$ implies $c\alpha_h \neq 1$ and $c\alpha_j\neq 1$, we have that

$$\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j}$$

has the same limit as

$$\frac{(1-c^{-1})^2}{(c\alpha_h-1)(c\alpha_j-1)}\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F^2_{k+1,n}(c^{-1})$$
$$+ \frac{(1-\alpha_h)(1-\alpha_j)}{(c\alpha_h-1)(c\alpha_j-1)}\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h)F_{k+1,n}(\alpha_j)$$
$$- \frac{(1-\alpha_h)(1-c^{-1})}{(c\alpha_h-1)(c\alpha_j-1)}\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h)F_{k+1,n}(c^{-1})$$
$$- \frac{(1-\alpha_j)(1-c^{-1})}{(c\alpha_h-1)(c\alpha_j-1)}\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_j)F_{k+1,n}(c^{-1}),$$

which is equal to

$$o(1) + \left(\frac{(\alpha_h-1)(\alpha_j-1)}{c^2\alpha_h\alpha_j}\right)\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h)F_{k+1,n}(\alpha_j).$$

Hence, we have that

$$\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j}[A^3_{k+1,n}]_{h,h}[A^3_{k+1,n}]_{j,j}$$
$$\xrightarrow{a.s} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty(1-Z_\infty)\begin{cases} \frac{(\alpha_h-1)(\alpha_j-1)}{\alpha_h\alpha_j} & \text{if } b_{\alpha_h}+b_{\alpha_j}=0,\\ 0 & \text{if } b_{\alpha_h}+b_{\alpha_j}\neq 0.\end{cases}$$

- $a.s. - \lim_n \frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 b_{k+1}(a^2_{k+1,n})^2$:
  Since the calculations are analogous to those in Subsection A.3.1, we have

$$\frac{n}{\ln(n)}\sum_{k=m_0}^{n-1} r_k^2 b_{k+1}(a^2_{k+1,n})^2 \xrightarrow{a.s} 0.$$

- $a.s. - \lim_n \frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [A^1_{k+1,n} B_{k+1} A^3_{k+1,n}]_{h,j}$:
Since $c(\alpha_h + \alpha_j) = 1$ implies $c\alpha_j \neq 1$, we have that

$$\frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} [A^1_{k+1,n}]_{h,h} [A^3_{k+1,n}]_{j,j}$$

has the same limit as

$$\begin{aligned} &\left(\frac{1-c^{-1}}{c\alpha_j - 1}\right) \frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(c^{-1}) \\ &\qquad - \left(\frac{1-\alpha_j}{c\alpha_j - 1}\right) \frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(\alpha_j) \\ &= o(1) - \left(\frac{1-\alpha_j}{c\alpha_j - 1}\right) \frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} F_{k+1,n}(\alpha_h) F_{k+1,n}(\alpha_j). \end{aligned}$$

Hence, we have

$$\begin{aligned} &\frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [B_{k+1}]_{h,j} [A^1_{k+1,n}]_{h,h} [A^3_{k+1,n}]_{j,j} \\ &\qquad \xrightarrow{a.s} (\mathbf{v}_h^\top \mathbf{v}_j) Z_\infty (1 - Z_\infty) \begin{cases} \frac{c^2(\alpha_j - 1)}{c\alpha_j - 1} = \frac{c(1-\alpha_j)}{\alpha_h} & \text{if } b_{\alpha_h} + b_{\alpha_j} = 0, \\ 0 & \text{if } b_{\alpha_h} + b_{\alpha_j} \neq 0. \end{cases} \end{aligned}$$

- $a.s. - \lim_n \frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 a^2_{k+1,n} [\mathbf{b}^\top_{k+1} A^1_{k+1,n}]_j$:
Since the calculations are analogous to those in Subsection A.3.1, we have

$$\frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j a^2_{k+1,n} [A^1_{k+1,n}]_{jj} \xrightarrow{a.s} 0.$$

- $a.s. - \lim_n \frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 a^2_{k+1,n} [\mathbf{b}^\top_{k+1} A^3_{k+1,n}]_j$:
Since the calculations are analogous to those in Subsection A.3.1, we have

$$\frac{n}{\ln(n)} \sum_{k=m_0}^{n-1} r_k^2 [\mathbf{b}_{k+1}]_j a^2_{k+1,n} [A^3_{k+1,n}]_{jj} \xrightarrow{a.s} 0.$$

B. Stable convergence and its variants

This brief appendix contains some basic definitions and results concerning stable convergence and its variants. For more details, we refer the reader to Crimaldi (2009, 2016); Crimaldi, Letta and Pratelli (2007); Hall and Heyde (1980) and the references therein.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let S be a Polish space, endowed with its Borel σ-field. A *kernel* on S, or a random probability measure on S, is a collection $K = \{K(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ-field of S such that, for each bounded Borel real function f on S, the map

$$\omega \mapsto Kf(\omega) = \int f(x)\, K(\omega)(dx)$$

is $\mathcal{A}$-measurable. Given a sub-σ-field $\mathcal{H}$ of $\mathcal{A}$, a kernel K is said $\mathcal{H}$-measurable if all the above random variables Kf are $\mathcal{H}$-measurable.

On $(\Omega, \mathcal{A}, P)$, let $(Y_n)_n$ be a sequence of S-valued random variables, let $\mathcal{H}$ be a sub-σ-field of $\mathcal{A}$, and let K be a $\mathcal{H}$-measurable kernel on S. Then we say that Y_n converges $\mathcal{H}$-*stably* to K, and we write $Y_n \longrightarrow K$ $\mathcal{H}$-stably, if

$$P(Y_n \in \cdot \,|\, H) \overset{weakly}{\longrightarrow} E\left[K(\cdot) \,|\, H\right] \qquad \text{for all } H \in \mathcal{H} \text{ with } P(H) > 0,$$

where $K(\cdot)$ denotes the random variable defined, for each Borel set B of S, as $\omega \mapsto KI_B(\omega) = K(\omega)(B)$. In the case when $\mathcal{H} = \mathcal{A}$, we simply say that Y_n converges *stably* to K and we write $Y_n \longrightarrow K$ stably. Clearly, if $Y_n \longrightarrow K$ $\mathcal{H}$-stably, then Y_n converges in distribution to the probability distribution $E[K(\cdot)]$. Moreover, the $\mathcal{H}$-stable convergence of Y_n to K can be stated in terms of the following convergence of conditional expectations:

$$E[f(Y_n) \,|\, \mathcal{H}] \overset{\sigma(L^1,\, L^\infty)}{\longrightarrow} Kf \tag{B.1}$$

for each bounded continuous real function f on S.

In Crimaldi, Letta and Pratelli (2007) the notion of $\mathcal{H}$-stable convergence is firstly generalized in a natural way replacing in (B.1) the single sub-σ-field $\mathcal{H}$ by a collection $\mathcal{G} = (\mathcal{G}_n)_n$ (called conditioning system) of sub-σ-fields of $\mathcal{A}$ and then it is strengthened by substituting the convergence in $\sigma(L^1, L^\infty)$ by the one in probability (i.e. in L^1, since f is bounded). Hence, according to Crimaldi, Letta and Pratelli (2007), we say that Y_n converges to K *stably in the strong sense*, with respect to $\mathcal{G} = (\mathcal{G}_n)_n$, if

$$E\left[f(Y_n) \,|\, \mathcal{G}_n\right] \overset{P}{\longrightarrow} Kf \tag{B.2}$$

for each bounded continuous real function f on S.

Finally, a strengthening of the stable convergence in the strong sense can be naturally obtained if in (B.2) we replace the convergence in probability by the almost sure convergence: given a conditioning system $\mathcal{G} = (\mathcal{G}_n)_n$, we say that Y_n converges to K in the sense of the *almost sure conditional convergence*, with respect to $\mathcal{G}$, if

$$E\left[f(Y_n) \,|\, \mathcal{G}_n\right] \overset{a.s.}{\longrightarrow} Kf$$

for each bounded continuous real function f on S. The almost sure conditional convergence has been introduced in Crimaldi (2009) and, subsequently, employed by others in the urn model literature (e.g. Aletti, Ghiglietti and Vidyashankar (2018); Aletti, May and Secchi (2009); Zhang (2014)).
We now conclude this section recalling two convergence results that we need in our proofs.
From (Crimaldi and Pratelli, 2005, Proposition 3.1), we can get the following result.

Theorem B.1. *Let $(\mathbf{T}_{n,k})_{n\geq 1, 1\leq k\leq k_n}$ be a triangular array of d-dimensional real random vectors, such that, for each fixed n, the finite sequence $(\mathbf{T}_{n,k})_{1\leq k\leq k_n}$ is a martingale difference array with respect to a given filtration $(\mathcal{G}_{n,k})_{k\geq 0}$. Moreover, let $(t_n)_n$ be a sequence of real numbers and assume that the following conditions hold:*

(c1) $\mathcal{G}_{n,k} \subseteq \mathcal{G}_{n+1,k}$ *for each n and $1 \leq k \leq k_n$;*
(c2) $\sum_{k=1}^{k_n} (t_n \mathbf{T}_{n,k})(t_n \mathbf{T}_{n,k})^\top = t_n^2 \sum_{k=1}^{k_n} \mathbf{T}_{n,k} \mathbf{T}_{n,k}^\top \overset{P}{\longrightarrow} \Sigma$, *where Σ is a random positive semidefinite matrix;*
(c3) $\sup_{1\leq k\leq k_n} |t_n \mathbf{T}_{n,k}| \overset{L^1}{\longrightarrow} 0$.

Then $t_n \sum_{k=1}^{k_n} \mathbf{T}_{n,k}$ converges stably to the Gaussian kernel $\mathcal{N}(\mathbf{0}, \Sigma)$.

The following result combines together a stable convergence and a stable convergence in the strong sense.

Theorem B.2. *(Berti et al., 2011, Lemma 1) Suppose that C_n and D_n are S-valued random variables, that M and N are kernels on S, and that $\mathcal{G} = (\mathcal{G}_n)_n$ is a filtration satisfying for all n*

$$\sigma(C_n) \subseteq \mathcal{G}_n \quad \textit{and} \quad \sigma(D_n) \subseteq \sigma\left(\textstyle\bigcup_n \mathcal{G}_n\right)$$

If C_n stably converges to M and D_n converges to N stably in the strong sense, with respect to $\mathcal{G}$, then

$$(C_n, D_n) \longrightarrow M \otimes N \qquad \textit{stably.}$$

(Here, $M \otimes N$ is the kernel on $S \times S$ such that $(M \otimes N)(\omega) = M(\omega) \otimes N(\omega)$ for all ω.)

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