On the Similarity Between Two Popular Tube MPC Formulations

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***Abstract*— Tube Model Predictive Control (TMPC) guarantees that a set of prescribed constraints is satisfied for all possible realizations of a bounded disturbance acting on the controlled system. In this paper, we compare two popular TMPC schemes, highlight the many similarities and few differences and close a theoretical gap by proving the existence of a unifying Lyapunov function. We demonstrate the theoretical results with the help of a simple example in simulations which highlights a case in which the two approaches differ.**

I. Introduction

Model Predictive Control (MPC) has become popular thanks to its ability to stabilize dynamical systems while enforcing a set of constraints and minimizing a prescribed cost function. Of the many developed variants, robust MPC formulations explicitly account for bounded disturbances and ensure that the constraints are satisfied for all possible realizations of the disturbances acting on the system. While some approaches robustly minimize the cost by minmax formulations which are arguably difficult to solve, in tube MPC (TMPC) schemes only the constraints are enforced robustly and some form of nominal cost is minimized.

The many variants of TMPC developed over the years, can essentially be classified as variations of two approaches which have been developed almost simultaneously in [1] and [2], [3]. Generalizations of tube MPC to homotetic, elastic and parametrized tubes [4], [5], [6] are out of the scope of this paper. Excellent surveys on MPC and TMPC can be found in [7], [8], [9].

In this paper, we focus on the two approaches in [1] and [2], [3] and we refer to them as CTMPC and MTMPC respectively, referring to the names of the first authors of the paper proposing them. For CTMPC, [1] proves attractivity and [10] proves asymptotic stability to the minimum robust positive invariant (mRPI) set. For MTMPC, [3] proves asymptotic stability to the mRPI set. From the formulation point of view, the main difference between the two approaches is that in CTMPC a time-varying constraint tightening procedure is required, while in MTMPC constraint tightening is time-invariant and performed with respect to the mRPI set, such that either the mRPI or an outer approximation needs to be computed explicitly. Additionally, in MTMPC the initial state does not need to coincide with the current system state.

In this paper, we aim at closing a theoretical gap. While it is well-known that the two approaches are very similar, we prove that there exists a Lyapunov function for CTMPT which is closely related to the one for MTMPC (see Corollary 3, Theorem 1). Moreover, we prove that while in most cases the controls computed by the two approaches coincide, they differ in the few cases in which the constraint tightening used in MTMPC is slightly more conservative than the one used on CTMPC (see Lemmas 1, 2). While the proposed results are arguably not unexpected, they are—to our best knowledge—lacking in the existing literature.

This paper is structured as follows. We introduce TMPC in Section II, we compare the domain of CTMPC and MTMPC in Section III, and propose a Lyapunov function for CTMPC in Section IV. Finally, we provide a simulation example in Section V and conclude the paper in Section VI.

II. Tube MPC Formulations

In this section, we introduce the two TMPC formulations and recall how constraint tightening is performed. A thorough review of all methods developed for TMPC is beyond the scope of this paper and we refer the interested reader to [7], [8] and references therein.

Consider the linear system

$$\mathbf{s}_{k+1} = A\mathbf{s}_k + B\mathbf{a}_k + \mathbf{w}_k, \tag{1}$$

with state $\mathbf{s}$, control $\mathbf{a}$ and disturbance $\mathbf{w} \in \mathbb{W}$. We assume that $\mathbb{W}$ is a compact polytope and aim at stabilizing system (1) while enforcing constraints

$$C\mathbf{s}_k + D\mathbf{a}_k + \hat{\mathbf{c}} \leq 0, \qquad \forall\, \mathbf{w}_j \in \mathbb{W},\ j \in \mathbb{I}_0^{k-1},\ k \in \mathbb{I}_0^{\infty}, \tag{2}$$

where C, D, and $\hat{\mathbf{c}}$ are known data of the problem.

A. CTMPC

We define CTMPC as

$$\begin{aligned}
\bar{V}(\mathbf{s}_i) = \min_{\mathbf{z}} \quad & \sum_{k=0}^{N-1} \ell(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{x}_N^\top P \mathbf{x}_N && \text{(3a)}\\
\text{s.t.} \quad & \mathbf{x}_0 = \mathbf{s}_i, && \text{(3b)}\\
& \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \qquad k \in \mathbb{I}_0^{N-1}, && \text{(3c)}\\
& C\mathbf{x}_k + D\mathbf{u}_k + \mathbf{c}_k \leq \mathbf{0}, \qquad k \in \mathbb{I}_0^{N-1}, && \text{(3d)}\\
& G\mathbf{x}_N + \mathbf{g} \leq \mathbf{0}, && \text{(3e)}
\end{aligned}$$

where $\mathbf{z} := (\mathbf{x}_0, \mathbf{u}_0, \ldots, \mathbf{u}_{N-1}, \mathbf{x}_N)$ and

$$\ell(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}^\top H \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \qquad H = \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \succ 0. \tag{4}$$

We will denote the optimal solution of (3) as $\mathbf{z}^i := (\mathbf{x}_0^i, \mathbf{u}_0^i, \ldots, \mathbf{u}_{N-1}^i, \mathbf{x}_N^i)$. At each time instant i, the control $\mathbf{a}_i = \mathbf{u}_0^i$ is applied to the system. Then the state is measured again and the CTMPC problem solved again.

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Note that in the original paper [1], the problem was formulated using a different cost, which we will define in Equation (23), based on stage cost (20). In the following, we will refer to feedback matrix K as the optimal LQR feedback for stage cost (4) and we define the corresponding minimum RPI (mRPI) set as [11]

$$\mathbb{X}_{\mathrm{m}}^K := \bigoplus_{j=0}^{\infty} A_K^j \mathbb{W}. \tag{5}$$

In order to compensate for the disturbance acting on the system, the path constraints (3d) do not coincide with the constraints (2). To perform the required constraint tightening, i.e., the computation of $\mathbf{c}_k$, we define the prediction error $\mathbf{e}_k := \mathbf{s}_k - \mathbf{x}_k$, satisfying $\mathbf{e}_k \in \mathbb{E}_k$, with

$$\mathbb{E}_{k+1} = (A - BK)\mathbb{E}_k \oplus \mathbb{W}, \qquad \mathbb{E}_0 = \{\mathbf{0}\}.$$

Set $\mathbb{E}_k$ predicts the tube around the predicted trajectory, i.e., the true system state will satisfy $\mathbf{s}_k \in \mathbf{x}_k \oplus \mathbb{E}_k$, $k = 0, \ldots, N$, when using the control law $\mathbf{a}_k = \mathbf{u}_k - K(\mathbf{s}_k - \mathbf{x}_k)$.

Feedback matrix K is used in order to model the fact that any closed-loop strategy will compensate for perturbations on the nominal model. For ease of notation, we define

$$C_K := C - DK, \qquad A_K := A - BK.$$

Robust constraint satisfaction is obtained if (2) holds, i.e.,

$$C\mathbf{x}_k + D\mathbf{u}_k + C_K\mathbf{e}_k + \hat{\mathbf{c}} \leq \mathbf{0}, \qquad \forall\ \mathbf{e}_k \in \mathbb{E}_k, \tag{6}$$

such that $\mathbf{c}_k$ is computed in a worst-case scenario as follows. For each component i of the path constraint at time k let

$$\begin{aligned}\mathbf{d}_{i,k} &:= \max_{\mathbf{e}}\ (C_K)_i\mathbf{e} && \text{s.t. } \mathbf{e} \in \mathbb{E}_k \\ &= \max_{\mathbf{w}}\ (C_K)_i \sum_{j=0}^{k-1} A_K^{k-1-j}\mathbf{w}_j && \text{s.t. } \mathbf{w}_j \in \mathbb{W}.\end{aligned} \tag{7}$$

We lump all components $\mathbf{d}_{i,k}$ in vector $\mathbf{d}_k$. Then, constraint satisfaction is obtained for all $\mathbf{w}_k \in \mathbb{W}$ if $\mathbf{c}_k = \hat{\mathbf{c}} + \mathbf{d}_k$.

In order to guarantee that CTMPC remains feasible at all times for all $\mathbf{w}_k \in \mathbb{W}$, one needs to impose ad-hoc terminal conditions: the terminal set $\mathbb{X}_{\mathrm{f}} := \{\, \mathbf{x} \,|\, G\mathbf{x} + \mathbf{g} \leq 0 \,\}$ must be such that there exists a terminal control law κ_{f} which guarantees that the state error $\mathbf{e}$ does not diverge and all path constraints are always satisfied. We consider $\kappa_{\mathrm{f}}(\mathbf{x}) = -K\mathbf{x}$, and define

$$\begin{aligned}&\mathbb{X}_0 := \{\ \mathbf{x} \mid C_K\mathbf{x} + \mathbf{c}_0 \leq 0\ \}, \\ &\mathbb{X}_k := \{\ \mathbf{x} \in A_K\mathbb{X}_{k-1} \oplus \mathbb{W} \mid C_K\mathbf{x} + \mathbf{c}_k \leq 0\ \}.\end{aligned}$$

Set $\mathbb{X}_\infty$ is Maximum Robust Positive Invariant (MRPI) and output admissible [11]. Additionally, whenever A_K is stable and the origin is in the interior of the constraint set, the MRPI is finitely determined [11, Theorem 6.3], i.e., $\exists\ k' < \infty$ s.t. $\mathbb{X}_{k'} \equiv \mathbb{X}_{k'+i}$, $i = 1, 2, \ldots$.

Consistently with the previously used notation, we define

$$G := \begin{bmatrix} C_K \\ C_K A_K \\ \vdots \\ C_K A_K^{k'} \end{bmatrix}, \qquad \hat{\mathbf{g}} := \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_{k'} \end{bmatrix}. \tag{8}$$

The condition $G\mathbf{x}_N + \hat{\mathbf{g}} \leq 0$ then guarantees robust constraint satisfaction for all future times provided that $\mathbf{s}_N = \mathbf{x}_N$. However, $\mathbf{s}_N = \mathbf{x}_N + \mathbf{e}_N$, such that also the terminal constraint must be tightened. Analogously to the case of path constraints, we define $\mathbf{g} = \hat{\mathbf{g}} + \mathbf{h}$ with

$$\mathbf{h}_{i,k} := \max_{\mathbf{w}}\ G_i \sum_{j=0}^{N-1} A_K^j \mathbf{w}_j \qquad \text{s.t. } \mathbf{w}_j \in \mathbb{W}. \tag{9}$$

Proposition 1 ([1]): Provided that the initial state $\mathbf{s}_0$ is feasible, system (1) under feedback $\mathbf{a}_i = \mathbf{u}_0^i$ satisfies the following properties: (i) constraints (2) are satisfied at all times; (ii) $\lim_{i\to\infty} \mathbf{a}^i = -K\mathbf{s}_i$; and (iii) $\mathbf{s}_i \to \mathbb{X}_{\mathrm{m}}^K$, with $\mathbb{X}_{\mathrm{m}}^K$ from (5).

This result proves recursive feasibility and convergence, but not asymptotic stability. Additionally, the Bellman function used in [1] to prove convergence is not a Lyapunov function. We remark that the stage cost used to define this Bellman function is indefinite and violates the observability assumption typically used to prove nominal asymptotic stability in linear MPC. Finally, asymptotic stability is proven in [10] where, however, no Lyapunov function is explicitly given and no comparison with MTMPC is made.

B. MTMPC

We define MTMPC as

$$\begin{aligned}\tilde{V}(\mathbf{s}_i) := \min_{\tilde{\mathbf{z}}}\quad & \sum_{k=0}^{N-1} \ell(\tilde{\mathbf{x}}_k, \tilde{\mathbf{u}}_k) + \tilde{\mathbf{x}}_N^\top P \tilde{\mathbf{x}}_N && \text{(10a)} \\ \text{s.t.}\quad & \mathbf{s}_i \in \tilde{\mathbf{x}}_0 \oplus \mathbb{X}_{\mathrm{m}}^K, && \text{(10b)} \\ & \tilde{\mathbf{x}}_{k+1} = A\tilde{\mathbf{x}}_k + B\tilde{\mathbf{u}}_k, \quad k \in \mathbb{I}_0^{N-1}, && \text{(10c)} \\ & C\tilde{\mathbf{x}}_k + D\tilde{\mathbf{u}}_k + \tilde{\mathbf{c}} \leq \mathbf{0}, \quad k \in \mathbb{I}_0^{N-1}, && \text{(10d)} \\ & G\tilde{\mathbf{x}}_N + \tilde{\mathbf{g}} \leq \mathbf{0}, && \text{(10e)}\end{aligned}$$

where (10b) constrains $\tilde{\mathbf{x}}_0$, $\tilde{\mathbf{c}} := \hat{\mathbf{c}} + \tilde{\mathbf{d}}$, and $\tilde{\mathbf{g}} := \hat{\mathbf{g}} + \tilde{\mathbf{h}}$ with

$$\tilde{\mathbf{d}}_i := \max_{\Delta\tilde{\mathbf{x}}}\ (C_K)_i\Delta\tilde{\mathbf{x}} \qquad \text{s.t. } \Delta\tilde{\mathbf{x}} \in \mathbb{X}_{\mathrm{m}}^K, \tag{11}$$

$$\tilde{\mathbf{h}}_i := \max_{\Delta\tilde{\mathbf{x}}}\ G_i\Delta\tilde{\mathbf{x}} \qquad \text{s.t. } \Delta\tilde{\mathbf{x}} \in \mathbb{X}_{\mathrm{m}}^K. \tag{12}$$

We denote the optimal solution of (10) for a given $\mathbf{s}_i$ as $\tilde{\mathbf{z}}^i := (\tilde{\mathbf{x}}_0^i, \tilde{\mathbf{u}}_0^i, \ldots, \tilde{\mathbf{u}}_{N-1}^i, \tilde{\mathbf{x}}_N^i)$, and we use $\tilde{\cdot}$ to distinguish MTMPC from CTMPC. The control applied to the system is

$$\mathbf{a}_i = \tilde{\mathbf{u}}_0^i - K(\mathbf{s}_i - \tilde{\mathbf{x}}_0^i), \tag{13}$$

and the MTMPC problem is solved again at each time instant i based on the new state measurement.

Asymptotic stability for MTMPC has been proven in [3] using $\tilde{V}$ as a Lyapunov function, as we recall next.

Proposition 2 ([3]): Function $\tilde{V}$ is a Lyapunov function for system (1) with respect to the mRPI set $\mathbb{X}_{\mathrm{m}}^K$.

III. CTMPC and MTMPC Domain

In this section, we compare the feasible domain (which coincides the region of attraction, i.e., the set of states for which the closed loop is asymptotically stable) of CTMPC and MTMPC. To that end, observe that for a given initial state $\mathbf{s}_i$ the MTMPC prediction $\tilde{\mathbf{z}}$ has the corresponding CTMPC prediction $\mathbf{z}$ (and vice-versa) given by:

$$\mathbf{u}_k = \tilde{\mathbf{u}}_k - K(\mathbf{x}_k - \tilde{\mathbf{x}}_k) \tag{14a}$$
$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\left(\tilde{\mathbf{u}}_k - K(\mathbf{x}_k - \tilde{\mathbf{x}}_k)\right), \quad \mathbf{x}_0 = \mathbf{s}_i. \tag{14b}$$

We prove in the next two lemmas that, while all trajectories that are feasible for MTMPC are feasible for CTMPC, the converse is not true.

Lemma 1: The state and control trajectory $\mathbf{x}_k$, $\mathbf{u}_k$ given by applying (14) to any trajectory $\tilde{\mathbf{x}}_k$, $\tilde{\mathbf{u}}_k$ which is feasible for the MTMPC Problem (10) is always feasible for the CTMPC Problem (3).

Proof: By applying $\mathbf{u}_k$ to the real system (1) we obtain that for any process noise $\mathbf{w}_k$, the true state is given by

$$\begin{aligned}\mathbf{s}_{i+k+1} &= A\mathbf{s}_{i+k} + B\left(\tilde{\mathbf{u}}_k - K(\mathbf{s}_{i+k} - \tilde{\mathbf{x}}_k)\right) + \mathbf{w}_k \\ &= (A - BK)\mathbf{s}_{i+k} + B\left(\tilde{\mathbf{u}}_k + K\tilde{\mathbf{x}}_k\right) + \mathbf{w}_k,\end{aligned} \tag{15}$$

and (14) equivalently reads

$$\mathbf{x}_{k+1} = (A - BK)\mathbf{x}_k + B\left(\tilde{\mathbf{u}}_k + K\tilde{\mathbf{x}}_k\right).$$

By (15), the prediction error $\mathbf{e}_k = \mathbf{s}_{i+k} - \mathbf{x}_k$ satisfies

$$\mathbf{e}_{k+1} = (A - BK)\mathbf{e}_k + \mathbf{w}_k.$$

For all $\mathbf{w}_j \in \mathbb{W}$, the trajectory is feasible if it satisfies

$$\begin{aligned}0 \geq C\mathbf{s}_{i+k} &+ D\left(\mathbf{u}_k - K(\mathbf{s}_{i+k} - \mathbf{x}_k)\right) + \hat{\mathbf{c}} \\ &= C(\mathbf{e}_k + \mathbf{x}_k) + D\left(\mathbf{u}_k - K\mathbf{e}_k\right) + \hat{\mathbf{c}} \\ &= C\mathbf{x}_k + D\mathbf{u}_k + C_K\mathbf{e}_k + \hat{\mathbf{c}} \\ &= C\mathbf{x}_k + D\mathbf{u}_k + C_K\sum_{j=0}^{k-1} A_K^{k-1-j}\mathbf{w}_j + \hat{\mathbf{c}}.\end{aligned}$$

By definition of $\mathbf{d}_k$, this condition is equivalent to

$$C\mathbf{x}_k + D\mathbf{u}_k + \mathbf{d}_k + \hat{\mathbf{c}} \leq 0,$$

such that feasibility holds by construction, provided that $\mathbf{w}_k \in \mathbb{W}$. ■

We prove next that the converse of Lemma 1 does not hold in general.

Lemma 2: The state and control trajectories $\tilde{\mathbf{x}}_k$, $\tilde{\mathbf{u}}_k$ given by applying (14) to a trajectory which is feasible for the CTMPC Problem (3) with

$$\tilde{\mathbf{x}}_k = \mathbf{x}_k + \Delta\mathbf{x}_k, \quad \tilde{\mathbf{u}}_k = \mathbf{u}_k - K\Delta\mathbf{x}_k, \quad -\Delta\mathbf{x}_0 \in \mathbb{X}_{\mathrm{m}}^K,$$

might not be feasible for the MTMPC Problem (10).

Proof: By applying $\mathbf{u}_k$ to the real system (1) we obtain

$$\begin{aligned}\mathbf{s}_{i+k+1} &= A\mathbf{s}_{i+k} + B\left(\mathbf{u}_k - K(\mathbf{s}_{i+k} - \mathbf{x}_k)\right) + \mathbf{w}_k, \\ &= (A - BK)\mathbf{s}_{i+k} + B\left(\mathbf{u}_k + K\mathbf{x}_k\right)) + \mathbf{w}_k.\end{aligned} \tag{16}$$

We remind that

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= A\tilde{\mathbf{x}}_k + B\left(\mathbf{u}_k - K(\tilde{\mathbf{x}}_k - \mathbf{x}_k)\right), \\ &= (A - BK)\tilde{\mathbf{x}}_k + B\left(\mathbf{u}_k + K\mathbf{x}_k)\right),\end{aligned}$$

such that, the prediction error $\tilde{\mathbf{e}}_k = \mathbf{s}_{i+k} - \tilde{\mathbf{x}}_k$ satisfies

$$\tilde{\mathbf{e}}_{k+1} = (A - BK)\tilde{\mathbf{e}}_k + \mathbf{w}_k, \qquad \tilde{\mathbf{e}}_0 = \mathbf{x}_k - \tilde{\mathbf{x}}_k,$$

since $\mathbf{x}_k = \mathbf{s}_{i+k}$. By definition, we have $\tilde{\mathbf{e}}_0 = -\Delta\mathbf{x}_0 \in \mathbb{X}_{\mathrm{m}}^K$.

For all $\mathbf{w}_j \in \mathbb{W}$, the trajectory is feasible if it satisfies

$$\begin{aligned}0 \geq C\mathbf{s}_{i+k} &+ D\left(\tilde{\mathbf{u}}_k - K(\mathbf{s}_{i+k} - \tilde{\mathbf{x}}_k)\right) + \hat{\mathbf{c}} \\ &= C(\mathbf{e}_k + \tilde{\mathbf{x}}_k) + D\left(\tilde{\mathbf{u}}_k - K\tilde{\mathbf{e}}_k\right) + \hat{\mathbf{c}} \\ &= C\tilde{\mathbf{x}}_k + D\tilde{\mathbf{u}}_k + C_K\tilde{\mathbf{e}}_k + \hat{\mathbf{c}} \\ &= C\tilde{\mathbf{x}}_k + D\tilde{\mathbf{u}}_k + C_K\left(\Delta\mathbf{x}_0 + \sum_{j=0}^{k-1} A_K^{k-1-j}\mathbf{w}_j\right) + \hat{\mathbf{c}}.\end{aligned} \tag{17}$$

However, differently from the case of Lemma 1, we cannot establish an equivalence in this case, since

$$C\tilde{\mathbf{x}}_k + D\tilde{\mathbf{u}}_k + \tilde{\mathbf{d}} + \hat{\mathbf{c}} \leq 0 \tag{18}$$

implies that (17) holds for all $-\Delta\mathbf{x}_0 \in \mathbb{X}_{\mathrm{m}}^K$. However, since (17) holds for a specific $-\Delta\mathbf{x}_0 \in \mathbb{X}_{\mathrm{m}}^K$, we have

$$(18) \Rightarrow (17),$$

but the converse does in general not hold. ■

We will show in Section V through an example that there do exist cases in which the solution of CTMPC is not feasible for MTMPC, such that CTMPC is in general less conservative than MTMPC and has a larger domain.

IV. A Lyapunov Function for CTMPC

In this section, we define a Lyapunov function $V(\mathbf{s}_i)$ for CTMPC using ideas similar to the approach used in [3]. To that end, using the optimal trajectory $\mathbf{z}^i$ from (3), we define the auxiliary linear quadratic (LQ) problem

$$\begin{aligned}V(\mathbf{s}_i) := \min_{\Delta\mathbf{x}} &\sum_{k=0}^{N-1} \ell(\mathbf{x}_k^i + \Delta\mathbf{x}_k, \mathbf{u}_k^i - K\Delta\mathbf{x}_k) \\ &+ (\mathbf{x}_N^i + \Delta\mathbf{x}_N)^\top P(\mathbf{x}_N^i + \Delta\mathbf{x}_N) \quad (19a) \\ \text{s.t. } &\mathbf{0} \in \Delta\mathbf{x}_0 + \mathbb{X}_{\mathrm{m}}^K, \quad (19b) \\ &\Delta\mathbf{x}_{k+1} = (A - BK)\Delta\mathbf{x}_k, \ \ k \in \mathbb{I}_0^{N-1}, \quad (19c)\end{aligned}$$

where the mRPI set $\mathbb{X}_{\mathrm{m}}^K$ is defined in (5). We denote the optimal solution of (19) as $\Delta\mathbf{x}^i = (\Delta\mathbf{x}_0^i, \ldots, \Delta\mathbf{x}_N^i)$.

It should be noted that the auxiliary Problem (19) shares some strong similarities with the MTMPC problem. In both cases the initial state is constrained to lie in the mRPI set and does not need to coincide with the actual system state. Moreover, the two problems minimize the same cost, as we will prove in Corollary 3. The main difference is highlighted by Lemma 2, which proves that the domain of MTMPC is smaller than the one of CTMPC. Finally, it ought to be noted that since all $\Delta\mathbf{x}_k$, $k \in \mathbb{I}_1^N$ are fixed by constraints (19c), the only remaining degree of freedom in (19) is $\Delta\mathbf{x}_0$, as we will prove in Lemma 4.

In order to prove that V is a Lyapunov function, we establish first some useful intermediate results. To that end, we first define the following stage cost, which penalizes deviations from the optimal LQR feedback

$$\ell^{\mathrm{d}}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}^\top \begin{bmatrix}K^\top\Gamma K & K^\top\Gamma \\ \Gamma K & \Gamma\end{bmatrix} \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix} = \|\mathbf{u} + K\mathbf{x}\|_\Gamma^2, \tag{20}$$

where $\Gamma := R + B^\top PB$, with K and P the optimal LQR feedback and cost-to-go matrix satisfying the Discrete Algebraic Riccati Equation (DARE)

$$K = (R + B^\top PB)^{-1}(S + B^\top PA), \tag{21a}$$
$$P = Q + A^\top PA - (S^\top + A^\top PB)K. \tag{21b}$$

We then recall the following known equivalence.

Lemma 3: For any state and input trajectory $\mathbf{x}_k, \mathbf{u}_k$ satisfying $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, the following holds

$$\sum_{k=0}^{N-1} \ell(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{x}_N^\top P\mathbf{x}_N = \mathbf{x}_0^\top P\mathbf{x}_0 + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k, \mathbf{u}_k). \tag{22}$$

Proof: We use the DARE (21) to obtain

$$K^\top \Gamma K = Q + A^\top PA - P,$$

which we use together with (21) to write

$$\ell^{\mathrm{d}}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix}^\top \begin{bmatrix} Q + A^\top PA - P & S^\top + A^\top PB \\ S + B^\top PA & R + B^\top PB \end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix}$$
$$= \ell(\mathbf{x}, \mathbf{u}) - \mathbf{x}^\top P\mathbf{x} + (A\mathbf{x} + B\mathbf{u})^\top P(A\mathbf{x} + B\mathbf{u}).$$

Consequently, using $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, we obtain

$$\ell(\mathbf{x}_k, \mathbf{u}_k) = \ell^{\mathrm{d}}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{x}_k^\top P\mathbf{x}_k - \mathbf{x}_{k+1}^\top P\mathbf{x}_{k+1},$$

and the proof is completed by simplifying the terms in the obtained telescopic sum. ■

Corollary 1 (Of Lemma 3): The primal solution of Problem (3) is also optimal if (3) is formulated using cost

$$\sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k, \mathbf{u}_k). \tag{23}$$

Proof: By Lemma 3, the two costs only differ by the term $\mathbf{x}_0^\top P\mathbf{x}_0 = \mathbf{s}^\top P\mathbf{s}$, which is constant. ■

Note that CTMPC was formulated in [1] using (23).

Lemma 4: It holds that

$$V(\mathbf{s}_i) = (\mathbf{s}_i + \Delta\mathbf{x}_0^i)^\top P(\mathbf{s}_i + \Delta\mathbf{x}_0^i) + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k^i, \mathbf{u}_k^i), \tag{24}$$

$$\Delta\mathbf{x}_0^i = \arg \min_{-\Delta\mathbf{x}_0 \in \mathbb{X}_{\mathrm{m}}^K} (\mathbf{s}_i + \Delta\mathbf{x}_0)^\top P(\mathbf{s}_i + \Delta\mathbf{x}_0). \tag{25}$$

Proof: We use Lemma 3 to rewrite (19a) as

$$V(\mathbf{s}_i) = (\mathbf{s}_i + \Delta\mathbf{x}_0^i)^\top P(\mathbf{s}_i + \Delta\mathbf{x}_0^i) + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k^i + \Delta\mathbf{x}_k^i, \mathbf{u}_k^i - K\Delta\mathbf{x}_k^i).$$

Then, Equation (24) follows from

$$\ell^{\mathrm{d}}(\mathbf{x}_k^i + \Delta\mathbf{x}_k^i, \mathbf{u}_k^i - K\Delta\mathbf{x}_k^i) = \|\mathbf{u}_k^i - K\Delta\mathbf{x}_k^i + K(\mathbf{x}_k^i + \Delta\mathbf{x}_k^i)\|_\Gamma = \|\mathbf{u}_k^i + K\mathbf{x}_k^i\|_\Gamma = \ell^{\mathrm{d}}(\mathbf{x}_k^i, \mathbf{u}_k^i).$$

Equation (25) is obtained by noting that the second term in the right hand side of (24) does not depend on $\Delta\mathbf{x}_0^i$. ■

Lemma 4 entails that $\mathbf{s}_i \in \mathbb{X}_{\mathrm{m}}^K \Rightarrow \Delta\mathbf{x}_0^i = -\mathbf{s}_i$: this fact will be exploited to prove that $V(\mathbf{s}_i) = 0$ for all $\mathbf{s}_i \in \mathbb{X}_{\mathrm{m}}^K$.

Corollary 2: It holds that

$$V(\mathbf{s}_i) = \bar{V}(\mathbf{s}_i) - \mathbf{s}_i^\top P\mathbf{s}_i + (\mathbf{s}_i + \Delta\mathbf{x}_0^i)^\top P(\mathbf{s}_i + \Delta\mathbf{x}_0^i). \tag{26}$$

Proof: We use Lemma 3 to write

$$\bar{V}(\mathbf{s}_i) = \mathbf{s}_i^\top P\mathbf{s}_i + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k^i, \mathbf{u}_k^i),$$

and use (24) to conclude the proof. ■

Corollary 3: The optimal solutions $\mathbf{z}^i$, $\Delta\mathbf{x}^i$ of Problems (3), (19) are also the optimal solution of

$$\min_{\mathbf{z}, \Delta\mathbf{x}} \ \sum_{k=0}^{N-1} \ell(\mathbf{x}_k + \Delta\mathbf{x}_k, \mathbf{u}_k - K\Delta\mathbf{x}_k) + (\mathbf{x}_N + \Delta\mathbf{x}_N)^\top P(\mathbf{x}_N + \Delta\mathbf{x}_N) \tag{27a}$$
$$\text{s.t.} \quad (3\text{b}) - (3\text{e}), (19\text{b}) - (19\text{c}). \tag{27b}$$

If additionally $\tilde{\mathbf{x}}_k = \mathbf{x}_k + \Delta\mathbf{x}_k$, $\tilde{\mathbf{u}}_k = \mathbf{u}_k - K\Delta\mathbf{x}_k$ is feasible for MTMPC, then the trajectory $\tilde{\mathbf{x}}_k$, $\tilde{\mathbf{u}}_k$ is also optimal for MTMPC, hence implying

$$V(\mathbf{s}_i) \leq \tilde{V}(\mathbf{s}_i).$$

Proof: By Lemma 4 we observe in (24) that the cost (27a) is composed of the two terms $(\mathbf{s}_i + \Delta\mathbf{x}_0)^\top P_0(\mathbf{s}_i + \Delta\mathbf{x}_0)$ and $\sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k, \mathbf{u}_k)$: the first term only depends on variable $\Delta\mathbf{x}$ and the second one only depends on variable $\mathbf{z}$. By Equation (25), the first term is minimized by $\Delta\mathbf{x}^i$. By Corollary 1 the second term is minimized by $\mathbf{z}^i$.

Optimality for MTMPC is obtained by observing that (a) by Lemma 1 the feasible domain of MTMPC is no larger than that of CTMPC; (b) the cost (27a) coincides with that of MTMPC. ■

We can now prove the main result.

Theorem 1: Assume $H \succ 0$. Then V is a Lyapunov function for system (1) with respect to the mRPI set $\mathbb{X}_{\mathrm{m}}^K$.

Proof: By Corollary 2 we have that $\forall\, \mathbf{s}_i \in \mathbb{X}_{\mathrm{m}}^K$,

$$V(\mathbf{s}_i) = 0,$$

since then $\Delta\mathbf{x}_0^i = -\mathbf{s}_i$ and no constraint is active, such that $\bar{V}(\mathbf{s}_i) = \mathbf{s}_i^\top P\mathbf{s}_i$. Moreover, $\exists\, \alpha \in \mathcal{K}_\infty$ such that

$$V(\mathbf{s}_i) \geq \alpha(|\mathbf{s}_i|_{\mathbb{X}_{\mathrm{m}}^K}), \ \forall\, \mathbf{s}_i \quad \Rightarrow \quad V(\mathbf{s}_i) > 0, \ \forall\, \mathbf{s}_i \notin \mathbb{X}_{\mathrm{m}}^K.$$

This fact, and continuity of V imply that the upper bound $V(\mathbf{s}) \leq \bar{\alpha}(|\mathbf{s}_i|_{\mathbb{X}_{\mathrm{m}}^K})$ holds [12] for some $\bar{\alpha} \in \mathcal{K}_\infty$. Note that V is continuous since (3) and (19) are convex QPs [13].

In order to prove decrease, we define the initial guess at time $i+1$, based on the optimal solution of (3) at time i as

$$\bar{\mathbf{x}}_0^{i+1} := \mathbf{s}_{i+1}, \qquad \mathbf{e}_0^{i+1} := \mathbf{w}_i := \mathbf{s}_{i+1} - \mathbf{x}_1^i,$$
$$\bar{\mathbf{x}}_{k+1}^{i+1} := A\bar{\mathbf{x}}_k^{i+1} + B\bar{\mathbf{u}}_k^{i+1}, \qquad \bar{\mathbf{u}}_k^{i+1} := \mathbf{u}_{k+1}^i - K\mathbf{e}_k^{i+1},$$
$$\mathbf{e}_{k+1}^{i+1} := (A - BK)\mathbf{e}_k^{i+1}, \qquad \Delta\bar{\mathbf{x}}_0^{i+1} := \Delta\mathbf{x}_1^i - \mathbf{w}_i,$$
$$\Delta\bar{\mathbf{x}}_{k+1}^{i+1} := (A - BK)\Delta\bar{\mathbf{x}}_k^{i+1}.$$

Since we used the same stabilizing feedback matrix K used for constraint tightening, $\bar{\mathbf{x}}_k^{i+1}$, $\bar{\mathbf{u}}_k^{i+1}$ is a feasible initial guess for Problem (3) at time $i+1$. Moreover, by construction $-\Delta\bar{\mathbf{x}}_0^{i+1} \in \mathbb{X}_{\mathrm{m}}^K$ holds $\forall\, \mathbf{w}_i \in \mathbb{W}$. Therefore, the initial guess $\Delta\bar{\mathbf{x}}_k^{i+1}$ is feasible for Problem (19) at time $i+1$.

We observe that

$$\bar{\mathbf{x}}_k^{i+1} = \mathbf{x}_{k+1}^i + \mathbf{e}_k^{i+1} \qquad \bar{\mathbf{u}}_k^{i+1} = \mathbf{u}_{k+1}^i - K\mathbf{e}_k^{i+1},$$

such that, by Equation (20), we have

$$\begin{aligned}\ell^{\mathrm{d}}(\bar{\mathbf{x}}_{k-1}^{i+1}, \bar{\mathbf{u}}_{k-1}^{i+1}) &= \|\bar{\mathbf{u}}_{k-1}^{i+1} + K\bar{\mathbf{x}}_{k-1}^{i+1}\|_\Gamma \\ &= \|\mathbf{u}_k^i - K\mathbf{e}_{k-1}^{i+1} + K(\mathbf{x}_k^i + \mathbf{e}_{k-1}^{i+1})\|_\Gamma \\ &= \|\mathbf{u}_k^i + K\mathbf{x}_k^i\|_\Gamma = \ell^{\mathrm{d}}(\mathbf{x}_k^i, \mathbf{u}_k^i). \qquad (28)\end{aligned}$$

We define $\mathbf{x}_{N+1}^i := A\mathbf{x}_N^i + B\mathbf{u}_N^i$, $\mathbf{u}_N^i := -K\mathbf{x}_N^i$, and note that this implies that

$$\ell^{\mathrm{d}}(\mathbf{x}_N^i, \mathbf{u}_N^i) = 0. \qquad (29)$$

Then, we can write the cost of the initial guess as

$$\begin{aligned}&J(\mathbf{s}_{i+1}) \\ &= \sum_{k=0}^{N-1} \ell(\bar{\mathbf{x}}_k^{i+1} + \Delta\bar{\mathbf{x}}_k^{i+1}, \bar{\mathbf{u}}_k^{i+1} - K\Delta\bar{\mathbf{x}}_k^{i+1}) \\ &\quad + (\bar{\mathbf{x}}_N^{i+1} + \Delta\bar{\mathbf{x}}_N^{i+1})^\top P(\bar{\mathbf{x}}_N^{i+1} + \Delta\bar{\mathbf{x}}_N^{i+1}) \\ &\overset{(22)}{=} (\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1})^\top P(\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1}) \\ &\qquad + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\bar{\mathbf{x}}_k^{i+1}, \bar{\mathbf{u}}_k^{i+1}) \\ &\overset{(28)}{=} (\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1})^\top P(\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1}) + \sum_{k=1}^{N} \ell^{\mathrm{d}}(\mathbf{x}_k^i, \mathbf{u}_k^i) \\ &\overset{(29)}{=} (\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1})^\top P(\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1}) + \sum_{k=1}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k^i, \mathbf{u}_k^i).\end{aligned} \qquad (30)$$

We observe that, by Equation (22),

$$\begin{aligned}V(\mathbf{s}_i) &= \ell(\mathbf{s}_i + \Delta\mathbf{x}_0^i, \mathbf{u}_0^i - K\Delta\mathbf{x}_0^i) \\ &\quad + (\mathbf{x}_1^i + \Delta\mathbf{x}_1^i)^\top P(\mathbf{x}_1^i + \Delta\mathbf{x}_1^i) + \sum_{k=1}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k^i, \mathbf{u}_k^i).\end{aligned}$$

Since $\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1} = \mathbf{x}_1^i + \mathbf{w} + \Delta\mathbf{x}_1^i - \mathbf{w}$, this entails

$$J(\mathbf{s}_{i+1}) = V(\mathbf{s}_i) - \ell(\mathbf{s}_i + \Delta\mathbf{x}_0^i, \mathbf{u}_0^i - K\Delta\mathbf{x}_0^i). \qquad (31)$$

To conclude the proof, we need to prove that $V(\mathbf{s}_{i+1}) \leq J(\mathbf{s}_{i+1})$. To that end we observe that, by Lemma 4, the optimizer of Problem (19) is independent of the solution of Problem (3) and, by optimality, we have

$$\begin{aligned}\bar{V}(\mathbf{s}_{i+1}) &= \mathbf{s}_{i+1}^\top P\mathbf{s}_{i+1} + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k^{i+1}, \mathbf{u}_k^{i+1}) \\ &\leq \mathbf{s}_{i+1}^\top P\mathbf{s}_{i+1} + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\bar{\mathbf{x}}_k^{i+1}, \bar{\mathbf{u}}_k^{i+1}). \qquad (32)\end{aligned}$$

The definition of V and (30), (31) and (32) imply

$$\begin{aligned}&V(\mathbf{s}_{i+1}) \\ &\overset{(24)}{=} (\mathbf{s}_{i+1} + \Delta\mathbf{x}_0^{i+1})^\top P(\mathbf{s}_{i+1} + \Delta\mathbf{x}_0^{i+1}) + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\mathbf{x}_k^{i+1}, \mathbf{u}_k^{i+1}) \\ &\overset{(25)(32)}{\leq} (\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1})^\top P(\mathbf{s}_{i+1} + \Delta\bar{\mathbf{x}}_0^{i+1}) + \sum_{k=0}^{N-1} \ell^{\mathrm{d}}(\bar{\mathbf{x}}_k^{i+1}, \bar{\mathbf{u}}_k^{i+1}) \\ &\overset{(30)}{=} J(\mathbf{s}_{i+1}) \overset{(31)}{=} V(\mathbf{s}_i) - \ell(\mathbf{s}_i + \Delta\mathbf{x}_0^i, \mathbf{u}_0^i - K\Delta\mathbf{x}_0^i).\end{aligned}$$

■

This result entails that, like MTMPC, CTMPC is asymptotically stabilizing to $\mathbb{X}_{\mathrm{m}}^K$, and by Lemma 1 it has a region of attraction which is no smaller than the one of MTMPC.

V. Numerical Example

In this section, we use the simple system introduced in [1] in order to be able to visualize all sets and trajectories in a simple way and, hence, validate the theoretical results in simulations. As one can expect, similar conclusions are obtained with the example introduced in [3], which we omit due to space limitations.

Consider the system, constraints and stage cost

$$\mathbf{s}_{k+1} = \begin{bmatrix} 1.1 & 1.0 \\ 0 & 1.3 \end{bmatrix} \mathbf{s}_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_k + \mathbf{w}_k,$$

$$\left| \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{s}_k \right| \leq 2, \quad |a_k| \leq 1, \quad \|\mathbf{w}_k\|_\infty \leq 0.1,$$

$$\ell(\mathbf{s}_k, a_k) = \begin{bmatrix} \mathbf{s}_k \\ a_k \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} \mathbf{s}_k \\ a_k \end{bmatrix},$$

such that the LQR cost-to-go and feedback matrices are

$$P = \begin{bmatrix} 1.9992 & -0.2629 \\ -0.2629 & 1.0859 \end{bmatrix}, \quad K = \begin{bmatrix} 0.7434 & 1.0922 \end{bmatrix}.$$

An approximate mRPI set $\bar{\mathbb{X}}_{\mathrm{m}}^K$ was computed using the algorithm proposed in [14] which guarantees

$$\bar{\mathbb{X}}_{\mathrm{m}}^K \subseteq \mathbb{X}_{\mathrm{m}}^K \oplus \mathbb{B}(\epsilon),$$

where $\mathbb{B}(\epsilon)$ denotes the ball of radius ϵ. In our simulations we selected $\epsilon = 10^{-6}$. We compute $V(\mathbf{s}_i)$ by solving (3) and (25), i.e., by exploiting Lemma 4 and Corollary 2.

We formulate both TMPC problems with $N = 4$ in order to better visualize the results, but we remark that with larger N all conclusions remain valid, though less visible in the plots. We perform set calculations using the MPT3 toolbox [15], we formulate the TMPC problems using YALMIP [16] and solve them using Mosek [17]. Starting from $\mathbf{s}_0 = \begin{bmatrix} -0.34 & 1.32 \end{bmatrix}^\top$, we display the closed-loop evolution of the state, the control and the Lyapunov function for CTMPC and MTMPC in Figures 1 and 2. It can be seen that the two schemes yield different trajectories because $\mathbf{a}_0 = -1$ is infeasible for MTMPC. Consequently, the resulting Lyapunov function $\tilde{V}$ for MTMPC has a higher value than the one of CTMPC, i.e., V: this validates the results of Corollary 3. Additionally, the feasible domains of the two schemes displayed in Figure 1 confirm the result of Lemma 2, as CTMPC has a larger domain than MTMPC.

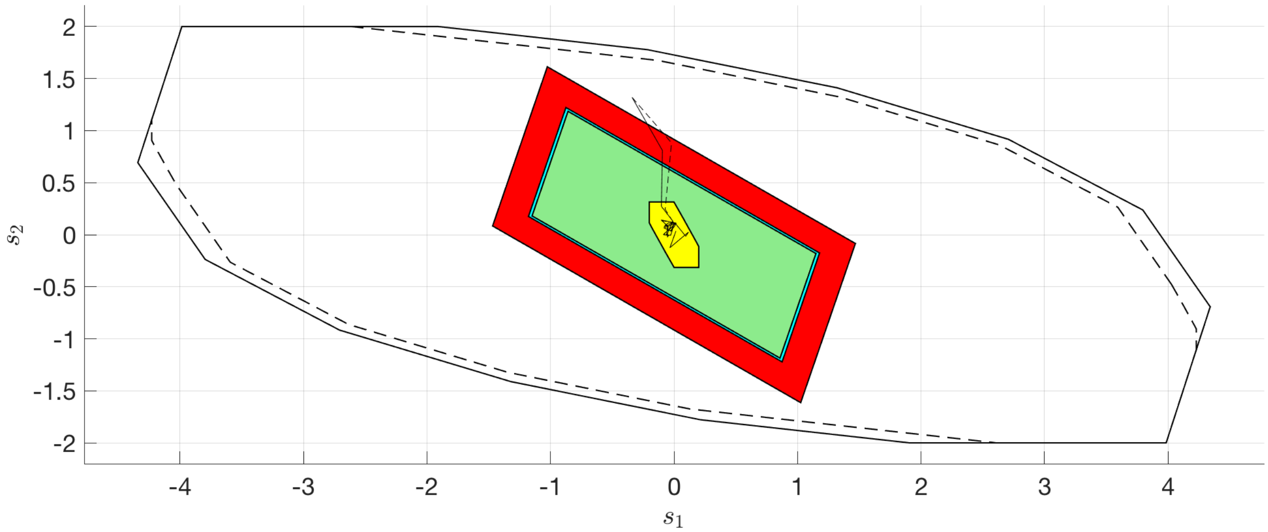

Fig. 1: MRPI set $\mathbb{X}_\infty$ (red), terminal set for CTMPC with $N = 4$ (cyan) and MTMPC (green), mRPI set approximation $\bar{\mathbb{X}}_{\mathrm{m}}^{K}$ (yellow). Closed-loop state trajectories for CTMPC (continuous line) and MTMPC (dashed line). Feasible domain for CTMPC (continuous line) and MTMPC (dashed line) with $N = 4$.

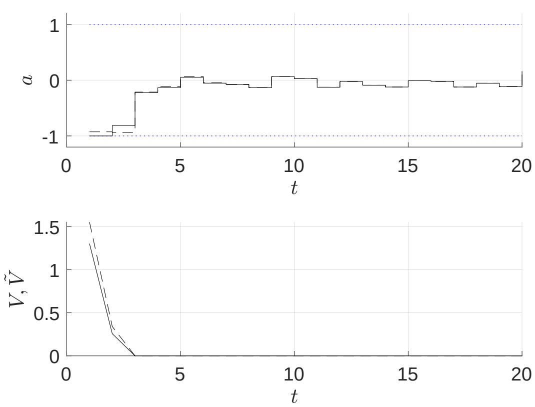

Fig. 2: Closed-loop control and Lyapunov function for CTMPC (continuous line) and MTMPC (dashed line).

VI. Conclusions

We presented a comparison of the two popular CTMPC and MTMPC formulations and we proved the existence of a Lyapunov function for CTMPC which is constructed in the same way as the one for MTMPC. We demonstrated the theoretical developments with a simple example in simulations, where we also showed that, the domain of MTMPC being smaller, CTMPC can yield a lower cost.

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