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Stabilization of strictly pre-dissipative receding horizon linear quadratic control by terminal costs☆

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ABSTRACT

Asymptotic stability in economic receding horizon control can be obtained under a strict dissipativity assumption and through the use of suitable terminal cost and constraints. In this paper, we analyze how terminal constraints can be replaced by suitable terminal costs. We restrict to the linear-quadratic setting as that allows us to obtain stronger results, while we analyze the fully nonlinear case in a separate contribution.

1. Introduction

Receding horizon control (often used synonymously with model predictive control) is a control technique in which a finite horizon Optimal Control Problem (OCP) is solved in each time step and only the first element of the resulting optimal control sequence is used, while in the next time step the state is measured and the problem solved again to update the control (Grüne & Pannek, 2017a; Rawlings et al., 2017). Under suitable stabilizability and OCP regularity conditions, this scheme yields a practically asymptotically stable closed loop if the system is strictly dissipative with supply function defined via the stage cost of the finite horizon optimal control problem (Grüne & Pannek, 2017a, Chapter 8). In this case, we call the optimal control problem *strictly dissipative*. Here, the size of the "practical" neighborhood of the equilibrium to which the closed-loop solution converges is determined by the length of the finite optimization horizon. True (as opposed to practical) asymptotic stability can be achieved by using suitable terminal constraints and costs, see Amrit et al. (2011), Diehl et al. (2011a) or Theorem 8.13 in Grüne and Pannek (2017a). In these approaches the terminal cost is typically a local control Lyapunov function for the system and the terminal constraints are needed because the design of a global control Lyapunov function is usually a very difficult task. As a simpler alternative, it was shown in Faulwasser and Zanon (2018) and Zanon and Faulwasser (2018) that linear terminal costs can also be used to obtain true asymptotic stability, provided that the local linear quadratic approximation of the problem at steady-state is stabilizing for a sufficiently long prediction horizon. The present paper presents an approach to achieve this property by adding a suitable quadratic terminal cost.

The strict dissipativity property that is at the heart of all these results requires the existence of a so-called storage function λ mapping the state space into the reals. It is a strengthened version of the system theoretic dissipativity property introduced by Willems in his seminal papers (Willems, 1972a, 1972b) and also featured in his slightly earlier paper (Willems, 1971) on linear quadratic optimal control and the algebraic Riccati equation. Readers familiar with Lyapunov's stability theory can see the storage function λ as a generalization of a Lyapunov function. However, unlike Lyapunov functions, λ need not be lower bounded by a monotonically increasing radially unbounded function. However, it must be bounded from below, and this property is crucial for deriving the (practical) stability properties for receding horizon control cited above.

For generalized linear quadratic problems (by which we mean problems with linear dynamics and a cost function containing quadratic and linear terms) with state space $\mathbb{R}^{n_x}$, a standard construction for a storage function results in a function of the form $\lambda(x) = x^T Px + \nu^T x$, for $P \in \mathbb{R}^{n_x \times n_x}$ and $\nu \in \mathbb{R}^{n_x}$, see Damm et al. (2014, Proposition 4.5). Clearly, such a function λ is in general *not* bounded from below and Example 2.3 in Damm et al. (2014), which we also present as Example 2.1, below, shows that storage functions unbounded from below may occur even for very simple

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scalar problems. While the potential unboundedness of λ has been handled somewhat informally in Damm et al. (2014), later in Grüne and Guglielmi (2018) the variant of strict dissipativity with storage function not bounded from below has been termed *strict pre-dissipativity*. For strictly pre-dissipative problems, one way to restore strict dissipativity and thus (practical) asymptotic stability is to suitably restrict the state space by means of state constraints, e.g., to a compact set, on which λ is bounded from below.

Since such a restriction of the state space may not always be desirable, in this paper we will look at an alternative way to regain (practical) asymptotic stability. More precisely, we want to answer the following question: Given a receding horizon control scheme with strictly pre-dissipative optimal control problem, what conditions must the terminal cost satisfy in order to guarantee asymptotic stability? The answer we are looking for is conceptually similar to the one given in Faulwasser and Zanon (2018) and Zanon and Faulwasser (2018), where the focus was on a linear terminal cost. However, we will show that a quadratic terminal cost is necessary. This quadratic cost can then be combined with the linear one proposed in Faulwasser and Zanon (2018) and Zanon and Faulwasser (2018). We emphasize that no terminal constraints are needed in our approach. While we consider the general nonlinear case in the companion paper (Grüne & Zanon, 2025), in this paper we focus mainly on the linear-quadratic case, for which we are able to provide stronger results, which we also connect to the generic results for the nonlinear case towards the end of the paper.

The analysis in this paper will deal with linear quadratic problems. After delivering the main results, we will briefly comment on the connections with the general nonlinear case. For this linear-quadratic problems, the question about asymptotic stability is closely linked with the existence of particular solutions to algebraic Riccati equations. Hence, we will make ample use of results from this area. The remainder of this paper is organized as follows. In Section 2 we introduce the problem and provide some definitions. We provide preliminary results in Section 3, which are instrumental for our main results, delivered in Section 4. We discuss the connections with other results available in the literature in Section 5. We illustrate our theory with a numerical example in Section 6 and we draw our conclusions in Section 7.

2. Problem statement

We consider discrete-time systems of the form

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$ denote the states and controls respectively.

Model predictive control consists in minimizing a given stage cost $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ over a given finite prediction horizon N, possibly subject to constraints and with the addition of a terminal cost. The receding horizon optimal control problem (RH-OCP) reads

$$\min_{x_0,u_0,\ldots,x_N} \quad \sum_{k=0}^{N-1} \ell(x_k,u_k) + V_f(x_N) \tag{2a}$$

$$\text{s.t.} \quad x_0 = \hat{x}_j, \tag{2b}$$

$$x_{k+1} = Ax_k + Bu_k, \qquad k \in \mathbb{I}_0^{N-1}, \tag{2c}$$

$$Cx_k + Du_k \le e, \qquad k \in \mathbb{I}_0^{N-1}, \tag{2d}$$

where (2d) defines the state and input constraints and the inequality is to be understood componentwise.

While we study the nonlinear case in the companion paper (Grüne & Zanon, 2025), in this paper we focus on the linear-quadratic case, i.e.,

$$\ell(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top H \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \tag{3a}$$

$$V^f(x) = x^\top P_f x. \tag{3b}$$

Note that, as the case of additional constant terms in (1) and additional linear terms in (3a)–(3b) has been fully analyzed in Faulwasser and Zanon (2018) and Zanon and Faulwasser (2018) under the assumption that the problem without such terms is stabilizing, in this section we assume for simplicity that these quantities are zero.

Starting in the initial value $\hat{x}_0$ at time instant $j = 0$, at every time instant $j \ge 0$ the state $\hat{x}_j$ is measured, Problem (2) is solved, and the first optimal input $u_0^\star$ is applied to the system to obtain

$$\hat{x}_{j+1} = A\hat{x}_j + Bu_0^\star. \tag{4}$$

This procedure is repeated iteratively for all $j \ge 0$.

In this paper, we are interested in obtaining stability properties of the closed loop system (4). While stability results are abundant for the case of suitably formulated terminal constraints and Lyapunov function terminal costs (Angeli et al., 2012; Diehl et al., 2011b; Faulwasser et al., 2018; Müller et al., 2013, 2015), we focus next on the case of no terminal constraint. This case has been analyzed, e.g., in Faulwasser and Zanon (2018), Grüne (2013), Müller and Grüne (2016) and Zanon and Faulwasser (2018), where we can further distinguish between formulations without terminal cost and formulations with simple terminal costs that need not be Lyapunov functions, as Lyapunov functions are usually difficult to design in the general nonlinear case. This last approach has in particular been taken in Faulwasser and Zanon (2018) and Zanon and Faulwasser (2018) by using a linear terminal cost and the present paper can be seen as a continuation of this research. As in these references, our analysis is based on dissipativity concepts.

We define the infinite-horizon Optimal Control Problem (OCP) related to (3), known as the Linear Quadratic Regulator (LQR), as

$$V(x_0) = \min_{u_0,u_1,\ldots} \lim_{N\to\infty} \sum_{k=0}^{N-1} \ell(x_k,u_k) + x_N^\top P_f x_N \tag{5a}$$

$$\text{s.t.} \quad x_{k+1} = Ax_k + Bu_k. \tag{5b}$$

Under suitable assumptions, if $P_f = 0$, the LQR solution is characterized by a solution P with associated K of the Discrete Algebraic Riccati Equation (DARE)

$$P = Q + A^\top PA - (S^\top + A^\top PB)K, \tag{6a}$$

$$K = (R + B^\top PB)^{-1}(S + B^\top PA), \tag{6b}$$

which provides both the optimal feedback law

$$F(x) = -Kx, \tag{7}$$

defining the optimal control via $u_k^\star = -Kx_k^\star$, and the associated quadratic value function

$$V(x) = x^\top Px.$$

We will generalize this conclusion in Theorem 4.3 to the case of suitably selected $P_f \neq 0$, not necessarily positive semidefinite. For a finite horizon N, the cost-to-go and the optimal feedback law are time-varying and read respectively $V_n(x) = x^\top P_n x$, and $F_k(x) = -K_n x$, where

$$P_{n+1} = Q + A^\top P_n A - (S^\top + A^\top P_n B)K_{n+1}, \tag{8a}$$

$$K_{n+1} = (R + B^\top P_n B)^{-1}(S + B^\top P_n A), \tag{8b}$$

defines $P_1, \ldots, P_N$ and $K_0, \ldots, K_N$ inductively with $P_0 = P_f$. Note that the index n we use here is different from the usual indexing in finite horizon optimal control, but convenient in what follows as it coincides with the prediction horizon related to the quantities involved. For an RH-OCP with data of the linear quadratic form (3) the receding horizon control in (4) is given by

$$u_0^\star = -K_N(\hat{x}_j).$$

In the context of this paper, it is paramount to clarify the fact that matrix P defining the optimal value function of the infinite-horizon LQR problem and the (symmetric) solutions P of the DARE do not necessarily coincide, as we explain next. It is well-known that the DARE has in general infinitely many symmetric solutions (Lancaster et al., 1986; Molinari, 1975; Ran & Trentelman, 1993; Willems, 1971), which can be interpreted as solutions of different OCPs. Note that the DARE might also have non-symmetric solutions, but these are not relevant in the context of this paper. Therefore, whenever referring to a solution of the DARE we will implicitly restrict to the set of symmetric solutions.

We call a solution P_s with associated K_s *stabilizing*, if all eigenvalues μ of $A - BK_s$ satisfy $|\mu| < 1$, i.e., if $A - BK_s$ is Schur stable. If there exists a stabilizing solution to the DARE, then it is unique, i.e., all other solutions are not stabilizing (Lancaster et al., 1986; Ran & Trentelman, 1993). Unfortunately, even in case the DARE does have a stabilizing solution, this one need not be the solution to the LQR (Ran & Trentelman, 1993), as also illustrated by the next example.

Example 2.1. Consider an LQR problem defined by $A = 2$, $B = 1$, $Q = 0$, $S = 0$, $R = 1$, and $P_f = 0$. The DARE reads

$$P = 4P - \frac{4P^2}{1+P}, \qquad K = \frac{2P}{1+P},$$

and has the two solutions

$$P = \{0, 3\}, \qquad K = \{0, 1.5\}.$$

Both solutions of the DARE correspond to the solution of an OCP. The first one is

$$\begin{aligned} \min_{u_0,u_1,\ldots} \ & \sum_{k=0}^{\infty} u_k^2 \\ \text{s.t.} \ & x_{k+1} = Ax_k + Bu_k, \qquad k \in \mathbb{I}_0^{\infty}, \end{aligned}$$

which has the trivial solution $u_k^\star = 0$, for all k and corresponds to $P = 0$, $K = 0$, which does not stabilize the system. The second OCP is

$$\begin{aligned} \min_{u_0,u_1,\ldots} \ & \sum_{k=0}^{\infty} u_k^2 \\ \text{s.t.} \ & x_{k+1} = Ax_k + Bu_k, \qquad k \in \mathbb{I}_0^{\infty}, \\ & \lim_{k\to\infty} x_k = 0, \end{aligned}$$

which corresponds to the stabilizing solution $P = 3$ and $K = 1.5$. Thus, the asymptotic state constraint $\lim_{k\to\infty} x_k = 0$ leads to a stabilizing optimal solution.

Furthermore, one can see that for the following OCP with terminal cost

$$\begin{aligned} \min_{u_0,u_1,\ldots} \ & \lim_{N\to\infty} P_f x_N^2 + \sum_{k=0}^{\infty} u_k^2 \\ \text{s.t.} \ & x_{k+1} = Ax_k + Bu_k, \qquad k \in \mathbb{I}_0^{\infty}, \end{aligned}$$

any $P_f > 0$ yields asymptotic stability. In the remainder of this paper, we will provide conditions on P_f that yield asymptotic stability in the general case.

In order to provide further arguments, we discuss next a simple modification of the example above, which fits the common LQR setting with positive definite cost satisfying the assumption that (Q, A) is detectable. We show that even in this case, by selecting a sufficiently small terminal cost, the infinite-horizon LQR can be destabilizing or fail to exist.

Example 2.2. Consider the system of Example 2.1, but with $Q = 1$. The DARE reads

$$P = 4P + 1 - \frac{4P^2}{1+P}, \qquad K = \frac{2P}{1+P},$$

and has the two solutions

$$P = \{2 - \sqrt{5}, 2 + \sqrt{5}\}, \qquad K = \left\{ \frac{4 - 2\sqrt{5}}{3 - \sqrt{5}}, \frac{4 + 2\sqrt{5}}{3 + \sqrt{5}} \right\}.$$

One can see that the first solution is destabilizing, while the second one is stabilizing.

By choosing $P_f = 2 - \sqrt{5}$ we have $P_f + R > 0$ and the one-step-ahead LQR is well defined, but yields a destabilizing feedback with cost-to-go matrix $P_- = P_f = 2 - \sqrt{5} < 0$. This entails that the infinite-horizon LQR is also destabilizing and has a negative-definite cost-to-go matrix.

It is easy to see that, by choosing $P_f = -0.5 \leq 2 - \sqrt{5}$ we have $P_f + R > 0$ and the one-step-ahead LQR is well defined, but yields a destabilizing feedback with cost-to-go matrix $P_- = -3$. The two-steps-ahead LQR is then not well defined, as $P_- + R < 0$ and the optimal control is $\pm\infty$.

Finally, one can verify that by selecting $P_f > 2 - \sqrt{5}$ the LQR feedback is stabilizing for a sufficiently long prediction horizon, e.g., for $P_f > 2 - \sqrt{5} + 10^{-8}$ the feedback is stabilizing for $N \geq 11$.

3. Preliminary results

In this section we provide some useful results that we will exploit in order to prove our main contributions.

3.1. Cost rotation and pre-stabilization

In order to derive our results, let us provide a few useful facts next.

3.1.1. Strict (x, u)-pre-dissipativity and rotated cost

For a given matrix Λ and storage function $\lambda(x) = x^\top \Lambda x$, we define the rotated cost

$$L(x, u) := \begin{bmatrix} x \\ u \end{bmatrix}^\top H_\Lambda \begin{bmatrix} x \\ u \end{bmatrix}, \tag{9a}$$

with

$$H_\Lambda := \begin{bmatrix} Q + \Lambda - A^\top \Lambda A & S^\top - A^\top \Lambda B \\ S - B^\top \Lambda A & R - B^\top \Lambda B \end{bmatrix}. \tag{9b}$$

Definition 3.1. Quadratic strict (x, u)-pre-dissipativity holds if there exists a scalar $\epsilon > 0$ such that for all x, u

$$L(x, u) \geq \epsilon(\|x\|^2 + \|u\|^2), \qquad \text{or, equivalently,} \qquad H_\Lambda \succ 0. \tag{10}$$

Checking whether $H_\Lambda \succ 0$ is a practically feasible way of verifying quadratic strict (x, u)-pre-dissipativity. However, other characterizations including conditions on the unobservable eigenvalues of A exist and can be found in Grüne and Guglielmi (2018). Moreover, in Lemma 4.1 of this reference it has been proven that for linear quadratic problems the more generic concept of strict pre-dissipativity implies quadratic strict pre-dissipativity. In this paper we restrict to the slightly stronger case of quadratic strict (x, u)-pre-dissipativity, in order to guarantee the existence of a solution to the DARE. In practice, this excludes singular LQ problems. Note that, differently from the general case, checking whether a linear system with quadratic cost is quadratically strictly (x, u)-pre-dissipative requires one to solve the Linear Matrix Inequality (LMI) $H_\Lambda \succ 0$: the answer is positive if a solution exists and negative otherwise.

Remark 3.2. Because the wording "quadratic strict (x, u)-pre-dissipativity" might seem overcomplicated, let us briefly explain their meaning: the term "pre" refers to the fact that the storage function is not bounded from below; the term "strict" refers to the fact that the rotated cost is not only lower bounded by 0, but additionally $L(x, u) > 0$ holds for all $x \neq 0$; the term "(x, u)" refers to the fact that, additionally, $L(x, u) > 0$ for all $u \neq 0$; the term quadratic refers to the fact that the involved quantities are quadratic functions.

Remark 3.3. Note that the equivalence of condition $H_\Lambda \succ 0$ in (10) with quadratic strict (x, u)-pre-dissipativity is known as the Positive Real lemma, or Kalman Yakubovich Popov (KYP) lemma, see, e.g., Caverly and Forbes (2024) and Kottenstette et al. (2014). Differently from the literature on economic MPC, in the literature on the KYP lemma the dissipativity condition is called QSR dissipativity.

In contrast to the more common strict dissipativity concept, strict pre-dissipativity, introduced under this name in Grüne and Guglielmi (2018), does not require the storage function λ to be bounded from below. This implies that one cannot use arguments as, e.g., in Grüne (2013) and Grüne and Stieler (2014) in order to conclude (practical) stability properties of the closed loop (4), and in fact stability may fail to hold, as we will show by means of the following example.

Example 3.4. Consider the LQR problem from Example 2.1. One easily sees that for any initial condition x_0 and any horizon N the optimal control sequence is $u_k^\star \equiv 0$, as this is the only control that produces 0 cost, while all other control sequences produce positive costs. This implies that system (4) becomes

$$\hat{x}_{j+1} = 2\hat{x}_j,$$

for which the origin is exponentially unstable. Yet, one checks that this problem is quadratically strictly (x, u)-pre-dissipative with storage function $\lambda(x) = -\Lambda x^2$ for each $\Lambda \in]0, 3[$. For example, the storage function $\lambda(x) = -x^2$ yields the rotated cost matrix

$$H_\Lambda = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \succ 0,$$

whose positive definiteness confirms that strict (x, u)-pre-dissipativity holds. Yet, asymptotic stability cannot be achieved unless suitable constraints are introduced. This shows that quadratic strict (x, u)-pre-dissipativity does not imply asymptotic stability of the optimal equilibrium.

As already mentioned in the introduction and as also seen in this example, storage functions that are not bounded from below appear naturally already for linear quadratic problems. In order to achieve closed-loop stability, often a compact state constraint set is imposed, as compactness implies boundedness of the storage function provided it is continuous (which is often the case). For Example 3.4, it was shown in Damm et al. (2014, Example 2.3) that this indeed renders the origin practically asymptotically stable for the closed loop. Yet, imposing compact state constraints just for the sake of achieving stability may not always be desirable. As we will prove in this paper, stability can be alternatively achieved by a suitably defined quadratic terminal cost.

The next well-known lemma formalizes the interest in rotated costs of the form (9) in the context of this paper.

Lemma 3.5. *For any finite horizon N as well as for the infinite horizon problem, an LQ problem (of the form (2) with cost defined in (3) and without inequality constraints) with stage cost matrix H and terminal cost matrix P_f yields the same feedback law $u_k = -K_N x_k$ as the corresponding rotated LQ problem, i.e., the same LQ problem formulated with stage cost matrix H_Λ and terminal cost matrix $P_f + \Lambda$. Moreover, the matrices defining the optimal value functions of the original and rotated problems satisfy $P_{\lambda,N} = P_N + \Lambda$.*

Proof. The proof follows from the observation that

$$\begin{aligned}
&\sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top H_\Lambda \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^\top (P_f + \Lambda) x_N \\
&\quad = \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^\top P_f x_N + \sum_{k=0}^{N-1} x_k^\top \Lambda x_k \\
&\qquad - \sum_{k=0}^{N-1} (Ax_k + Bu_k)^\top \Lambda (Ax_k + Bu_k) + x_N^\top \Lambda x_N \\
&\quad = \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^\top P_f x_N + \sum_{k=0}^{N-1} x_k^\top \Lambda x_k \\
&\qquad - \sum_{k=0}^{N-1} x_{k+1}^\top \Lambda x_{k+1} + x_N^\top \Lambda x_N \\
&\quad = \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^\top P_f x_N + x_0^\top \Lambda x_0.
\end{aligned}$$

Consequently, the costs of the original and rotated LQ problems only differ by the constant term $x_0^\top \Lambda x_0$, hence their minimizers coincide. The same holds true if we consider the limit for $N \to \infty$ of the cost. □

This lemma entails that we can study the stability properties of any problem satisfying quadratic strict (x, u)-pre-dissipativity as an LQR problem with a positive-definite stage cost, provided that we suitably modify the terminal cost. However, while optimal control problems with positive-definite stage costs are always stabilizing (for sufficiently large N and in the infinite-horizon case, provided that (A, B) is stabilizable) if there is a positive-semidefinite terminal cost, this may no longer be true in the presence of a terminal cost which is not positive definite. Hence, since in the pre-dissipative case the rotated terminal cost matrix $P_f + \Lambda$ can be indefinite even in case $P_f \succ 0$, quadratic strict (x, u)-pre-dissipativity does not necessarily entail that the LQR problem will be stabilizing. An example of such a situation is given once more by the system in Example 2.1 with $P_f = 0$, for which the terminal cost matrix adapted to the rotated problem is $P_f + \Lambda = -1$ and the corresponding optimal feedback $K = 0$ is indeed not stabilizing.

3.1.2. Pre-stabilized system

Let us define $A_{\hat{K}} := A - B\hat{K}$ and consider the following reformulation of Problem (5), where we apply the change of variable $\bar{u}_k = u_k - \hat{K} x_k$:

$$\min_{\bar{u}_0, \bar{u}_1, \ldots} \lim_{N \to \infty} \sum_{k=0}^{N-1} \ell(x_k, \bar{u}_k - \hat{K} x_k) + x_N^\top P_f x_N \tag{11a}$$

$$\text{s.t.} \quad x_{k+1} = A_{\hat{K}} x_k + B\bar{u}_k. \tag{11b}$$

In this case, we have that the stage cost matrix reads

$$H_{\hat{K}} = \begin{bmatrix} Q_{\hat{K}} & S_{\hat{K}}^\top \\ S_{\hat{K}} & R \end{bmatrix}, \tag{12a}$$

$$Q_{\hat{K}} := Q - S^\top \hat{K} - \hat{K}^\top S + \hat{K}^\top R \hat{K}, \tag{12b}$$

$$S_{\hat{K}} := S - R\hat{K}. \tag{12c}$$

With slight abuse of terminology, we will refer to this problem as pre-stabilized, even though $A_{\hat{K}}$ need not be Schur stable.

By Lemma 3.5 if Problem (11) is strictly pre-dissipative then its optimal solution is also optimal for

$$\min_{\bar{u}_0,\bar{u}_1,\dots} \lim_{N\to\infty} \sum_{k=0}^{N-1} L(x_k, \bar{u}_k - \hat{K}x_k) + x_N^\top (P_f + \Lambda) x_N \tag{13a}$$

$$\text{s.t.} \ \ x_{k+1} = A_{\hat{K}} x_k + B\bar{u}_k, \tag{13b}$$

where the rotated stage cost matrix reads

$$H_{\hat{K},\Lambda} := \begin{bmatrix} Q_{\hat{K}} + \Lambda - A_{\hat{K}}^\top \Lambda A_{\hat{K}} & S_{\hat{K}}^\top - A_{\hat{K}}^\top \Lambda B \\ S_{\hat{K}} - B^\top \Lambda A_{\hat{K}} & R - B^\top \Lambda B \end{bmatrix} \succ 0.$$

Finally, as we prove next, quadratic strict (x, u)-pre-dissipativity is not affected by pre-stabilizing the system.

Lemma 3.6. *Assume that quadratic strict (x, u)-pre-dissipativity holds for a linear quadratic problem (5). Then for any $\hat{K} \in \mathbb{R}^{n_u \times n_x}$ quadratic strict (x, u)-pre-dissipativity also holds for the pre-stabilized problem (11). Moreover, any solution of the DARE associated with the original problem (5) is also a solution of the DARE associated with the pre-stabilized problem (11).*

Proof. With a few algebraic manipulations, one can show that

$$M_{\hat{K}}^\top H M_{\hat{K}} = H_{\hat{K}}, \tag{14}$$

$$M_{\hat{K}}^\top G M_{\hat{K}} = \begin{bmatrix} \Lambda - A_{\hat{K}}^\top \Lambda A_{\hat{K}} & -A_{\hat{K}}^\top \Lambda B \\ -B^\top \Lambda A_{\hat{K}} & -B^\top \Lambda B \end{bmatrix} =: G_{\hat{K}}, \tag{15}$$

where

$$G := \begin{bmatrix} \Lambda - A^\top \Lambda A & -A^\top \Lambda B \\ -B^\top \Lambda A & -B^\top \Lambda B \end{bmatrix}, \qquad M_{\hat{K}} := \begin{bmatrix} I & 0 \\ -\hat{K} & I \end{bmatrix}.$$

This entails $M_{\hat{K}}^\top H_\Lambda M_{\hat{K}} = H_{\hat{K},\Lambda}$. Because $M_{\hat{K}}$ is full rank by construction, this yields that quadratic strict (x, u)-pre-dissipativity for the original problem, i.e., $H_\Lambda \succ 0$, implies $H_{\hat{K},\Lambda} \succ 0$, i.e., quadratic strict (x, u)-pre-dissipativity for the pre-stabilized problem.

In order to prove the second claim, we write the DARE associated with the pre-stabilized Problem (11):

$$P = Q_{\hat{K}} + A_{\hat{K}}^\top P A_{\hat{K}} - (S_{\hat{K}}^\top + A_{\hat{K}}^\top PB) K_{\hat{K}},$$

$$K_{\hat{K}} = (R + B^\top PB)^{-1} (S_{\hat{K}} + B^\top P A_{\hat{K}}).$$

We then observe that

$$K_{\hat{K}} = (R + B^\top PB)^{-1}(S + B^\top PA) - \hat{K} \\ = K - \hat{K},$$

such that, after few algebraic manipulations one obtains

$$P = Q + A^\top PA - (S^\top + A^\top PB)K,$$

i.e., the DARE associated with (5). □

We prove next some additional useful results.

Lemma 3.7. *Assume that (A, B) is controllable and quadratic strict (x, u)-pre-dissipativity holds. Then any symmetric solution P of the DARE (6) satisfies $R + B^\top PB \succ 0$.*

Proof. Because quadratic strict (x, u)-pre-dissipativity holds, by Zanon et al. (2014, Theorem 1) and Zanon et al. (2016, Corollary 8) a stabilizing solution exists, i.e., there exists a P solving the DARE (6) with $R + B^\top PB \succ 0$. The proof then follows from Lancaster et al. (1986, Theorem 2.5), (see also (Stoorvogel & Saberi, 1998)), which states that, under the given assumption, if there exists one Hermitian solution P such that $R + B^\top PB \succ 0$, then the condition holds for all Hermitian solutions. □

This result can be then extended to the case of stabilizable systems as follows.

Theorem 3.8. *Assume that (A, B) is stabilizable and quadratic strict (x, u)-pre-dissipativity holds. Then any symmetric solution P of the DARE (6) satisfies*

$$R + B^\top PB \succ 0. \tag{16}$$

Proof. Note that, since we assume stabilizability instead of controllability, the result of Lemma 3.7 needs to be sharpened.

Let us assume for simplicity and without loss of generality that the system is already in the form of the Kalman controllability decomposition:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{17}$$

with (A_{11}, B_1) controllable. This allows us to separate the solution to the DARE into components. In particular, through few algebraic manipulations one can see that the DARE can be split in components, each defining one component of P, split as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}.$$

Through few algebraic manipulations, one can see that the controllable part of the DARE reads

$$P_{11} = Q_{11} + A_{11}^\top P_{11} A_{11} - (S_1^\top + A_{11}^\top P_{11} B_1) K_1,$$

$$K_1 = (R + B_1^\top P_{11} B_1)^{-1} (S_1 + B_1^\top P_{11} A_{11}),$$

i.e., it is a DARE in P_{11} which is independent of P_{12} and P_{22}.

It is worth noting that the term $R + B^\top PB$ reads

$$R + \begin{bmatrix} B_1^\top & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = R + B_1^\top P_{11} B_1,$$

such that positive-definiteness of $R + B^\top PB$ can be assessed by considering the controllable part of the system only. The result then directly follows from Lemma 3.7. □

Since the computations in the proof only characterize part of the solution to the DARE, let us briefly discuss the full solution. By few algebraic manipulations one can see that: (a) the sub-equation defining P_{12} does depend on P_{11} but not on P_{22}, and the sub-equation is linear in P_{12}, such that the solution is unique; (b) once P_{11} and P_{12} have been solved for, also the sub-equation in P_{22} is linear in P_{22} and, consequently, has a unique solution.

Indeed, we have

$$P_{12} = \tilde{Q}_{12} + \tilde{A}_{12}^\top P_{12} A_{22},$$

$$\tilde{Q}_{12} := Q_{12} + A_{11}^\top P_{11} A_{12} - (S_2^\top + A_{11}^\top P_{11} B_1) K_{12}$$

$$K_{12} := (R + B_1^\top P_{11} B_1)^{-1} (S_1 + B_1^\top P_{11} A_{12} + B_1^\top P_{12} A_{22}),$$

$$\tilde{A}_{12}^\top := A_{12}^\top - (S_2^\top + A_{11}^\top P_{11} B_1)(R + B_1^\top P_{11} B_1)^{-1} B_1^\top.$$

and, remembering that $P_{21} = P_{12}^\top$,

$$P_{22} = \tilde{Q}_{22} + A_{22}^\top P_{22} A_{22},$$

$$\tilde{Q}_{22} := Q_{22} + A_{12}^\top P_{11} A_{12} + A_{22}^\top P_{21} A_{12} + A_{12}^\top P_{12} A_{22}$$

$$- (S_2^\top + A_{12}^\top P_{11} B_1 + A_{22}^\top P_{21} B_1) K_{22},$$
$$K_{22} := (R + B_1^\top P_{11} B_1)^{-1} (S_2 + B_1^\top P_{11} A_{12} + B_1^\top P_{12} A_{22}).$$

These results allow us to formulate the following conclusion.

Remark 3.9. The stability properties of any stabilizable linear system with quadratic stage cost satisfying the quadratic strict (x, u)-pre-dissipativity condition (10) can be studied by considering the corresponding fully controllable part with the corresponding rotated cost. Moreover, whenever matrix A_{11} is singular, it will be convenient to pre-stabilize it so as to make it nonsingular. Since pre-stabilization does not alter the results, this approach remains general. □

Before establishing our results, we need to provide some further definitions and preliminary results.

3.2. The reverse discrete-time algebraic Riccati equation

As we will discuss next, among the symmetric solutions of the DARE, when they exist, two of them play a fundamental role in the context of this paper: the stabilizing solution P_s with corresponding feedback K_s such that $A - BK_\mathrm{s}$ has all the eigenvalues strictly inside the unit circle; and the antistabilizing solution P_a with corresponding feedback K_a such that $A - BK_\mathrm{a}$ has all the eigenvalues strictly outside the unit circle. Note that in Examples 2.1 and 2.2 the DARE has exactly two solutions, coinciding with P_s and P_a.

Unfortunately, even in case the DARE has a stabilizing solution, existence of the antistabilizing solution is not guaranteed in general. We will discuss next that the existence of P_a is guaranteed under the assumption that matrix $R - SA^{-1}B$ is nonsingular. In this case, P_a coincides with the stabilizing solution $\bar{P}_\mathrm{s}$ to the so-called Reverse Discrete-time Algebraic Riccati Equation (RDARE). Since we will prove that $\bar{P}_\mathrm{s}$ exists whenever P_s exists, we will then also prove that it can be used as a substitute for P_a in case P_a does not exist.

The Reverse Discrete-time Algebraic Riccati Equation (RDARE) reads:

$$\bar{P} = \bar{Q} + \bar{A}^\top \bar{P} \bar{A} - (\bar{S}^\top + \bar{A}^\top \bar{P} \bar{B}) \bar{K},$$
$$\bar{K} = (\bar{R} + \bar{B}^\top \bar{P} \bar{B})^{-1} (\bar{S} + \bar{B}^\top \bar{P} \bar{A}),$$

with

$$\bar{A} := A^{-1}, \qquad \bar{B} := A^{-1}B,$$
$$\bar{Q} := -\bar{A}^\top Q \bar{A}, \qquad \bar{S} := S\bar{A} - \bar{B}^\top Q \bar{A},$$
$$\bar{R} := -R + S\bar{B} + \bar{B}^\top S^\top - \bar{B}^\top Q \bar{B}.$$

Note that we assumed without loss of generality that A^{-1} exists, see Remark 3.9.

Proposition 3.10 (Ionescu, 1996, Proposition 2, Remark 2)**.** *Assume that A is nonsingular. The DARE and RDARE share the same solutions if and only if $R - SA^{-1}B$ is nonsingular. In particular, if they exist, then $P_\mathrm{a} = \bar{P}_\mathrm{s}$ and $P_\mathrm{s} = \bar{P}_\mathrm{a}$, where $\bar{P}_\mathrm{a}$ is the antistabilizing solution of the RDARE. Moreover, if $R - SA^{-1}B$ is singular, then $R + B^\top \bar{P} B$ is singular for any symmetric $\bar{P}$ solving the RDARE.*

We observe that the second claim follows from the observation that

$$R + B^\top \bar{P} B = (R - SA^{-1}B)^\top (\bar{R} + \bar{B}^\top \bar{P} \bar{B})^{-1} (R - SA^{-1}B),$$

for any symmetric $\bar{P}$ solving the RDARE. This provides a clear intuition as to why the antisymmetric solution exists if and only if $R - SA^{-1}B$ is nonsingular. Finally, note that also

$$\bar{R} + \bar{B}^\top P \bar{B} = (R - SA^{-1}B)^\top (R + B^\top PB)^{-1} (R - SA^{-1}B)$$

holds. Consequently, if $R - SA^{-1}B$ is singular, then no solution P of the DARE can be a solution $\bar{P}$ of the RDARE and viceversa.

Example 3.11. Consider the scalar system with $A = 1$, $B = 1$, and stage cost matrices $Q = 1$, $R = 1$, $S = 1$. The DARE reduces to $P = 1$, which corresponds to $K = 1$, such that $A - BK = 0$ is stable. Moreover, we have $\bar{A} = 1$, $\bar{B} = 1$, $\bar{Q} = -1$, $\bar{R} = 0$, $\bar{S} = 0$, such that the RDARE reduces to $\bar{P} + 1 = 0$, which corresponds to $\bar{K} = 1$, such that $\bar{A} - \bar{B}\bar{K} = 0$ is stable. However, $R + B^\top \bar{P} B = 0$, and the stabilizing solution of the RDARE does not solve the DARE.

We prove next that, assuming that (A, B) is controllable, quadratic strict (x, u)-pre-dissipativity is not only necessary and sufficient for the existence of the stabilizing solution of the DARE (a proof can be found in Zanon et al. (2016, Corollary 8)), but it also entails existence of the stabilizing solution to the RDARE.

Lemma 3.12. *Assume that A is full rank, (A, B) is controllable, and quadratic strict (x, u)-pre-dissipativity holds. Then the RDARE does have a stabilizing solution.*

Proof. Without loss of generality we can assume that A is invertible. Otherwise, since (A, B) is controllable, we choose a feedback $\hat{K}$ such that $A - B\hat{K}$ is invertible. Then the pre-stabilized system is still controllable and by Lemma 3.6 it is still pre-dissipative and the corresponding DARE (and thus also the RDARE) has the same solution as for the original system.

Assuming full rank of A, we first observe that the original DARE and the rotated DARE deliver solution matrices satisfying $P_\Lambda = P + \Lambda$. Consequently, we can assume without loss of generality that quadratic strict (x, u)-pre-dissipativity holds for $\Lambda = 0$, i.e., $H \succ 0$. Then we observe that

$$\begin{bmatrix} -\bar{A}^\top & 0 \\ -\bar{B}^\top & I \end{bmatrix} \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} -\bar{A} & -\bar{B} \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} -\bar{A}^\top Q & -\bar{A}^\top S^\top \\ -\bar{B}^\top Q + S & -\bar{B}^\top S^\top + R \end{bmatrix} \begin{bmatrix} -\bar{A} & -\bar{B} \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} \bar{A}^\top Q \bar{A} & \bar{A}^\top Q \bar{B} - \bar{A}^\top S^\top \\ \bar{B}^\top Q \bar{A} - S\bar{A} & \bar{B}^\top Q \bar{B} - S\bar{B} - \bar{B}^\top S^\top + R \end{bmatrix}$$
$$= - \begin{bmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{bmatrix},$$

such that, since the cost matrix H is pre- and post-multiplied by a full-rank matrix, it holds that $\bar{H} \prec 0$. We observe that controllability of (A, B) implies controllability of $(\bar{A}, \bar{B})$, as

$$A^{n_x} \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n_x - 1}\bar{B} \end{bmatrix}$$
$$= A^{n_x} \begin{bmatrix} A^{-1}B & A^{-2}B & \cdots & A^{-n_x}B \end{bmatrix}$$
$$= \begin{bmatrix} A^{n_x - 1}B & \cdots & AB & B \end{bmatrix}.$$

In case A^{n_x} is singular, since the system is controllable, we can pre-stabilize it by selecting a feedback matrix $\hat{K}$ which results in $(A - B\hat{K})^{n_x}$ nonsingular.

Moreover, for any DARE/RDARE, changing the sign of the stage cost H or $\bar{H}$ entails changing the sign of the solution P or $\bar{P}$, while K and $\bar{K}$ remain unchanged. Consequently, the RDARE does have a stabilizing solution $\bar{P}_\mathrm{s} \prec 0$. □

As anticipated above, however, the fact that the RDARE does have a stabilizing solution does not entail that the DARE has an antistabilizing solution. We provide a sufficient condition in the next theorem.

Theorem 3.13. *Assume that (A, B) is controllable and quadratic strict (x, u)-pre-dissipativity holds. Select $\hat{K}$ such that $A_{\hat{K}} := A - B\hat{K}$ is invertible. Define $D := R - S_{\hat{K}} A_{\hat{K}}^{-1} B$, with $S_{\hat{K}} = S - R\hat{K}$. Assume that D is invertible. Then, the antistabilizing solution of the DARE exists.*

Before proving the theorem, let us briefly comment on the implications of pre-stabilizing the system.

Remark 3.14. By Lemma 3.5 and Zanon et al. (2014, Lemma 2), since quadratic strict (x, u)-pre-dissipativity holds, we can assume without loss of generality $H \succ 0$ and, hence, $R \succ 0$, i.e., it is invertible. Consequently, the nonsingularity of $R - SA^{-1}B$ is equivalent to the nonsingularity of

$$\begin{bmatrix} R & S \\ B & A \end{bmatrix}. \tag{18}$$

For the same reason, the nonsingularity of $R - S_{\hat{K}} A_{\hat{K}}^{-1} B$ is equivalent to the nonsingularity of

$$\begin{bmatrix} R & S - R\hat{K} \\ B & A - B\hat{K} \end{bmatrix} = \begin{bmatrix} R & S \\ B & A \end{bmatrix} \begin{bmatrix} I & -\hat{K} \\ 0 & I \end{bmatrix}. \tag{19}$$

Since the last matrix is full rank, nonsingularity of $R - SA^{-1}B$ is equivalent to nonsingularity of $R - S_{\hat{K}} A_{\hat{K}}^{-1} B$, i.e., that property is independent of any pre-stabilization of the system, but allows us to characterize the existence of the minimal solution also in case A is not invertible. Finally, the condition $A - BR^{-1}S = A_{\hat{K}} - BR^{-1}S_{\hat{K}}$ nonsingular is another equivalent condition which is sometimes found in the literature, see, e.g., Ran and Trentelman (1993).

Proof (*Proof of Theorem 3.13*). We first prove that the claim can be verified by checking it for a pre-stabilized system instead of the original one. This is particularly important in case A is not invertible. Controllability ensures that there exist a feedback matrix $\hat{K}$ such that the eigenvalues of $A - B\hat{K}$ can be chosen arbitrarily. In particular, this entails that $A - B\hat{K}$ can be made invertible.

By Zanon et al. (2014, Lemma 1), we know that the DARE formulated with system matrices $(A_{\hat{K}}, B)$ and cost matrix $H_{\hat{K}}$ defined as per (12) yields the same matrix as the original DARE, i.e., $P_{\hat{K}} = P$, while the feedback matrix is given by $K_{\hat{K}} = K - \hat{K}$. This entails that the solutions of the pre-stabilized DARE coincide with those of the original DARE.

We can therefore exploit Lemma 3.12 to conclude existence of a stabilizing solution to the RDARE. Moreover, because $R - S_{\hat{K}} A_{\hat{K}}^{-1} B$ is nonsingular the stabilizing solution to the RDARE coincides with the antistabilizing solution to the DARE (Ionescu, 1996, Proposition 2). □

We establish next a relation between $\bar{P}_s$ or P_a and cost rotations.

3.3. Positive-semidefinite cost rotations

Next, we need to establish one additional useful result. To that end, we first prove the following lemma.

Lemma 3.15. *Consider a DARE and rotate the stage cost matrix using matrix Λ. The corresponding RDARE coincides with the RDARE associated with the original cost rotated using matrix Λ.*

Proof. We observe that the rotated cost yields the following stage cost matrices for the RDARE:

$$\begin{aligned} \bar{Q}_\lambda &= -\bar{A}^\top Q_\lambda \bar{A}, \\ &= -\bar{A}^\top (Q + \Lambda - A^\top \Lambda A)\bar{A} \\ &= \bar{Q} - \bar{A}^\top \Lambda \bar{A} + \Lambda, \end{aligned}$$

$$\begin{aligned} \bar{S}_\lambda &= S_\lambda \bar{A} - \bar{B}^\top Q_\lambda \bar{A}, \\ &= (S - B^\top \Lambda A)\bar{A} - \bar{B}^\top (Q + \Lambda - A^\top \Lambda A)\bar{A}, \\ &= S\bar{A} - B^\top \Lambda - \bar{B}^\top Q \bar{A} - \bar{B}^\top \Lambda \bar{A} + \underbrace{\bar{B}^\top A^\top}_{=B^\top} \Lambda, \\ &= \bar{S} - \bar{B}^\top \Lambda \bar{A}, \end{aligned}$$

$$\begin{aligned} \bar{R}_\lambda &= -R_\lambda + S_\lambda \bar{B} + \bar{B}^\top S_\lambda^\top - \bar{B}^\top Q_\lambda \bar{B} \\ &= -(R - B^\top \Lambda B) + (S - B^\top \Lambda A)\bar{B} \\ &\quad + \bar{B}^\top (S^\top - A^\top \Lambda B) - \bar{B}^\top (Q + \Lambda - A^\top \Lambda A)\bar{B} \\ &= -R + B^\top \Lambda B + S\bar{B} - B^\top \Lambda \underbrace{A\bar{B}}_{=B} \\ &\quad + \bar{B}^\top S^\top - \underbrace{\bar{B}^\top A^\top}_{=B^\top} \Lambda B - \bar{B}^\top Q \bar{B} - \bar{B}^\top \Lambda \bar{B} \\ &\quad + \underbrace{\bar{B}^\top A^\top}_{=B^\top} \Lambda \underbrace{A\bar{B}}_{=B} \\ &= \bar{R} - \bar{B}^\top \Lambda \bar{B}. \end{aligned}$$

Consequently, we obtain

$$\bar{H}_\Lambda = \begin{bmatrix} \bar{Q} + \Lambda - \bar{A}^\top \Lambda \bar{A} & \bar{S}^\top - \bar{A}^\top \Lambda \bar{B} \\ \bar{S} - \bar{B}^\top \Lambda \bar{A} & \bar{R} - \bar{B}^\top \Lambda \bar{B} \end{bmatrix},$$

which, by definition, is the stage cost matrix obtained by rotating the original RDARE cost with matrix Λ. □

We are now ready to prove the following.

Theorem 3.16. *Assume that A is full rank, (A, B) is controllable and quadratic strict (x, u)-pre-dissipativity holds. Rotate the cost using $\Lambda = -\bar{P}_s$. Then the rotated stage cost matrix H_Λ is positive semidefinite, i.e., $H_\Lambda \succeq 0$.*

Proof. In order to obtain the proof we first focus on the stage cost matrix of the RDARE, which reads

$$\bar{H}_\Lambda := \begin{bmatrix} \bar{Q} - \bar{P}_s + \bar{A}^\top \bar{P}_s \bar{A} & \bar{S}^\top + \bar{A}^\top \bar{P}_s \bar{B} \\ \bar{S} + \bar{B}^\top \bar{P}_s \bar{A} & \bar{R} + \bar{B}^\top \bar{P}_s \bar{B} \end{bmatrix}.$$

By Lemma 3.12 we know that if $H \succ 0$ the stage cost of the RDARE is negative definite, such that $\bar{R} \prec 0$, $\bar{P}_s \prec 0$. This entails

$$\bar{R}_\lambda = \bar{R} + \bar{B}^\top \bar{P}_s B \prec 0.$$

Then we observe that the Schur complement of $\bar{R}$ in $\bar{H}_\Lambda$ yields the RDARE, which is solved by $\bar{P}_s$. Consequently, $\bar{H}_\Lambda \preceq 0$.

Finally, we recall that

$$-\bar{H}_\Lambda = \begin{bmatrix} -\bar{A}^\top & 0 \\ -\bar{B}^\top & I \end{bmatrix} H_\Lambda \begin{bmatrix} -\bar{A} & -\bar{B} \\ 0 & I \end{bmatrix} \succeq 0.$$

Since in the equation above H_Λ is pre- and post-multiplied by a full rank matrix and its transpose, we conclude that $H_\Lambda \succeq 0$. □

Note that, as a result of Lemma 3.5, after rotating with $\Lambda = -\bar{P}_s$ we have that the new solution is $\bar{P}_s = 0$, consequently, if it exists, $P_a = 0$.

Finally, we discuss next further useful reformulations and properties.

3.4. Properties of the cost-to-go and DARE solutions

Finite-horizon LQR problems are characterized by the Riccati iterations, defined by (8), which is fully equivalent to

$$\begin{aligned} P_{n+1} &= A_{K_{n+1}}^\top P_n A_{K_{n+1}} \\ &\quad + \underbrace{Q - S^\top K_{n+1} - K_{n+1}^\top S + K_{n+1}^\top R K_{n+1}}_{=:Q_{K_{n+1}}}, \end{aligned}$$

where K_{n+1} is defined as per (8b).

The following lemma is known, see e.g. Freiling and Ionescu (1999, Theorem 3.2) (which even considers the time-varying setting). Since the proof in our setting is very simple, we provide it for convenience of the readers.

Lemma 3.17. *Assume that $P_{n+1} \succeq P_n$. Then, $P_{n+2} \succeq P_{n+1}$.*

Proof. We observe that

$$\begin{aligned} P_{n+2} &= Q_{K_{n+2}} + A_{K_{n+2}}^\top P_{n+1} A_{K_{n+2}} \\ &\succeq Q_{K_{n+2}} + A_{K_{n+2}}^\top P_n A_{K_{n+2}} \\ &\succeq Q_{K_{n+1}} + A_{K_{n+1}}^\top P_n A_{K_{n+1}} = P_{n+1}, \end{aligned}$$

where we used optimality of the one-step-ahead problem with terminal cost matrix P_n to obtain the second inequality, while the first inequality stems from the assumption $P_{n+1} \succeq P_n$. □

This lemma entails that, if the LQR problem over a horizon $n = 1$ yields a cost-to-go which is no smaller than the terminal cost, then the sequence of matrices P_n yielded by (8) is monotonic.

Let us denote the set of solutions to the DARE as $\mathcal{P}$ and the set of the non stabilizing solutions as $\bar{\mathcal{P}} := \mathcal{P} \setminus \{P_s\}$.

Lemma 3.18 (*Ran & Trentelman, 1993, Lemmas 3.1–3.2*). *Suppose that P is an arbitrary solution to the DARE. Provided the stabilizing and the antistabilizing solution to this DARE exist, define*

$$R_s := R + B^\top P_s B R_a := R + B^\top P_a B. \tag{20}$$

Then, $\Delta_a := P - P_a$ and $\Delta_s := P - P_s$ satisfy the algebraic Riccati equations

$$\Delta_a = A_a^\top \Delta_a A_a - A_a^\top \Delta_a B(R_a + B^\top \Delta_a B)^{-1} B^\top \Delta_a A_a, \tag{21}$$

$$\Delta_s = A_s^\top \Delta_s A_s - A_s^\top \Delta_s B(R_s + B^\top \Delta_s B)^{-1} B^\top \Delta_s A_s. \tag{22}$$

Lemma 3.19. *Suppose that P_+, P solve the Riccati iteration*

$$P_+ = Q + A^\top P A - (S^\top + A^\top P B)K, \tag{23a}$$

$$K = (R + B^\top P B)^{-1}(S + B^\top P A). \tag{23b}$$

Provided the stabilizing and the antistabilizing solution to this DARE exist, define

$$R_s := R + B^\top P_s B R_a := R + B^\top P_a B. \tag{24}$$

Then, $\Delta_a := P - P_s$, $\Delta_a^+ := P_+ - P_s$ and $\Delta_a := P - P_a$, $\Delta_a^+ := P_+ - P_a$ satisfy the algebraic Riccati equations

$$\Delta_a^+ = A_a^\top \Delta_a A_a - A_a^\top \Delta_a B(R_a + B^\top \Delta_a B)^{-1} B^\top \Delta_a A_a, \tag{25}$$

$$\Delta_s^+ = A_s^\top \Delta_s A_s - A_s^\top \Delta_s B(R_s + B^\top \Delta_s B)^{-1} B^\top \Delta_s A_s. \tag{26}$$

Proof. The proof follows along the same lines as Lemma 3.18 and Ran and Trentelman (1993, Lemmas 3.1–3.2). □

Remark 3.20. Note that these lemmas entail that the differences between solutions of a DARE can be interpreted as solutions to an optimal control problem which only penalizes the input with matrix R_a or R_s, respectively. Consequently, we can apply Theorem 4.2 to prove some useful properties of all solutions to the DARE. □

We recall next that $P_s \succeq P \succeq P_a$. A proof is provided in Ran and Trentelman (1993), but we provide an alternative and shorter one next.

Lemma 3.21. *Assume that (A, B) is stabilizable and quadratic strict (x, u)-pre-dissipativity holds. Then, it holds that $P_s \succeq P$, for all symmetric P solving the DARE.*

Proof. The claim can equivalently be formulated as $\Delta_s \preceq 0$. We observe that, by Theorem 3.8, we have

$$R_s + B^\top \Delta_s B = R + B^\top P B \succ 0.$$

This entails that

$$M_s := A_s^\top \Delta_s B(R_s + B^\top \Delta_s B)^{-1} B^\top \Delta_s A_s \succeq 0.$$

We observe that (22) also reads

$$\Delta_s = A_s^\top \Delta_s A_s - M_s,$$

such that, since $M_s \succeq 0$, this directly entails

$$\Delta_s = -\sum_{k=0}^{\infty} \underbrace{A_s^\top M_s A_s}_{\succeq 0} \preceq 0. \quad \square$$

Lemma 3.22. *Whenever P_a exists, it holds that $P_a \preceq P$, for all symmetric P solving the DARE.*

Proof. The claim can equivalently be formulated as $\Delta_a \succeq 0$. In order to prove the result, we first observe that A_a has all its eigenvalues outside the unit circle, such that A_a^{-1} exists and has all its eigenvalues inside the unit circle. Then, (21) can be rewritten as

$$A_a^{-\top} \Delta_a A_a^{-1} = \Delta_a - \Delta_a B(R_a + B^\top \Delta_a B)^{-1} B^\top \Delta_a.$$

By rearranging terms we have

$$\Delta_a = A_a^{-\top} \Delta_a A_a^{-1} + \Delta_a B(R_a + B^\top \Delta_a B)^{-1} B^\top \Delta_a.$$

By Theorem 3.8 we have

$$R_a + B^\top \Delta_a B = R + B^\top P B \succ 0.$$

This entails that

$$M_a := \Delta_a B(R_a + B^\top \Delta_a B)^{-1} B^\top \Delta_a \succeq 0.$$

The remainder of the proof follows along the same lines as for Lemma 3.21. □

Lemma 3.23. *Assume that both P_s and P_a exist and define $\Xi_s := P_s - P_a$. Then $\Xi_s \succ 0$, i.e., $P_s \succ P_a$.*

Proof. Lemma 3.22 already established $P_s \succeq P_a$, so we are left with proving that the inequality is strict.

In order to prove this result, it is convenient to use the reformulation proposed in Lemmas 3.18–3.19, i.e., use Δ_a instead of P, and observe that Ξ_s can be interpreted as the solution of an infinite-horizon problem with a positive semidefinite stage cost and a suitably defined terminal cost/constraint for a linear system defined by matrices (A_a, B). In particular, the terminal cost related to P_a is 0. Since A_a is unstable and the feedback corresponding to Ξ_s is stabilizing, this entails that, for any initial state $x \neq 0$, the optimal input cannot be 0 at all times, which, together with $R_a \succ 0$, entails that the optimal cost-to-go needs to be strictly positive. In turn, this entails $\Xi_s \succ 0$. □

4. Stabilizing terminal costs

We are now ready to prove the main results. We proceed in three steps in the next three sections: We first establish asymptotic stability of the optimal solutions of the infinite-horizon LQR problem assuming existence of the antistabilizing solution P_a, then extend this result to the case in which P_a does not exist and finally address the receding horizon feedback law.

4.1. The antistabilizing solution does exist

Before discussing the case of indefinite stage costs, let us first consider the simpler case of positive semidefinite costs. To that end, we first need to prove the following lemma, which establishes some form of monotonicity of the sequence of cost-to-go matrices P_n.

Lemma 4.1. *Consider an LQR problem formulated with terminal cost matrix $P_0 = P_a + \alpha \Xi_s$, with scalar $0 < \alpha < 1$ and $\Xi_s := P_s - P_a$. Then, $P_1 \neq P_0$ and $P_1 \succeq P_0$.*

Proof. In order to prove the result it will be convenient to look at the solutions Δ_n of the Riccati iteration (25) for Δ_a, which is initialized with terminal cost matrix $\Delta_0 = \alpha \Xi_s$. The claim then reads $\Delta_1 \neq \Delta_0$, $\Delta_1 \succeq \Delta_0$.

We first prove the second claim. We use Lemma 3.19 and the Woodbury matrix identity to obtain

$$\begin{aligned}\Delta_1 &= A_a^\top \Delta_0 A_a - A_a^\top \Delta_0 B(R_a + B^\top \Delta_0 B)^{-1} B^\top \Delta_0 A_a,\\ &= A_a^\top (\Delta_0^{-1} + BR_a^{-1}B^\top)^{-1} A_a\\ &= A_a^\top (\alpha^{-1}\Xi_s^{-1} + BR_a^{-1}B^\top)^{-1} A_a\\ &= \alpha A_a^\top (\Xi_s^{-1} + \alpha BR_a^{-1}B^\top)^{-1} A_a.\end{aligned}$$

We observe that, by Lemma 3.18 and the Woodbury matrix identity we have

$$\begin{aligned}\alpha \Xi_s &= \alpha \left(A_a^\top \Xi_s A_a - A_a^\top \Xi_s B(R_a + B^\top \Xi_s B)^{-1} B^\top \Xi_s A_a\right),\\ &= \alpha A_a^\top (\Xi_s^{-1} + BR_a^{-1}B^\top)^{-1} A_a.\end{aligned}$$

Because $\Xi_s^{-1} + BR_a^{-1}B^\top \succeq \Xi_s^{-1} + \alpha BR_a^{-1}B^\top \succ 0$, then we have $(\Xi_s^{-1} + \alpha BR_a^{-1}B^\top)^{-1} \succeq (\Xi_s^{-1} + BR_a^{-1}B^\top)^{-1} \succ 0$, which entails $\Delta_1 \succeq \Delta_0$, i.e., $P_1 \succeq P_0$.

The first claim is directly obtained by contradiction, as $\Delta_1 = \Delta_0$ implies that $BR_a^{-1}B^\top = \Xi_s^{-1} + \alpha BR_a^{-1}B^\top$, which is clearly impossible, since by Theorem 3.8 we have $R_a \succ 0$. □

Theorem 4.2. *Consider Problem (5) with $H \succeq 0$, and $P_f \succ 0$ and let (A, B) be stabilizable. Then, if quadratic strict (x, u)-pre-dissipativity holds, the DARE has a stabilizing solution and the corresponding optimal value function and feedback coincide with those of the infinite horizon problem (5).*

Proof. Since quadratic strict (x, u)-pre-dissipativity holds the rotated cost is positive definite. Together with the stabilizability of (A, B) this implies that the rotated DARE has a stabilizing solution (Anderson & Moore, 1990, Section 3.3) which by Lemma 3.6 has a corresponding unique stabilizing solution of the DARE for the original system. This shows the first claim.

In order to prove the second claim, proceeding backwards in time starting from the final time, we write the dynamic programming recursion

$$W_{n+1}(x) = \min_v \ \ell(x, v) + W_n(Ax + Bv),$$

with $W_0(x) = x^\top P_f x$. We observe that, due to the quadratic form of ℓ, for all time instants n the functions $W_n(x)$ are quadratic, i.e., of the form $W_n(x) = x^\top P_n x$ for symmetric and positive definite matrices P_n. Since the pair (A, B) is stabilizable, the W_n are also uniformly bounded on compact sets. We prove next that the sequence P_n converges. By construction $P_n \succeq 0$. We consider first the case $P_f \prec P_s$, which entails $P_n \prec P_s$ for all $n < \infty$. Moreover, as discussed in Lemma 3.12, we have $P_a = \bar{P}_s \preceq 0$, such that $P_f \succ P_a$. Lemma 4.1 then entails that P_n is bounded from below by the sequence $\hat{P}_n$ obtained using terminal cost $P_a + \alpha \Xi_s \preceq P_f$. Note that such terminal cost always exists for a sufficiently small $\alpha > 0$, since $P_f \succ P_a$. The sequence $\hat{P}_n$ is nondecreasing and bounded, such that it must converge to a fixed value. The case $P_f \not\prec P_s$ is obtained as a direct extension to the argument above. Indeed, we know that for any $\tilde{P}_0 \succ P_s$ we obtain a sequence $\tilde{P}_{n+1} \preceq \tilde{P}_n$ which converges to P_s, see, e.g., Grüne and Pannek (2017a) and Rawlings et al. (2017). This immediately provides an upper bound for the case $P_f \not\prec P_s$, while the lower bound remains unaltered and we obtain once more that the sequence P_n converges to a fixed value. Because we will prove next that $\lim_{n\to\infty} \hat{P}_n = P_s$, this will also entail that $\lim_{n\to\infty} P_n = P_s$. Consequently, we will assume without loss of generality that $P_f = P_a + \alpha \Xi_s$, such that $P_n = \hat{P}_n$.

Since the sequence converges, using dynamic programming we obtain that there exists a matrix K such that the infinite horizon optimal control is given by the feedback law $u_k^\star = -Kx_k^\star$, i.e., by $F(x) = -Kx$. In order to prove the claim, we need to show that all eigenvalues $A - BK$ are inside the unit circle, i.e., that P_∞ is the unique stabilizing solution of the DARE.

We observe that either the optimal linear feedback policy $F(x) = -Kx$ yields a closed-loop matrix $A - BK$ with stable eigenvalues (inside or on the boundary of the unit circle with those on the boundary being semi-simple); or one or more eigenvalues are outside the unit circle or at the boundary and not semi-simple. Because any feedback policy which does not stabilize the system yields a state trajectory for which at least one component of the state diverges to $\pm\infty$, all policies of the latter type incur an unbounded terminal cost as $N \to \infty$, which contradicts the fact that $V(x) = x^\top P_\infty x$ is finite for all x.

Since this excludes the case of diverging solutions, it remains to exclude the possibility that the optimal closed-loop matrix has one or more eigenvalues on the unit circle, i.e., $|\mu| = 1$ for some eigenvalue μ of $A_K := A - BK$.

To that end, consider the reformulation of problem (5) given in (13), where we apply the change of variable $\bar{u}_k = u_k - Kx_k$. Since $-K$ is the infinite-horizon optimal feedback law, the optimal control of the rotated and pre-stabilized problem (13) is $\bar{u}_k = 0$. Consequently, the optimal cost is given by

$$\begin{aligned}V(x_0) = \lim_{N\to\infty} &\sum_{k=0}^{N-1} (A_K^k x_0)^\top (Q_K + \Lambda - A_K^\top \Lambda A_K) A_K^k x_0\\ &+ x_N^\top (P_f + \Lambda) x_N.\end{aligned} \tag{27}$$

As we proved before, $\lim_{N\to\infty} \|x_N\| < \infty$, such that the terminal cost is bounded for $N \to \infty$.

In order to prove that $|\mu| < 1$ for all eigenvalues of A_K, let us proceed by contradiction: let us assume that there is an eigenvalue with $|\mu| = 1$, implying $\|A_K^k v_\mu\| = \|v_\mu\| > 0$ for the corresponding eigenvector v_μ. Since quadratic strict (x, u)-pre-dissipativity holds, the matrix $Q_K + \Lambda - A_K^\top \Lambda A_K$ is positive definite, implying the existence of a constant $c > 0$ such that

$$(v_\mu A_K^k)^\top (Q_K + \Lambda - A_K^\top \Lambda A_K) A_K^k v_\mu \geq c\|A_K^k v_\mu\|^2 \geq c\|v_\mu\|^2.$$

This implies that the sum in (27) diverges for $x_0 = v_\mu$, which again contradicts the finiteness of the optimal cost. □

We are now ready to extend the result above to indefinite stage costs.

Theorem 4.3. *Consider a strictly (x, u)-pre-dissipative LQR problem (5), and assume that the antistabilizing solution P_a of the corresponding DARE exists. Then, for any terminal cost matrix $P_0 = P_a + E$, with $E \succ 0$, the optimal solution of (5) is given by the unique stabilizing solution of the corresponding DARE. In particular, the origin is an asymptotically stable equilibrium of the optimally controlled system, i.e., of the closed-loop system with feedback (7).*

Proof. In order to prove the result it will be convenient to look at the problem in its transformed version using matrix Δ_a defined in (21), which defines the optimal cost for the infinite-horizon optimal control problem with zero penalization of the state in the stage cost and terminal cost matrix $\Delta_0 = E \succ 0$.

Lemmas 3.18–3.19 establish a direct connection between the solutions P of the DARE and the solutions Δ_a of the transformed DARE (21) and the corresponding Riccati iterations. Consequently, applying Theorem 4.2 to the transformed DARE (21) with the terminal cost being positive definite and the stage cost positive semidefinite, we have that the solution to the infinite-horizon LQR must coincide with the unique stabilizing solution of the transformed DARE (21). In turn, Lemmas 3.18–3.19 entail that the result also applies to the original DARE. □

Corollary 4.4. *Consider any $P \in \bar{\mathcal{P}}$, i.e., $P \neq P_s$ solving the DARE (6). Then, if P_a exists, $P \nsucc P_a$.*

Proof. Our proof will exploit uniqueness of the stabilizing solution, i.e., that $P \neq P_s$ cannot be a stabilizing solution.

Assume that $P \in \bar{\mathcal{P}}$ and $P_s \neq P \succ P_a$. Because P is a symmetric solution to the DARE and $R + B^\top PB \succ 0$, when using P as terminal cost matrix, the cost-to-go matrix of the LQR associated with the DARE is P for any horizon length. However, by Theorem 4.3, because $P \succ P_a$, the infinite-horizon LQR yields the unique stabilizing solution to the DARE (6). Since this contradicts uniqueness of the stabilizing solution, $P \nsucc P_a$. □

Remark 4.5. Note that the results above implicitly prove that the unique stabilizing solution is the only solution to the DARE satisfying $P \succ P_a$ and all other solutions can only satisfy $P \succeq P_a$. □

Remark 4.6. Note that, though we proved the result for terminal cost matrix $P_0 = P_a + E$, the result clearly holds for terminal cost matrix $P_0 = \tilde{P} + E$, with $\tilde{P}$ any solution of the DARE. However, referring the result to P_a yields the least restrictive condition as, by Lemma 3.22 we have $P_a \preceq \tilde{P}$. Moreover, this sufficient condition is also close to being necessary, as selecting $E = 0$ yields a cost-to-go matrix $P_n = P_a$ for all n, with a corresponding destabilizing feedback. A similar reasoning holds for $E = P - P_a$, where $P \neq P_s$ is any other non-stabilizing symmetric solution to the DARE, in which case the closed-loop system has both stable and unstable eigenvalues. □

4.2. The antistabilizing solution does not exist

In this subsection we extend the previous results to also cover the case in which the antistabilizing solution of the DARE does not exist.

Theorem 4.7. *Consider a strictly pre-dissipative LQR problem with controllable (A, B) and let $\bar{P}_s$ be the stabilizing solution of the RDARE (which exists according to Lemma 3.12). Choose the terminal cost matrix $P_f = \bar{P}_s + E$, with $E \succ 0$. Then the optimal solution is given by the unique stabilizing solution of the corresponding DARE. In particular, the origin is an asymptotically stable equilibrium of the optimally controlled system.*

Proof. We define H^ϵ as

$$0 \prec (1-\epsilon)H \preceq H^\epsilon := \begin{bmatrix} (1-\epsilon)Q & (1-\epsilon)S^\top \\ (1-\epsilon)S & (1-\epsilon)R + T^\epsilon \end{bmatrix} \preceq H,$$

with $1 > \epsilon > 0$ sufficiently small and T^ϵ such that $\epsilon R \succeq T^\epsilon \succ 0$. For this cost we have that the condition for the existence of the antistabilizing solution is the nonsingularity of

$$(1-\epsilon)(R - S_{\hat{K}} A_{\hat{K}}^{-1} B) + T^\epsilon.$$

Because T^ϵ can be chosen in the cone $\epsilon R \succeq T^\epsilon \succ 0$, for all ϵ there exists T^ϵ such that the matrix above is nonsingular. To prove that, we write

$$NWN^{-1} = (1-\epsilon)(R - S_{\hat{K}} A_{\hat{K}}^{-1} B),$$

in Jordan form and define

$$T^\epsilon := N\bar{W}N^{-1},$$

with $\bar{W}$ a diagonal matrix such that

$$\bar{W}_{ii} = \begin{cases} a & \text{if } W_{ii} = 0, \\ 0 & \text{otherwise} \end{cases},$$

with $0 < a < \epsilon \sigma_{\min}(R)$. By construction, we then have that $(1-\epsilon)(R - S_{\hat{K}} A_{\hat{K}}^{-1} B) + T^\epsilon = N(W + \bar{W})N^{-1}$ is nonsingular. Consequently, for all ϵ under consideration the antistabilizing solution P_a^ϵ exists and matches the stabilizing solution $\bar{P}_s^\epsilon$ of the RDARE. Note that because $H^\epsilon \preceq H$, we have $(1-\epsilon)\bar{P}_s \succeq P_a^\epsilon = \bar{P}_s^\epsilon \succeq \bar{P}_s$. Here $\bar{P}_s$ denotes the stabilizing solution of the RDARE for H, for which $P_s^\epsilon \to \bar{P}_s$ as $\epsilon \to 0$ holds. We recall that these are the RDARE stabilizing solutions, which decrease as the stage cost increases.

One can now select ϵ small enough such that $E \succ P_a^\epsilon - \bar{P}_s$. Then, for all such ϵ by Theorem 4.3 the optimal solution for the problem with cost H^ϵ is given by the stabilizing solution P_s^ϵ of the DARE for H^ϵ, i.e., the optimal value function is given by

$$V^\epsilon(x) = x^T P_s^\epsilon x.$$

Together with the ordering of the matrices H^ϵ this implies that

$$(1-\epsilon)x^T P_s x \leq V^\epsilon(x) \leq x^T P_s x.$$

Hence, for $\epsilon \to 0$ we get that $V(x) = x^T P_s x$ which shows the claim. □

Remark 4.8. Since by Lemma 3.5 the optimal solutions for the original and the rotated cost coincide, under the condition of Theorem 4.7, it follows that the origin is also exponentially stable for the infinite horizon optimal solutions for the rotated cost. This implies that according to (27) the optimal value function satisfies

$$V(x_0) = x_0^T P_\lambda x_0$$
$$= \lim_{N\to\infty} \sum_{k=0}^{N-1} (A_K^k x_0)^\top (Q_K + \Lambda - A_K^\top \Lambda A_K) A_K^k x_0,$$

as the terminal cost in (27) vanishes for $N \to \infty$. Hence, the matrix P_λ is positive definite, since the first term in the above sum is positive definite and all others are positive semidefinite, because of quadratic strict (x, u)-pre-dissipativity. As Lemma 3.5 states that $P_\lambda = P_s + \Lambda$, this implies that $P_s + \Lambda \succ 0$. □

4.3. Convergence implies exponential stability

With the results above, we have proven that the cost-to-go matrix converges to the stabilizing solution to the DARE, such that exponential stability is obtained for an infinite horizon LQR with a properly selected terminal cost. We prove next that exponential stability is also obtained by the receding horizon feedback law for a sufficiently long but finite horizon N.

Theorem 4.9. *Consider a strictly pre-dissipative LQR problem with controllable (A, B) and let $\bar{P}_s$ be the stabilizing solution of the RDARE (which exists according to Lemma 3.12). Select the terminal cost matrix as $P_f \succ \bar{P}_s$. Assume that RH-OCP (2) is formulated without constraints (2d). Then, for any sufficiently large finite horizon N the RH-OCP (2) yields a closed-loop system (4) for which the origin is globally exponentially stable.*

Proof. By standard Lyapunov function arguments, exponential stability follows if we show that for all sufficiently large N the iterate $P_{\lambda,N}$ of the rotated Riccati iteration is positive definite and satisfies

$$A_{K_N}^\top P_{\lambda,N} A_{K_N} - P_{\lambda,N} \prec 0. \quad (28)$$

We know that

$$P_{\lambda,n} = Q_{\lambda,K_n} + A_{K_n}^\top P_{\lambda,n-1} A_{K_n},$$

with $Q_{\lambda,K_n} = Q_{K_n} + \Lambda - A_{K_n}^\top \Lambda A_{K_n}$. This implies

$$A_{K_n}^\top P_{\lambda,n} A_{K_n} - P_{\lambda,n} = -Q_{\lambda,K_n} + A_{K_n}^\top (P_{\lambda,n} - P_{\lambda,n-1}) A_{K_n},$$

i.e., the decrease condition for stability (28) becomes

$$Q_{\lambda,K_n} \succ A_{K_n}^\top (P_{\lambda,n} - P_{\lambda,n-1}) A_{K_n}. \tag{29}$$

Now, as $n \to \infty$, the left hand side converges to $Q_{\lambda,K}$, which is positive definite because of quadratic strict (x, u)-pre-dissipativity, while the right hand side converges to 0, since $K_n \to K$ and $P_{\lambda,n} \to P_s + \Lambda$, which is positive definite as explained in Remark 4.8. Hence, there is $N > 0$ such that (29) holds and $P_{\lambda,n} \succ 0$ for all $n \geq N$, which shows the claim. □

Remark 4.10. Note that, while our result is valid for the case without path constraints (2d), the results can be extended to the constrained case by combining our results with, e.g., Boccia et al. (2014, Theorem 13). The full developments of such connection are left for future research.

Remark 4.11. Another known way to obtain stability by adding a terminal cost is described (for general nonlinear problems) in Amrit et al. (2011, Assumption 6). There, the terminal cost defines a control Lyapunov function that is compatible with the stage cost. For the LQ setting of this paper, computing such a terminal cost amounts to finding a stabilizing controller and then solving a Lyapunov equation. Hence, computationally this approach is simpler than finding the lower bound $\bar{P}_s$ in our approach, which involves solving a Riccati equation. Yet, we can find some $P_f \succ \bar{P}_s$ also by solving a Lyapunov equation for a stabilizing controller, hence the effort for computing some (i.e., not necessarily a small) stabilizing terminal cost has comparable effort in both approaches. An advantage of our result is that for LQ-RHC it gives a rigorous justification for the folklore result that adding a sufficiently large terminal cost (which may even be found by a trial-and-error procedure) stabilizes the RH closed loop and that it precisely defines what "sufficiently large" means.

Remark 4.12. For the LQ problem (5), imposing stabilizing terminal conditions means that the resulting solution is not the overall optimal solution anymore but the *optimal stabilizing solution*. With the same proof technique as for Grüne and Pannek (2017b, Theorem 8.21) one can show that the cost of the MPC closed-loop solution approximates the cost of this optimal stabilizing LQ solution. Hence, we obtain asymptotic stability of the RH closed loop with approximately optimal transient behavior.

5. Relation with known results

In this section we connect our results to similar ones available in the literature.

5.1. The nonlinear case

In the companion paper (Grüne & Zanon, 2025) we discuss the problem for the general nonlinear case. For the linear quadratic case the conditions derived therein require the existence of matrix Λ such that

$$P_f \succ -\Lambda, \qquad H_\Lambda \succeq 0. \tag{30}$$

We observe that, by Theorem 3.16 we have that the choice $\Lambda = -\bar{P}_s$ yields a positive semidefinite rotated stage cost. If P_a does not exist, then using $-\Lambda \prec \bar{P}_s$ entails that matrix $R - B^\top \Lambda B$ must have some negative eigenvalue, as $-B^\top \Lambda B \preceq B^\top \bar{P}_s B$, and $-B^\top \Lambda B \neq B^\top \bar{P}_s B$. In case P_a exists, using the same arguments as in Lemma 4.1 one can prove that, for $P_f = \bar{P}_s - \alpha \Xi_s$, with $\alpha > 0$ we have $P_1 \preceq P_f$. The Schur complement of H_Λ reads

$$\begin{aligned} &Q + \Lambda - A^\top \Lambda A \\ &\qquad - (S^\top - A^\top \Lambda B)(R - B^\top \Lambda B)^{-1}(S - B^\top \Lambda A) \\ &= P_1 + P_f \preceq 0. \end{aligned}$$

Since $P_1 \neq P_f$ the Schur complement must necessarily have some negative eigenvalue. Consequently, no choice $-\Lambda \prec \bar{P}_s$ can yield a positive semidefinite rotated stage cost matrix H_Λ. In turn, this entails that conditions (30) in their least restrictive form become

$$P_f \succ \bar{P}_s, \qquad H_\Lambda \succeq 0.$$

which are only marginally different from the conditions we derive in this paper, which require the slightly stronger condition $H_\Lambda \succ 0$. Note that this condition is required to make sure that the DARE does have a solution. In other words, this paper shows that the conditions in the companion paper (Grüne & Zanon, 2025) for general nonlinear problems are almost tight in the nonsingular linear quadratic case. "Almost" here refers to the fact that there is no general statement for the case $P_f \succeq \bar{P}_s$.

As a side remark, we also observe that, using the same arguments as for $\bar{P}_s$, one also obtains that no choice $-\Lambda \succ P_s$ can yield a positive semidefinite rotated stage cost matrix H_Λ.

5.2. Relation to first-order cost corrections

In this paper we focused on the quadratic term in the terminal cost, while in Faulwasser and Zanon (2018) and Zanon and Faulwasser (2018) the impact of the linear term in the quadratic cost was analyzed. In particular, it has been proven that a wrong gradient (in the context of this paper the correct gradient is $\nabla V_f(0) = 0$) impedes asymptotic stability, though under some technical assumptions one still obtains practical stability (Grüne, 2013). Even if the system starts at the optimal steady state (in our case the origin), the closed-loop system immediately leaves it. Our result allows us to further comment on this aspect. Indeed, in case the antistabilizing solution P_a exists, selecting $P_f = P_a$ yields a destabilizing controller for all initial states $x_0 \neq 0$. Clearly, this is a limit case of little practical interest, but it illustrates the fact that, while first-order conditions distinguish to which steady-state the system is stabilized, if at all, second-order conditions on the terminal cost, instead, make the difference between stability and instability, with the special limit case discussed above.

6. Simulation results

Consider the case of Example 2.1. We recall that $P_a = 0$ and we display in Fig. 1 the minimum horizon length yielding asymptotic stability, which we computed for several terminal costs by solving the Riccati equation and evaluating the eigenvalues of $A - BK_N$. Furthermore, we display in Fig. 2 the absolute value of the eigenvalue of the closed-loop system matrix for different N and $P_f = 10^{-4}$. One can see that stability is obtained for $N \geq 8$.

Let us introduce the constraint $|x| \leq 1$. Since this constraint ensures that the state remains bounded, also the storage function remains bounded and the results of Grüne (2013) guarantee that the system converges to a neighborhood of the origin whose size decreases with increased prediction horizons. The results of Faulwasser and Zanon (2018) guarantee convergence to the origin for a sufficiently long prediction horizon, under the condition that the unconstrained control law associated with the local linear-quadratic approximation of the system is stabilizing. For this example, unless one adds a terminal cost the unconstrained control law is not stabilizing. Indeed, we observe that in this setting, once the prediction horizon is long enough such that the

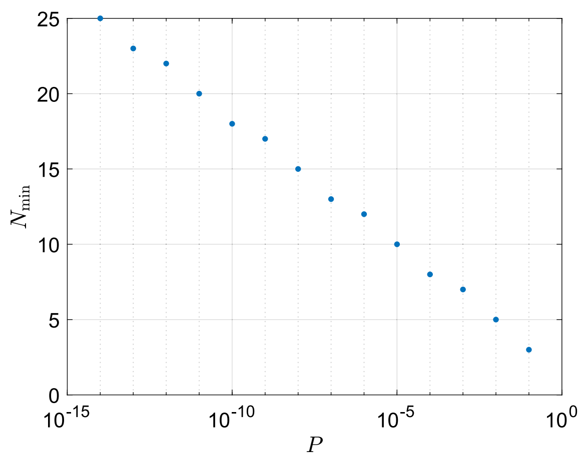

Fig. 1. Minimum stabilizing horizon length.

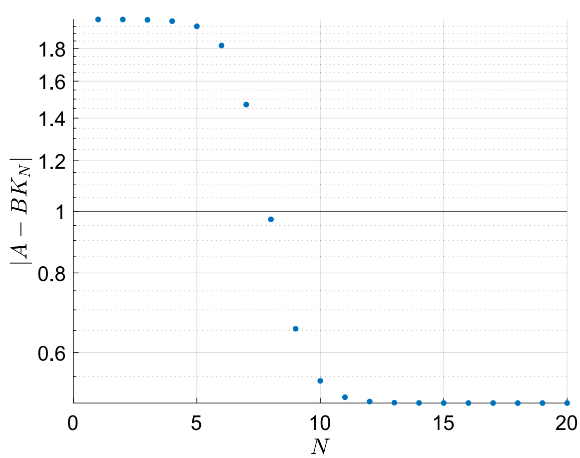

Fig. 2. Closed-loop matrix eigenvalues with $P_f = 10^{-4}$.

unconstrained control law becomes stabilizing, then the system is indeed stabilized to the origin. This is shown in Fig. 3, where, after starting from initial state $\hat{x}_0 = 1$, the final state after $N_{\text{sim}} = 500$ time steps is never exactly 0 is due to, on the one hand the finite simulation horizon, and, on the other hand, the fact that the RH QP is solved to finite precision. Nevertheless, a clear difference between the case $P_f = 10^{-4}$ and the case $P_f = 0$ is seen for $N > 8$, which is the horizon length starting from which the LQR is stabilizing also in the absence of constraints, as shown in Fig. 2.

7. Conclusions

We have discussed the role of suitably chosen terminal costs in yielding exponential stability in the linear-quadratic case by establishing a strong connection to the symmetric solutions of the associated (reverse) discrete-time algebraic Riccati equation. We have connected our results to those obtained for the full nonlinear case and we have provided simple examples to illustrate our results.

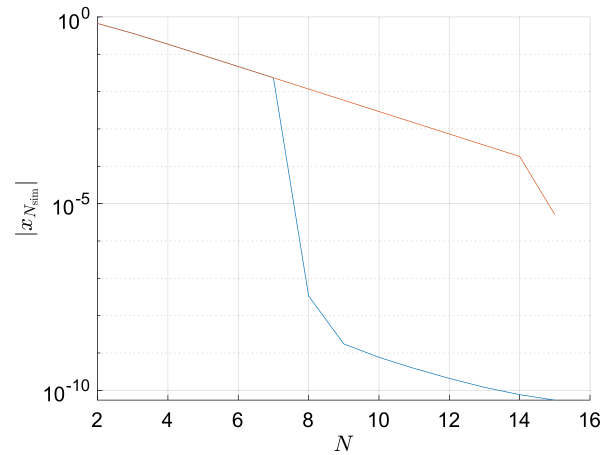

Fig. 3. State at time $k = 500$ obtained with $P_f = 0$ (red) and $P_f = 1e-4$ (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

References

Amrit, R., Rawlings, J., & Angeli, D. (2011). Economic optimization using model predictive control with a terminal cost. *Annual Reviews in Control*, *35*, 178–186.

Anderson, B. D. O., & Moore, J. B. (1990). *Optimal control - linear quadratic methods*. Dover.

Angeli, D., Amrit, R., & Rawlings, J. (2012). On average performance and stability of economic model predictive control. *IEEE Transactions on Automatic Control*, *57*, 1615–1626.

Boccia, A., Grüne, L., & Worthmann, K. (2014). Stability and feasibility of state constrained MPC without stabilizing terminal constraints. *Systems & Control Letters*, *72*, 14–21.

Caverly, R. J., & Forbes, J. R. (2024). LMI properties and applications in systems, stability, and control theory. arXiv https://arxiv.org/abs/1903.08599.

Damm, T., Grüne, L., Stieler, M., & Worthmann, K. (2014). An exponential turnpike theorem for dissipative discrete time optimal control problems. *SIAM Journal on Control and Optimization*, *52*(3), 1935–1957.

Diehl, M., Amrit, R., & Rawlings, J. B. (2011a). A Lyapunov function for economic optimizing model predictive control. *IEEE Transactions on Automatic Control*, *56*, 703–707.

Diehl, M., Amrit, R., & Rawlings, J. (2011b). A Lyapunov function for economic optimizing model predictive control. *IEEE Transactions of Automatic Control*, *56*(3), 703–707. http://dx.doi.org/10.1109/TAC.2010.2101291.

Faulwasser, T., Grüne, L., & Müller, M. (2018). Economic nonlinear model predictive control: Stability, optimality and performance. *Foundations and Trends in Systems and Control*, *5*(1), 1–98. http://dx.doi.org/10.1561/2600000014.

Faulwasser, T., & Zanon, M. (2018). Asymptotic stability of economic NMPC: The importance of adjoints. In *Proceedings of the IFAC nonlinear model predictive control conference*.

Freiling, G., & Ionescu, V. (1999). Time-varying discrete Riccati equation: some monotonicity results. *Linear Algebra and its Applications*, *286*(1?3), 135–148. http://dx.doi.org/10.1016/s0024-3795(98)10166-0.

Grüne, L. (2013). Economic receding horizon control without terminal constraints. *Automatica*, *49*, 725–734.

Grüne, L., & Guglielmi, R. (2018). Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems. *SIAM Journal on Control and Optimization*, *56*(2), 1282–1302.

Grüne, L., & Pannek, J. (2017a). *Communications and control engineering*, *Nonlinear model predictive control* (2nd ed.). Springer International Publishing.

Grüne, L., & Pannek, J. (2017b). *Nonlinear predictive control. Theory and algorithms* (2nd ed.). London: Springer-Verlag.

Grüne, L., & Stieler, M. (2014). A Lyapunov function for economic MPC without terminal conditions. In *Proc. of the 53rd IEEE conference on decision and control* (pp. 2740–2745).

Grüne, L., & Zanon, M. (2025). Stabilization of strictly pre-dissipative nonlinear receding horizon control by terminal costs. [Online]. Available: https://arxiv.org/abs/2412.13538, in press in Automatica.

Ionescu, V. (1996). Reverse discrete-time Riccati equation and extended Nehari's problem. *Linear Algebra and its Applications*, *236*, 59–94.

Kottenstette, N., McCourt, M. J., Xia, M., Gupta, V., & Antsaklis, P. J. (2014). On relationships among passivity, positive realness, and dissipativity in linear systems. *Automatica*, *50*(4), 1003–1016.

Lancaster, P., Ran, A. C. M., & Rodman, L. (1986). Hermitian solutions of the discrete algebraic Riccati equation. *International Journal of Control*, *44*, 777–802.

Molinari, B. P. (1975). The stabilizing solution of the discrete algebraic Riccati equation. *IEEE Transactions on Automatic Control*, *20*, 396–399.

Müller, M., Angeli, D., & Allgöwer, F. (2013). On convergence of averagely constrained economic MPC and necessity of dissipativity for optimal steady-state operation. In *Proceedings of the American control conference*.

Müller, M. A., Angeli, D., & Allgöwer, F. (2015). On necessity and robustness of dissipativity in economic model predictive control. *IEEE Transactions on Automatic Control*, *60*(6), 1671–1676.

Müller, M. A., & Grüne, L. (2016). Economic model predictive control without terminal constraints for optimal periodic behavior. *Automatica*, *70*, 128–139.

Ran, A. C. M., & Trentelman, H. L. (1993). Linear quadratic problems with indefinite cost for discrete time systems. *SIAM Journal on Matrix Analysis and Applications*, *14*, 776–797.

Rawlings, J. B., Mayne, D. Q., & Diehl, M. (2017). *Model predictive control: Theory, computation, and design* (2nd ed.). Nob Hill Publishing.

Stoorvogel, A. A., & Saberi, A. (1998). The discrete algebraic Riccati equation and linear matrix inequality. *Linear Algebra and its Applications*, *274*(1), 317–365.

Willems, J. C. (1971). Least squares stationary optimal control and the algebraic riccati equation. *IEEE Transactions on Automatic Control*, *AC-16*(6), 621–634.

Willems, J. (1972a). Dissipative dynamical systems Part I: General theory. *Archive for Rational Mechanics and Analysis*, *45*(5), 321–351.

Willems, J. (1972b). Dissipative dynamical systems Part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, *45*(5), 352–393.

Zanon, M., & Faulwasser, T. (2018). Economic MPC without terminal constraints: Gradient-correcting end penalties enforce asymptotic stability. *Journal of Process Control*, *63*, 1–14.

Zanon, M., Gros, S., & Diehl, M. (2014). Indefinite linear MPC and approximated economic MPC for nonlinear systems. *Journal of Process Control*, *24*, 1273–1281.

Zanon, M., Gros, S., & Diehl, M. (2016). A tracking MPC formulation that is locally equivalent to economic MPC. *Journal of Process Control*, *45*, 30–42.

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