Periodic optimal control, dissipativity and MPC

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***Abstract*—Recent research has established the importance of (strict) dissipativity for proving stability of economic MPC in the case of an optimal steady state. In many cases, though, steady state operation is not economically optimal and periodic operation of the system yields a better performance. In this paper, we propose ways of extending the notion of (strict) dissipativity for periodic systems. We prove that optimal P-periodic operation and MPC stability directly follow, similarly to the steady state case, which can be seen as a special case of the proposed framework. Finally, we illustrate the theoretical results with several simple examples.**

***Index Terms*—Periodic Economic MPC, Strict dissipativity**

I. Introduction

Economic MPC is a variant of model predictive control (MPC) in which the objective consists in directly optimising a given performance index as opposed to tracking a given reference. The main advantage of economic MPC over tracking MPC becomes apparent in transients, when the system is steered to steady state while minimising the given performance index.

Unfortunately, proving stability of economic MPC schemes is hard, as the stage cost $\ell(x,u)$ does in general not have a pointwise minimum on the trajectory the system converges to. The idea of rotating the cost using the Lagrange multipliers λ has been proposed in [6] in order to prove stability. The proof relies on an equivalent auxiliary MPC scheme with a rotated stage cost that has a stationary point at the optimal steady state. The rotated stage cost is obtained by adding the term $\lambda^\top x - \lambda^\top f(x,u)$ to the stage cost. In [3], [1] this idea has been extended to a nonlinear rotation, given by a function $\lambda(x)$. This generalisation is equivalent to the systems theoretic notion of strict dissipativity [14], [15] with λ as a storage function and allows one to rotate the stage cost such that it is bounded from below by a positive definite function. For a given system and stage cost, if there exists a storage function $\lambda(x)$ that satisfies a strict dissipativity property, then stability of the MPC scheme is guaranteed.

As opposed to previous techniques for periodic control [4], periodic economic MPC comes with performance guarantees [3, Theorem 2]. A first extension of the dissipativity framework has been proposed in [16] for time varying systems: the Lagrange multipliers λ_k of a periodic optimal trajectory are used to rotate the cost with a linear (time varying) term. In contrast to this reference, in this paper we consider optimal periodic trajectories for time invariant dynamics and stage costs. To this end, we propose and discuss two different ways of extending the definition of dissipativity based on the newly introduced notion of set-valued distance of a point from a periodic trajectory. We use this new dissipativity notion to prove optimality properties of periodic orbits and stability of periodic economic MPC schemes using appropriate terminal constraints and costs.

The paper is structured as follows. Section II introduces the basic notation and summarises previous results obtained for the steady state case. The newly proposed concept of P-periodic dissipativity is introduced in Section III, and in Section IV previous results on optimal operation at steady state are extended to the periodic case. The stability proof for periodic economic MPC is given in Section V. Some simple examples are presented in Section VI in order to illustrate the theory. Conclusions and a discussion on future research directions are given in Section VII.

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II. Setting

We consider discrete time nonlinear systems of the form

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

with $f : X \times U \to X$, with $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$. Solutions for initial value x and control sequence $\boldsymbol{u}$ are denoted by $x_k^{\boldsymbol{u}}(x)$.

We assume that f is continuous in (x,u) and the system is subject to state and input constraints $(x_k,u_k) \in \mathbb{Z} \subset \mathbb{X} \times \mathbb{U}$ for all $k \geq 0$. In the MPC framework, the system is equipped with a stage cost $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ which is assumed to be continuous.

For a given constraint set $\mathbb{Z}$, each initial value $x \in \mathbb{X}$ and any $N \geq 1$ we denote the set of admissible trajectories by $\mathbb{Z}^N(x) := \{(\boldsymbol{x},\boldsymbol{u}) \,|\, (x_k,u_k) \in \mathbb{Z},\ x_k = x_k^{\boldsymbol{u}}(x)\ \forall\ k = 0,\ldots,N-1\}$. Analogously we define $\mathbb{Z}^\infty(x)$. For simplicity of exposition we assume $\mathbb{Z}$ to be compact. An extension of our results to non compact $\mathbb{Z}$ would be possible but would require regional bounds on the involved functions and assumptions on the sets in which optimal trajectories evolve. As the corresponding technical overhead might obscure the main arguments of our paper, we prefer to work with the compactness assumption. We consider the finite horizon functional

$$J_N(\boldsymbol{x},\boldsymbol{u}) := \sum_{k=0}^{N-1} \ell(x_k,u_k) + V_\mathrm{f}(x_N),$$

where $V_\mathrm{f}(x_N)$ is the so-called terminal cost. Further, for an initial state x, we define the infinite horizon averaged functional

$$J_\infty^{\mathrm{av}}(\boldsymbol{x},\boldsymbol{u}) := \limsup_{N\to\infty} \frac{1}{N} J_N(\boldsymbol{x},\boldsymbol{u}).$$

Given an initial value $x \in \mathbb{X}$, the basic model predictive control (MPC) scheme with nominal system dynamics works as follows:

(i) set the time index $i := 0$ and initial state $x_0^{\mathrm{MPC,cl}} = x$

(ii) minimise $J_N(\boldsymbol{x},\boldsymbol{u})$ over all admissible trajectories $(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{Z}^N\left(x_i^{\mathrm{MPC,cl}}\right)$ and denote the optimal sequence by $\boldsymbol{u}^\star$

(iii) set $u_i^{\mathrm{MPC,cl}} := u_0^\star$, $x_{i+1}^{\mathrm{MPC,cl}} := f\left(x_i^{\mathrm{MPC,cl}}, u_i^{\mathrm{MPC,cl}}\right)$, $i := i+1$ and go to (ii)

Since the stage cost ℓ is not of tracking type (i.e. does not necessarily penalise the distance to a given equilibrium) this MPC scheme is often termed *economic MPC* [1], [2]. In this setting, the classical notion of (strict) dissipativity [14], [15] has recently gained renewed interest.

Definition 2.1 (Strict Dissipativity [1]): System (1) is dissipative with respect to a steady state $(x^\mathrm{s},u^\mathrm{s}) \in \mathbb{Z}$ of (1) for supply rate $\ell(x,u) - \ell(x^\mathrm{s},u^\mathrm{s})$ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ such that the inequality $L(x,u) := \ell(x,u) - \ell(x^\mathrm{s},u^\mathrm{s}) + \lambda(x) - \lambda(f(x,u)) \geq 0$ holds for all $(x,u) \in \mathbb{Z}$. If, in addition, there exists a function $\rho \in \mathcal{K}_\infty$ such that the inequality $L(x,u) \geq \rho\left(\|x - x^\mathrm{s}\|\right)$ holds, then the system (1) is strictly dissipative on $\mathbb{Z}$.

If a system equipped with a stage cost ℓ is (strictly) dissipative, then this has several consequences:

- The system is optimally operated at (uniformly suboptimally operated off) steady state [2], [10].
- For economic MPC with terminal constraint, the averaged performance $J_\infty^{\mathrm{av}}\left(\boldsymbol{x}^{\mathrm{MPC,cl}},\boldsymbol{u}^{\mathrm{MPC,cl}}\right)$ equals $\ell(x^{\mathrm{s}},u^{\mathrm{s}})$ and the steady state x^{s} is asymptotically stable for the closed loop solutions. This was shown for endpoint constraints in [6] for linear storage functions and in [2] for general storage functions as well as for regional constraints and terminal costs in [1].
- For economic MPC without terminal constraint, the averaged performance $J_\infty^{\mathrm{av}}\left(\boldsymbol{x}^{\mathrm{MPC,cl}},\boldsymbol{u}^{\mathrm{MPC,cl}}\right)$ equals $\ell(x^{\mathrm{s}},u^{\mathrm{s}})+\varepsilon(N)$ and the optimal equilibrium is practically asymptotically stable, cf. [9], [7]. Moreover, approximate transient optimality was shown in these references and — under an exponential turnpike property which in turn is implied by dissipativity and suitable controllability properties [5] — the error terms converge to 0 exponentially fast as $N\to\infty$.

For general discrete time optimal control problems, it is well known that the optimal value is not necessarily attained at an equilibrium. Particularly, it may happen that periodic orbits exhibit smaller average values than any feasible equilibrium, see, e.g., [2, Section VII] or our examples below. In this case, the existing theory based on dissipativity of an equilibrium is not applicable and does thus not ensure asymptotic stability of the optimal periodic orbit. For this reason, in the next section we discuss dissipativity notions which are adapted to characterising periodic orbits.

III. Periodic Dissipativity

In this section, we introduce concepts of P-periodic (strict) dissipativity. In the following sections we analyse how they relate to optimal P-periodic operation and periodic EMPC stability. Let us first give definitions of periodic orbits and periodic trajectories.

Definition 3.1 (Periodic Orbit): An ordered P-tuple of points $\Pi=(\bar{x}_0^{\mathrm{p}},\dots,\bar{x}_{P-1}^{\mathrm{p}})$, $P\geq 1$, is called a *feasible P-periodic orbit* with control sequence $(\bar{u}_0^{\mathrm{p}},\dots,\bar{u}_{P-1}^{\mathrm{p}})$ if $(\bar{x}_k^{\mathrm{p}},\bar{u}_k^{\mathrm{p}})\in\mathbb{Z}$, $k=0,\dots,P-1$,

$$\bar{x}_{k+1}^{\mathrm{p}}=f(\bar{x}_k^{\mathrm{p}},\bar{u}_k^{\mathrm{p}})\quad \text{for } k=0,\dots,P-2,$$

and $\bar{x}_0^{\mathrm{p}}=f(\bar{x}_{P-1}^{\mathrm{p}},\bar{u}_{P-1}^{\mathrm{p}})$. The number P is called the *period* of the orbit Π and if there is no $Q\geq 1$ with $Q<P$ such that $(\bar{x}_k^{\mathrm{p}},\bar{u}_k^{\mathrm{p}})=(\bar{x}_{k+Q}^{\mathrm{p}},\bar{u}_{k+Q}^{\mathrm{p}})$ for all $k=0,\dots,P-Q$, then P is called the *minimal period* of Π. Given the corresponding control sequence $\bar{u}_0^{\mathrm{p}},\dots,\bar{u}_{P-1}^{\mathrm{p}}$ we define the tuple of state-control pairs $\Pi_U:=\left((\bar{x}_0^{\mathrm{p}},\bar{u}_0^{\mathrm{p}}),\dots,(\bar{x}_{P-1}^{\mathrm{p}},\bar{u}_{P-1}^{\mathrm{p}})\right)$.

Note that in our terminology an equilibrium is a periodic orbit with period $P=1$. Moreover, for $P>1$, the periodic orbit is not unique, as phase shifts produce an orbit which is defined by the same states and controls, but in a shifted order. For this reason, we define in the following the periodic trajectory as a periodic orbit with a fixed phase, extended infinitely long into the future.

Definition 3.2 (Periodic Trajectory): (i) A sequence $X^P=(x_0,x_1,x_2,\dots)$, is called a *feasible P-periodic trajectory* with control sequence $U^P=(u_0,u_1,u_2,\dots)$ if $(x_k,u_k)\in\mathbb{Z}$, $x_k=x_{k+P}$, $u_k=u_{k+P}$ for all $k=0,1,\dots$, and

$$x_{k+1}=f(x_k,u_k)\quad \text{for } k=0,1,\dots.$$

(ii) Given a P-periodic orbit $\Pi=(\bar{x}_0^{\mathrm{p}},\dots,\bar{x}_P^{\mathrm{p}})$ and a *phase* $\phi\in\{0,\dots,P-1\}$, we define the infinite sequence

$$X_\phi^P(\Pi):=(\bar{x}_\phi^{\mathrm{p}},\dots,\bar{x}_{P-1}^{\mathrm{p}},\bar{x}_0^{\mathrm{p}},\dots,\bar{x}_{P-1}^{\mathrm{p}},\dots).$$

The points on $X_\phi^P(\Pi)$ will be denoted by x_k^ϕ, i.e, $x_k^\phi=\bar{x}^{\mathrm{p}}_{(k+\phi)\mod P}$, where mod is the common modulus operator, and the corresponding control values by u_k^ϕ.

For any P-periodic trajectory, the ordered tuple $(\bar{x}_0^{\mathrm{p}},\dots,\bar{x}_{P-1}^{\mathrm{p}})=(x_0^{\mathrm{p}},\dots,x_{P-1}^{\mathrm{p}})$ is a P-periodic orbit Π. Conversely, for every P-periodic orbit Π and any $\phi\in\{0,\dots,P-1\}$ the sequence $X_\phi^P(\Pi)$ from (ii) is a P-periodic trajectory in the sense of (i).

We extend the definition of (strict) dissipativity to periodic orbits as a generalisation of [1]. To this end, in what follows we denote the particular periodic orbit for which the system is dissipative by Π^* with corresponding control sequence $\boldsymbol{u}^*$. The corresponding elements will be denoted by $\bar{x}_k^{\mathrm{p}*}$ and $\bar{u}_k^{\mathrm{p}*}$. Given a phase ϕ, we denote the elements of the corresponding P-periodic trajectory $X_\phi^P(\Pi^*)$ by $(x_0^{\phi*},x_1^{\phi*},\dots)$ and the corresponding control values by $(u_0^{\phi*},u_1^{\phi*},\dots)$. Let us define the two notions of distance

$$|x|_{\Pi^*}:=\min_{\bar{x}_k^{\mathrm{p}*}\in\Pi}\|x-\bar{x}_k^{\mathrm{p}*}\|,$$

$$|(x,u)|_{\Pi_U^*}:=\min_{(\bar{x}_k^{\mathrm{p}*},\bar{u}_k^{\mathrm{p}*})\in\Pi_U}\|x-\bar{x}_k^{\mathrm{p}*}\|+\|u-\bar{u}_k^{\mathrm{p}*}\|.$$

Let us define functions $\sigma^\bullet(x,u)$ as

$$\sigma^{\mathrm{A}}(x,u):=\rho(|(x,u)|_{\Pi_U})\qquad (2)$$

$$\text{or}\qquad \sigma^{\mathrm{B}}(x,u):=\rho(|x|_\Pi),\qquad (3)$$

with ρ being a positive definite function. We remark that in case of (3) function $\sigma^{\mathrm{B}}(\cdot,\cdot)$ does not depend on u, but in order to obtain a uniform notation in what follows we always write $\sigma^\bullet(x,u)$.

Definition 3.3 (P-Periodic (Strict) Dissipativity): The system (1) is P-periodic dissipative on a set $\mathbb{Z}\subset\mathbb{X}\times\mathbb{U}$ with respect to the supply rate $\ell(x,u)-\ell(x_k^\phi,u_k^\phi)$ if there exists a feasible P-periodic orbit Π^* a phase ϕ and bounded storage functions $\lambda_0,\dots,\lambda_{P-1},\lambda_P,\dots:X\to\mathbb{R}$, with $\lambda_{k+P}=\lambda_k$ such that the inequalities

$$L_k(x,u):=\ell(x,u)-\ell(x_k^\phi,u_k^\phi)+\lambda_k(x)-\lambda_{k+1}(f(x,u))\geq 0\quad (4)$$

hold for all $(x,u)\in\mathbb{Z}$, where x_k^ϕ are the elements of the sequence $X_\phi^P(\Pi)$ and all $k=0,1,\dots$. If, in addition, there exist functions of the form (2) or (3) such that

$$L_k(x,u)\geq\sigma^\bullet(x,u),\quad \bullet\in\{\mathrm{A},\mathrm{B}\}\qquad (5)$$

holds, then the system (1) satisfies P-periodic strict dissipativity of type A or B, respectively, on $\mathbb{Z}$.

It is easily seen that for (3) this definition is equivalent to Definition 2.1 in case $P=1$. Moreover, for $P>1$, (strict) dissipativity might hold for more than one phase ϕ. It is however not true that if strict dissipativity holds for one phase ϕ then it holds for all phases. Indeed, while functions λ_k could be shifted in time, the phase ϕ fixes x_k^ϕ, u_k^ϕ, see e.g. Example 6.2. While this can be restrictive if one is interested in the actual computation of $L_k(x,u)$, this does not constitute any problem for the theoretical results that we aim at establishing next, i.e. optimal P-periodic operation (uniform suboptimal non P-periodic operation), and sufficiency of strict P-periodic dissipativity for P-periodic stability of EMPC.

Remark 3.4: As it holds that $|(x,u)|_{\Pi_U^*}\geq|x|_{\Pi^*}$, Definition (5) in the sense A implies Definition (5) in the sense B.

Remark 3.5: Note that, the time-varying and phase-dependent definition $\sigma_k^{\mathrm{C}}(x,u):=\rho(\|x-x_k^\phi\|)$ would at first look like the natural extension of the steady state case. However, in contrast to the time varying case in [16], this definition does not work in the time invariant setting of this paper. More precisely, if $L_k(x,u)\geq\sigma_k^{\mathrm{C}}(x,u)$ for phase ϕ_1 and the rotated cost of the P-periodic optimal trajectory is evaluated for phase $\phi_2\neq\phi_1$, then we obtain the inequality $\sum_{k=0}^{P-1}L_k(x_k^{\phi_2},u_k^{\phi_2})\geq\sum_{k=0}^{P-1}\rho(\|x_k^{\phi_2}-x_k^{\phi_1}\|)$, which can never be satisfied since $\sum_{k=0}^{P-1}L_k(x_k^{\phi_2},u_k^{\phi_2})=0$ and $\sum_{k=0}^{P-1}\rho(\|x_k^{\phi_2}-x_k^{\phi_1}\|)>0$.

IV. Optimal P-periodic Operation and Dissipativity

A P-periodic orbit Π^* with corresponding control sequence u^* is called *optimal* if it has minimal period P^* and corresponds to the state-control pairs Π_U^* defined as

$$(P^*, \Pi_U^*) \in \underset{P,\Pi_U}{\operatorname{argmin}} \ \frac{1}{P} \sum_{k=0}^{P-1} \ell(x_k, u_k), \tag{6}$$

where minimisation is carried out over all periods $P \geq 1$ and all periodic state-control sequences Π_U of minimal period P. We emphasise that, in general, the argmin is not unique. Also note that the minimum might not exist.

The average optimal P-periodic cost (which is independent of ϕ) is given by

$$\ell_P^* := \frac{1}{P} \sum_{k=0}^{P-1} \ell(x_k^{\phi*}, u_k^{\phi*}).$$

For a real vector valued sequence $\boldsymbol{v} = (v_0, v_1, \ldots)$ we define the set of P-step asymptotic averages as

$$\mathrm{Av}^P[\boldsymbol{v}] = \{\bar{v} \in \mathbb{R}^{n_v} : \exists t_n \to +\infty : \lim_{n\to\infty} \frac{\sum_{k=0}^{t_n} \sum_{j=0}^{P-1} v_{Pk+j}}{P(t_n+1)} = \bar{v}\},$$

noting that this set is actually independent of P if the sequence $\boldsymbol{v}$ is bounded.

Let us now define, analogously to [2] and [11], several optimal P-periodic operation concepts. In the following, we use the notation $\boldsymbol{\ell}(\boldsymbol{x}, \boldsymbol{u}) = (\ell(x_0, u_0), \ell(x_1, u_1), \ldots)$.

Definition 4.1 (Optimal P-Periodic Operation): The system (1) is *optimally P-periodically operated* at a periodic orbit Π^* with respect to the stage cost ℓ, if for each solution satisfying $(x_k, u_k) \in \mathbb{Z}$ for all $k = 0, 1, \ldots$, the following holds:

$$\mathrm{Av}^P[\boldsymbol{\ell}(\boldsymbol{x}, \boldsymbol{u})] \subset [\ell_P^*, \infty). \tag{7}$$

Definition 4.2 (Suboptimal non P-Periodic Operation): The system (1) is *suboptimally non P-periodically operated* at a periodic orbit Π^* with respect to the stage cost ℓ and the functions $\sigma^\bullet$ from (2) or (3), if it is optimally P-periodically operated and in addition one of the following two conditions holds:

$$\mathrm{Av}^P[\boldsymbol{\ell}(\boldsymbol{x}, \boldsymbol{u})] \subset (\ell_P^*, \infty), \tag{8a}$$

$$\liminf_{k\to\infty} \sigma^\bullet(x_k, u_k) = 0. \tag{8b}$$

Definition 4.3 (Uniform Suboptimal non P-Periodic Operation): The system (1) is *uniformly suboptimally non P-periodically operated* at a periodic orbit Π^* with respect to the stage cost ℓ and the functions $\sigma^\bullet$ from (2) or (3), if it is suboptimally non P-periodically operated and in addition for each $\delta > 0$ there exists an integer $\bar{t} \geq 1$ such that one of the following two conditions holds:

$$\sum_{k=0}^{t-1} \sum_{j=0}^{P-1} \frac{\ell(x_{Pk+j}, u_{Pk+j})}{Pt} \geq \ell_P^*, \text{for all } t \geq \bar{t}, \tag{9a}$$

$$\sigma^\bullet(x_k, u_k) \leq \delta, \text{ for } P \text{ consecutive } k \in [1, \bar{t}]. \tag{9b}$$

Remark 4.4: We note that the actual behaviour of the trajectories satisfying (9b) differs depending on $\sigma^\bullet$.

In case of σ^{A}, i.e. from (2), if Property (9b) holds for sufficiently small δ, then from the continuity of f and from $\rho(\|x_k - \bar{x}_k^{\mathrm{p}*}\| + \|u_k - \bar{u}_k^{\mathrm{p}*}\|) \leq \delta$ we obtain $f(x_k, u_k) \approx \bar{x}_{k^+}^{\mathrm{p}*}$ with $k^+ = k + 1 \,(\mathrm{mod}\, P^*)$. Since the periodic orbit consists of finitely many distinct points, for sufficiently small $\delta > 0$ this implies $\rho(\|f(x_k, u_k) - \bar{x}_j^{\mathrm{p}*}\| + \|u_{k+1} - \bar{u}_j^{\mathrm{p}*}\|) > \delta$ for all $j \neq k^+$. On the other hand, $\sigma^{\mathrm{A}}(f(x_k, u_k), u_{k+1}) \leq \delta$ implies that there must be some j with $\rho(\|f(x_k, u_k) - \bar{x}_j^{\mathrm{p}*}\| + \|u_{k+1} - \bar{u}_j^{\mathrm{p}*}\|) \leq \delta$ which yields $\rho(\|f(x_k, u_k) - \bar{x}_{k^+}^{\mathrm{p}*}\| + \|u_{k+1} - \bar{u}_{k^+}^{\mathrm{p}*}\|) \leq \delta$. As a consequence, any state-control sequence sufficiently close to Π_U^* and satisfying strict dissipativity with respect to the supply rate $\ell(x_k, u_k) - \ell(x_k^{\mathrm{p}*}, u_k^{\mathrm{p}*})$ and σ^{A}, approximately follows the periodic motion.

In contrast to this, in case of σ^{B}, i.e. from (3), we can only conclude that the solution stays near the set Π_U^* but it need not approximately follow the periodic motion. While it is possible to re-establish approximate periodicity in case Π_U^* is the unique minimiser of $\frac{1}{P}\sum_{k=0}^{P-1} \ell(x_k, u_k)$ over all (not necessarily periodic) orbits of length P, this will require additional arguments in the subsequent proofs and does not directly follow from (3), see also Remark 5.7.

We can now state the following theorem relating dissipativity and optimal operation of the system.

Theorem 4.5: Assume that system (1) is (strictly) P-periodically dissipative on $\mathbb{Z}$ with respect to the supply rate $\ell(x_k, u_k) - \ell(x_k^{\phi*}, u_k^{\phi*})$ and $\sigma^\bullet$ from (2) or (3). Assume, moreover, that the storage functions λ_k are bounded. Then system (1) is optimally P-periodically operated (uniformly suboptimally non P-periodically operated) at the optimal P-periodic trajectory $X_\phi^P(\Pi^*)$.

Proof: The proof follows with appropriate adaptations from the one given in [2, Proposition 6.4] and [11, Theorem 1] for the case $P = 1$. We have

$$\begin{aligned} 0 &= \lim_{T\to\infty} \frac{\lambda_{PT}(x_{PT}) - \lambda_0(x_0)}{PT} \\ &= \lim_{T\to\infty} \sum_{k=0}^{T-1} \sum_{j=0}^{P-1} \frac{\lambda_{Pk+j+1}(x_{Pk+j+1}) - \lambda_{Pk+j}(x_{Pk+j})}{PT} \\ &\leq \liminf_{T\to\infty} \sum_{k=0}^{T-1} \sum_{j=0}^{P-1} \frac{\ell(x_{Pk+j}, u_{Pk+j})}{PT} - \ell_P^*. \end{aligned}$$

This establishes the first claim. If strict P-periodic dissipativity holds

$$\begin{aligned} 0 \leq &\liminf_{T\to\infty} \sum_{k=0}^{T-1} \sum_{j=0}^{P-1} \frac{\sigma^\bullet(x_{Pk+j}, u_{Pk+j})}{PT} \\ &\leq \liminf_{T\to\infty} \sum_{k=0}^{T-1} \sum_{j=0}^{P-1} \frac{\ell(x_{Pk+j}, u_{Pk+j})}{PT} - \ell_P^*, \end{aligned}$$

and two cases are possible:

1) $\liminf_{T\to\infty} \sum_{k=0}^{T-1} \sum_{j=0}^{P-1} \frac{\ell(x_{Pk+j}, u_{Pk+j})}{PT} > \ell_P^*$, which implies $\mathrm{Av}^P[\ell(x, u)] \subset (\ell_P^*, \infty)$, or
2) $\liminf_{T\to\infty} \sum_{k=0}^{T-1} \sum_{j=0}^{P-1} \frac{\ell(x_{Pk+j}, u_{Pk+j})}{PT} = \ell_P^*$, hence $\liminf_{k\to\infty} \sigma^\bullet(x_k, u_k) = 0$.

This proves that strict P-periodic dissipativity entails suboptimal non P-periodic operation. It remains to prove uniform suboptimal non P-periodic operation.

For each feasible solution and $t \geq 0$, (5) and boundedness of functions λ_k entails that

$$\begin{aligned} -c &:= -2 \sup_{\substack{0\leq k\leq P-1 \\ x\in\mathbb{X}_\mathbb{Z}}} |\lambda_k(x)| \leq \lambda_{Pt}(x_{Pt}) - \lambda_0(x_0) \\ &\leq \sum_{k=0}^{t-1} \sum_{j=0}^{P-1} [\ell(x_{Pk+j}, u_{Pk+j}) - \sigma^\bullet(x_{Pk+j}, u_{Pk+j})] - Pt\ell_P^*. \end{aligned}$$

Let $\delta > 0$ be fixed and choose $\bar{t} := \lceil \frac{c}{\delta} \rceil + 1$. Then two cases are possible:

1) $\sum_{k=0}^{t-1} \sum_{j=0}^{P-1} \ell(x_{Pk+j}, u_{Pk+j}) > Pt\ell_P^*$ for all $t \geq \bar{t}$, or
2) $\sum_{j=0}^{P-1} \sigma(x_{Pk+j}, u_{Pk+j}) \leq c/\bar{t}$ for at least one $k \in [1, \bar{t}]$, implying $\sigma^\bullet(x_j, u_j) \leq c/\bar{t}$ and thus $\sigma^\bullet(x_j, u_j) \leq \delta$ for $j = Pk, \ldots, (P+1)k - 1$,

which concludes the proof. □

V. Periodic Stability of Economic MPC

Let us consider the following MPC problem

$$V_N^i(x) = \min_{x_0,u_0,\dots,x_N} \quad J_N^i(\boldsymbol{x},\boldsymbol{u}) \tag{10a}$$

$$\text{s.t.} \quad x_0 = x, \qquad x_{k+1} = f(x_k,u_k), \tag{10b}$$

$$(x_k,u_k)\in\mathbb{Z}, \quad x_N \in \mathbb{X}_\mathrm{f}^{N+i}, \tag{10c}$$

where we define $\boldsymbol{x} = (x_0,\dots,x_N), \boldsymbol{u} = (u_0,\dots,u_{N-1})$ and $J_N^i(\boldsymbol{x},\boldsymbol{u}) = \sum_{k=0}^{N-1}\ell(x_k,u_k) + V_\mathrm{f}^{N+i}(x_N)$. Note that the (periodic) terminal set and cost depend on the current time instant i. We note that this time-dependence can be used in order to induce a fixed phase for the EMPC closed loop trajectory. Note that the choice of terminal constraint may or may not fix the phase of the closed-loop trajectory. One also use terminal costs and constraints which are independent of i, in which case the phase is not fixed. We also remark that for non constant $\mathbb{X}_\mathrm{f}^{N+i}$ the feasible sets $\mathbb{X}_N^i$, i.e., the sets of all x for which the constraints in (10) can be satisfied, depend periodically on i.

Let us introduce the following assumptions. We remark that Assumptions 5.2, 5.4 and 5.5 are standard in stability theory for tracking MPC [13]. Assumption 5.1 is slightly more restrictive than the state of the art for tracking MPC. We remark, however, that for practical applications this assumption is not very restrictive. Finally, Assumption 5.3 is the most important one and the most difficult to check. For steady-state tracking MPC it is always satisfied with storage function $\lambda(x) = 0$.

Assumption 5.1: The sets $\mathbb{Z}$ and $\mathbb{X}_\mathrm{f}^{N+i}$ are compact.

Assumption 5.2: The stage cost $\ell(\cdot,\cdot)$ and system dynamics $f(\cdot,\cdot)$ are continuous on $\mathbb{Z}$. The terminal cost function $V_\mathrm{f}^{N+i}(\cdot)$ is continuous on the terminal region $\mathbb{X}_\mathrm{f}^{N+i}$.

Assumption 5.3 (P-Periodic Strict Dissipativity): System (1) is strictly dissipative at a periodic orbit Π^* with respect to the supply rate $\ell(x,u) - \ell(x_k^{\phi*},u_k^{\phi*})$ and $\sigma^\bullet$ from (2) or (3). Moreover, the storage functions λ_k are bounded and continuous in every point $x_k^{\mathrm{p}*}\in\Pi^*$.

Assumption 5.4: The value function $V_N^i(\cdot)$ is bounded on $\mathbb{X}_N^i$ and continuous in every point $x^{\mathrm{p}*}\in\Pi^*$.

Let us define the rotated MPC problem and the corresponding rotated value function as

$$\bar{V}_N^i(x) = \min_{x_0,u_0,\dots,x_N} \quad \bar{J}_N^i(\boldsymbol{x},\boldsymbol{u}) \qquad \text{s.t. (10b)–(10c)} \tag{11}$$

where we define $\bar{J}_N^i(\boldsymbol{x},\boldsymbol{u}) = \sum_{k=0}^{N-1} L_{k+i}(x_k,u_k) + \bar{V}_\mathrm{f}^{N+i}(x_N)$ and the rotated terminal and stage cost are phase-dependent and defined respectively as $\bar{V}_\mathrm{f}^{N+i}(x) := V_\mathrm{f}^{N+i}(x) + \lambda_{N+i}(x)$ and $L_k = L_{k(\mathrm{mod}P^*)}$ from (4). These definitions imply

$$\bar{J}_N^i(\boldsymbol{x},\boldsymbol{u}) = J_N^i(\boldsymbol{x},\boldsymbol{u}) + \lambda_i(x) - \sum_{k=0}^{N-1}\ell(x_{k+i}^{\phi*},u_{k+i}^{\phi*}) \tag{12}$$

and thus the rotated MPC Problem (11) delivers the same optimal trajectories and control sequences as the original Problem (10), see also [16]. Problem (11) therefore serves as an auxiliary problem for proving stability, even though the problem solved online is typically Problem (10), the two MPC formulations have the same stability properties. For this reason, one could decide to solve online Problem (11) instead, however, in this second case knowledge of the storage function is necessary in order to formulate the MPC problem.

Let us consider a family of periodic terminal regions $\mathbb{X}_\mathrm{f}^k\subset\mathbb{X}$ and terminal costs V_f^k satisfying the following assumption.

Assumption 5.5: The terminal regions are periodic, i.e., $\mathbb{X}_\mathrm{f}^{k+P} = \mathbb{X}_\mathrm{f}^k$ for all $k\geq 0$ and the periodic terminal regions $\mathbb{X}_\mathrm{f}^k$ contain the states $x_k^{\phi*}$ of the periodic trajectory $X_\phi^P(\Pi^*)$ from Assumption 5.3. Moreover, the terminal costs are periodic, i.e., $V_\mathrm{f}^{k+P} = V_\mathrm{f}^k$ for all $k\geq 0$ and there exists a terminal control law $\kappa_\mathrm{f}^k : \mathbb{X}_\mathrm{f}^k\to\mathbb{U}$ with $\kappa_\mathrm{f}^{k+P} = \kappa_\mathrm{f}^k$ for all $k\geq 0$ such that, at a given time instant i, for all $x\in\mathbb{X}_\mathrm{f}^{N+i}$ the inclusion $x^+ := f(x,\kappa_\mathrm{f}^{N+i}(x))\in\mathbb{X}_\mathrm{f}^{N+i+1}$ holds and

$$V_\mathrm{f}^{N+i+1}(x^+) \leq V_\mathrm{f}^{N+i}(x) - \ell(x,\kappa_\mathrm{f}^{N+i}(x)) + \ell(x_{N+i}^{\phi*},u_{N+i}^{\phi*}).$$

We remark that in case $\mathbb{X}_\mathrm{f}^k = \{x_k^{\phi*}\}$, Assumption 5.5 is satisfied with $\kappa_\mathrm{f}^k\equiv u_k^{\phi*}$ and $V_\mathrm{f}^k\equiv 0$. The simplest example for time invariant terminal conditions are $\mathbb{X}_\mathrm{f} = \{x^{\mathrm{p}*}\in\Pi^*\}$ with $\kappa_\mathrm{f}(x_k^{\mathrm{p}*}) = u_k^{\mathrm{p}*}$ and again $V_\mathrm{f}\equiv 0$. We also note that Assumption 5.5 is satisfied for the original MPC problem if and only if it is satisfied for the rotated problem, see e.g. [1]. For an analysis of a periodic EMPC scheme without any terminal conditions we refer to [12].

Theorem 5.6: Let Assumptions 5.1, 5.2, 5.3, 5.4 and 5.5 hold. Then the rotated optimal value function $\bar{V}_N(x)$ is a Lyapunov function and, for σ^A, i.e. from (2), there exists a phase ϕ such that the trajectory $x_k^{\phi*}$ corresponding to the optimal periodic orbit Π^* is asymptotically stable for the closed loop system. For σ^B, i.e. from (3), the optimal periodic orbit Π^* is an asymptotically stable set for the closed loop system.

Proof: The proof uses ideas similar to the steady state case [1] with appropriate adaptations. We define $\sigma^*(x) := \inf_{u\in U}\sigma^\mathrm{A}(x,u) = \sigma^\mathrm{B}(x,u)$. Assumptions 5.1 and 5.2, Formula (12) and the boundedness and continuity in every $x_k^{\mathrm{p}*}$ from Π^* of V_N^i and λ ensured by Assumptions 5.3 and 5.4 imply that $\bar{V}_N^i$ is also bounded on $\mathbb{X}_N^i$ and continuous in every $x_k^{\mathrm{p}*}$ from Π^*. Moreover, the strict dissipativity Assumption 5.3 implies $\bar{V}_N^i(x)\geq L_k(x,u)\geq\sigma^*(x)$ and $L_k(x_k^{\mathrm{p}*},u_k^{\mathrm{p}*}) = 0$ implies $\bar{V}_N^i(x_i^{\mathrm{p}*}) = 0$. Together, these properties ensure the existence of $\mathcal{K}$ functions $\hat{\alpha}$ and α such that

$$\hat{\alpha}(\sigma^*(x))\leq\bar{V}_N^i(x)\leq\alpha(\sigma^*(x)).$$

Note that local loss of controllability near the periodic optimal trajectory can entail a discontinuity of $V_N^i(\cdot)$ at Π^* and hence of $\bar{V}_N^i(\cdot)$. If the cost $\bar{V}_N^i(\cdot)$ is not continuous at the periodic optimal trajectory, we cannot establish the upper bound $\bar{V}_N^i(x)\leq\alpha(\sigma^*(x))$.

In order to prove descent of the rotated value function $\bar{V}_N^i(x)$, let us define the optimal (open loop) state and control trajectory as

$$\boldsymbol{x}_i^\mathrm{MPC} = (x_{0,i}^\mathrm{MPC},\dots,x_{N,i}^\mathrm{MPC}), \quad \boldsymbol{u}_i^\mathrm{MPC} = (u_{0,i}^\mathrm{MPC},\dots,u_{N-1,i}^\mathrm{MPC}).$$

Let us moreover define a feasible candidate trajectory for the MPC problem at the next time step as

$$\begin{aligned}\bar{\boldsymbol{x}}_{i+1} &= (x_{1,i}^\mathrm{MPC},\dots,x_{N,i}^\mathrm{MPC}, f(x_{N,i}^\mathrm{MPC},\kappa_{\mathrm{f},i}^k(x_{N,i}^\mathrm{MPC}))),\\ \bar{\boldsymbol{u}}_{i+1} &= (u_{1,i}^\mathrm{MPC},\dots,u_{N-1,i}^\mathrm{MPC},\kappa_{\mathrm{f},i}^k(x_{N,i}^\mathrm{MPC})).\end{aligned}$$

The rotated objective value associated with this trajectory is given by

$$\begin{aligned}\bar{J}_N^{i+1}(\bar{\boldsymbol{x}}_{i+1},\bar{\boldsymbol{u}}_{i+1}) \;&=\; \bar{V}_N^i(x_{0,i}^\mathrm{MPC}) - L_i(x_{0,i}^\mathrm{MPC}u_{0,i}^\mathrm{MPC})\\ &-\; V_\mathrm{f}^{N+i}(x_{N,i}^\mathrm{MPC}) + V_\mathrm{f}^{N+i}(f(x_{N,i}^\mathrm{MPC},\kappa_\mathrm{f}^{N+i}(x_{N,i}^\mathrm{MPC})))\\ &+\; \ell(x_N,\kappa_\mathrm{f}^{N+i}(x_N)) - \ell(x_{N+1+i}^{\phi*},u_{N+1+i}^{\phi*})\\ &\leq \bar{V}_N^i(x_{0,i}^\mathrm{MPC}) - L_i(x_{0,i}^\mathrm{MPC},u_{0,i}^\mathrm{MPC}),\end{aligned}$$

where the last inequality follows from Assumption 5.5. Optimality implies $\bar{V}_N^{i+1}(x_{1,i}^\mathrm{MPC})\leq\bar{J}_N^{i+1}(\bar{\boldsymbol{x}}_{i+1},\bar{\boldsymbol{u}}_{i+1})$ and hence

$$\begin{aligned}\bar{V}_N^{i+1}(x_{1,i}^\mathrm{MPC}) - \bar{V}_N^i(x_{0,i}^\mathrm{MPC}) &\leq -L_i(x_{0,i}^\mathrm{MPC},u_{0,i}^\mathrm{MPC})\\ &\leq -\sigma^\bullet(x_{0,i}^\mathrm{MPC},u_{0,i}^\mathrm{MPC}).\end{aligned}$$

The periodic family of rotated value functions is hence a family of Lyapunov functions for the nonlinear system; particularly, $\bar{V}_N^i$ converges to 0 along the closed loop trajectory. From this, for σ^B from (3), the claimed stability properties immediately follow.

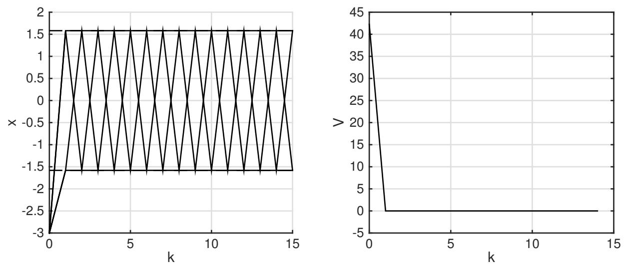

Fig. 1. Example 6.1. Left graph: closed-loop trajectory (continuous line) obtained starting from $\hat{x}_0 = -3$. The periodic optimal states are displayed in dashed line. Right graph: Value function of the rotated MPC problem.

For σ^{A} from (2), the lower bound $\hat{\alpha}(\sigma^*(x))$ of the Lyapunov functions only implies the convergence of the states of the closed loop to Π^* but not necessarily of the controls. Hence, the proof so far only shows asymptotic stability of the set Π^* but not of the periodic trajectory $x_k^{\phi*}$ corresponding to Π^*. However, from the last inequality, above, we obtain $\sigma^{\mathrm{A}}(x_{0,i}^{\mathrm{MPC}}, u_{0,i}^{\mathrm{MPC}}) \leq \bar{V}_N^i(x_{0,i}^{\mathrm{MPC}})$ implying that since V^i tends to 0 the value $\sigma^{\mathrm{A}}(x_{0,i}^{\mathrm{MPC}}, u_{0,i}^{\mathrm{MPC}})$ also tends to 0. By (2) this yields that $|(x_{0,i}^{\mathrm{MPC}}, u_{0,i}^{\mathrm{MPC}})|_{\Pi_u}$ tends to 0 as V_N^i tends to 0 and thus asymptotic stability of the trajectory corresponding to the periodic orbit Π^* follows by similar arguments as in Remark 4.4. □

Remark 5.7: In case of strict dissipativity of type B, i.e. with σ^{B} from (3), asymptotic stability of the periodic trajectory $x_k^{\phi*}$ follows if the optimal periodic orbit Π^* is the unique minimiser of $J_P(\boldsymbol{x}, \boldsymbol{u})$ over all (not necessarily periodic) orbits of length P. Indeed, in this case for $x_{0,i}^{\mathrm{MPC}}$ sufficiently close to Π^*, due to continuity X_i^{MPC} must approximately follow $x_k^{\phi*}$ because otherwise we would obtain a contradiction to the optimality of $\boldsymbol{x}_i^{\mathrm{MPC}}$.

Remark 5.8: A straightforward adaptation of the proof of [1, Theorem 18] shows that under the conditions of Theorem 5.6 the averaged infinite horizon functional along the closed loop satisfies

$$J_\infty^{\mathrm{av}}(\boldsymbol{x}^{\mathrm{MPC,cl}}, \boldsymbol{u}^{\mathrm{MPC,cl}}) \leq \ell_P^*.$$

VI. Examples

The following examples illustrate the proposed concepts.

Example 6.1 (Strict Dissipativity of type B):
Consider the 1d dynamics $f(x,u) = -x + u$ and stage cost

$$\ell(x,u) = (x-2)(x-1)(x+1)(x+2).$$

The optimal trajectory can either be of period $P = 1$, i.e. one of the two steady states $x_s^{1,2} = \pm\frac{\sqrt{10}}{2}$, or of period $P = 2$, with $\Pi^* = \left(\frac{\sqrt{10}}{2}, -\frac{\sqrt{10}}{2}\right)$ and $u_1^{\mathrm{p}*} = u_0^{\mathrm{p}*} = 0$. Using $\lambda_0(x) = \lambda_1(x) = 0$, it can be verified that $L_0(x,u) = L_1(x,u)$ satisfy the strict dissipation of type B, i.e. with $\sigma^{\mathrm{B}}(\cdot,\cdot)$ from (3).

As the control is not constrained and it does not enter the cost, MPC will stabilise the system in one step. The solution of the MPC problem is not unique and we can conclude from Definition (3) that the system will be stabilised to the set of states included in the periodic optimal trajectory. However, both staying at one of the steady states and moving to the other one is optimal. Using the initial condition $\hat{x}_0 = -3$ and the terminal constraint $x_N = (-1)^{i+N+1}\sqrt{10}/2$, all possible closed-loop trajectories and the value of the rotated problem are displayed in Figure 1. Note that, as expected, the system is stabilised to the optimal operation in one step. Moreover, using the terminal constraints $x_N = (-1)^{i+N}\sqrt{10}/2$, $x_N = -\sqrt{10}/2$ or $x_N = \sqrt{10}/2$ yields the exact same closed-loop result. The same holds if one uses no terminal constraints but the terminal cost $V_{\mathrm{f}}(x) = \ell(x,0)$, i.e. the cost of stabilising the system in one step.

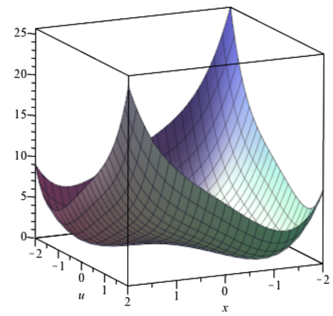

Fig. 2. Example 6.2: graph of the rotated stage costs $L_1 = L_2$.

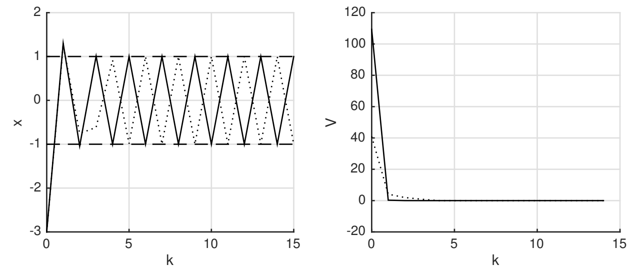

Fig. 3. Example 6.2. Left: closed-loop trajectory starting from $\hat{x}_0 = -3$ by using terminal constraint TC1 (solid line) and TC2 (dotted line). The periodic optimal states are displayed in dashed line. Right: Value function of the rotated MPC problem for terminal constraint TC1 (solid line) and TC2 (dotted line).

Example 6.2 (Strict Dissipativity of type A): Consider the 1d system with dynamics $f(x,u) = u$ and stage cost

$$\ell(x,u) = x^4 - x^3/3 - 2x^2 + x + (x+u)^2.$$

Using $\lambda_1(x) = -x^4/2 + x^3/6 + x^2 - x/2 + 2/3$, $\lambda_2(x) = -x^4/2 + x^3/6 + x^2 - x/2$, $\Pi = (1,-1)$ and $\Pi_U = ((1,-1),(-1,1))$, and phase $\phi = 0$ one obtains

$$L_1(x,u) = L_2(x,u) = \frac{x^4}{2} - \frac{x^3}{6} + \frac{x}{2} + 1 + \frac{u^4}{2} - \frac{u^3}{6} + \frac{u}{2} + 2xu.$$

One checks that this polynomial has exactly two local minima at $(1,-1)$ and $(-1,1)$ at which its value is 0, cf. Figure 2. Hence, it is positive elsewhere and since it grows unboundedly for $|x|, |u| \to \infty$, we can find $\rho \in \mathcal{K}_\infty$ such that (5) holds with $\sigma^\bullet(\cdot,\cdot)$, $\bullet = \{\mathrm{A}, \mathrm{B}\}$. Note that using the wrong phase, i.e. $\phi = 1$, leads to a function L_2 which attains negative values so that (5) can never hold.

We consider an MPC scheme with horizon $N = 5$, initial condition $\hat{x}_0 = -3$ and terminal constraint $x_N = (-1)^{i+N+1}$ (TC1), terminal constraint $x_N = (-1)^{i+N}$ (TC2), or terminal cost $V_{\mathrm{f}}(x) = \ell(x, \kappa(x))$, with $\kappa(x) = -1$ for $x \geq 0$ and $\kappa(x) = 1$ otherwise.

The closed-loop trajectories obtained by using the two proposed terminal point constraints and the value of the rotated problem are displayed in Figure 3. In this example the phase of the terminal constraint does determine the phase of the closed-loop trajectory. The closed-loop results obtained with the terminal cost formulation differ only very slightly from the one obtained using TC1 and are indistinguishable by eye inspection. □

Example 6.3 (Strict Dissipativity of type B for a 2d system): Define $x = [z,\, y]^T$ and consider the system dynamics and stage cost

$$f(x,u) = [-0.9\,z + y\,u,\; -y]^\top,$$
$$\ell(x,u) = (z-1.9)(z-0.9)(z+1.1)(z+2.1) + (u-20)^2,$$

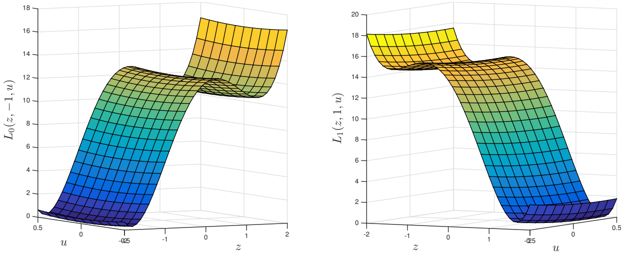

Fig. 4. Example 6.3: Rotated stage cost $L_0(z, y, u)$ with $y = -1$ fixed (left) and $L_1(z, y, u)$ with $y = 1$ fixed (right).

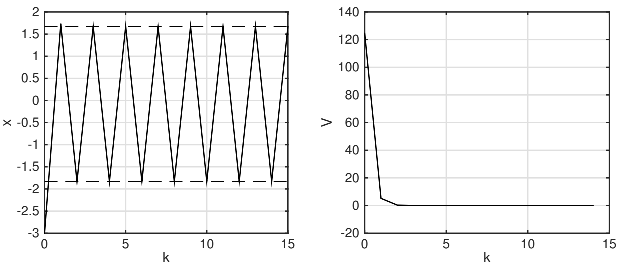

Fig. 5. Example 6.3. Left: closed-loop trajectory (continuous line) obtained starting from $\hat{x}_0 = (-3, 1)$. The periodic optimal states are displayed in dotted line. Right: Value function of the rotated MPC problem.

with constraint $y \in \{-1, 1\}$.

Let us consider the case $\{y_0^{\mathrm{p}*}, y_1^{\mathrm{p}*}\} = \{1, -1\}$. The optimal trajectory is periodic with period $P = 2$ and can be computed numerically: $\Pi^* = \{(z_0^{\mathrm{p}*}, y_0^{\mathrm{p}*})^T, (z_1^{\mathrm{p}*}, y_1^{\mathrm{p}*})^T\}$ with $z_0^{\mathrm{p}*} \approx -1.8294$, $z_1^{\mathrm{p}*} \approx 1.6719$, $\{u_0^{\mathrm{p}*}, u_1^{\mathrm{p}*}\} \approx \{0.0254, 0.3247\}$ and Lagrange multipliers associated to the z-variable of the dynamic constraints $\{\lambda_0^{\mathrm{p}}, \lambda_1^{\mathrm{p}}\} \approx \{39.9492, -39.3506\}$. Using $\lambda_0(x) = (ay - b)(z - z_0) + (1 - y)e$ and $\lambda_1(x) = (ay - b)(z - z_1)$, with $a = \frac{\lambda_0^{\mathrm{p}} - \lambda_1^{\mathrm{p}}}{2}$ and $b = \frac{\lambda_0^{\mathrm{p}} + \lambda_1^{\mathrm{p}}}{2}$, we obtain

$$L_0(x, u) = \ell(x, u) - \ell(x_0, u_0) + \lambda_0(x) - \lambda_1(f(x, u)) + (1 - y)e,$$
$$L_1(x, u) = \ell(x, u) - \ell(x_1, u_1) + \lambda_1(x) - \lambda_0(f(x, u)) - (1 + y)e.$$

Computing the minima of these functions reveals that for $e \approx 6.89763344$ the functions L_k satisfy the strict dissipation inequalities (5) for $\sigma^{\mathrm{A}}(\cdot, \cdot)$, cf. Figure 4.

For an MPC scheme with horizon $N = 5$ and an initial condition $\hat{x}_0 = (-3, 1)$ with terminal constraint $x_N = (\bar{z} + (-1)^{i+N}\Delta\bar{z}, (-1)^{i+N})$, with $\bar{z} = \frac{z_0^{\mathrm{p}} + z_1^{\mathrm{p}}}{2}$, $\Delta\bar{z} = \frac{z_0^{\mathrm{p}} - z_1^{\mathrm{p}}}{2}$, the closed-loop trajectory and the value of the rotated problem are displayed in Figure 5. Note that any terminal constraint having $y_{N+i} = (-1)^{i+N+1}$ would be infeasible, while any constraint having $x_N = (\bar{z} + (-1)^{i+N+1}\Delta\bar{z}, (-1)^{i+N})$ would not contain the periodic optimal trajectory and therefore violate Assumption 5.5.

We have implemented the same scheme using the terminal cost given by $V_{\mathrm{f}}(x) = \ell(x, \kappa(x))$, with $x = (z, y)$ and $\kappa(x) = z_1^{\mathrm{p}} + 0.9z$ for $y \geq 0$ and $\kappa(x) = -(z_0^{\mathrm{p}} + 0.9z)$ otherwise. The resulting closed loop trajectory is very close to the one obtained using the terminal point constraint.

VII. Discussion and Conclusions

In this paper, we have presented an extension of strict dissipativity to the case of optimal periodic operation. We have proven that several previous results obtained for the steady state case extend to our setting for periodic operation. These theoretical results have been illustrated using several numerical examples. In particular, analogously to the steady state case (see Section II), if a system equipped with a stage cost ℓ is P-periodically (strictly) dissipative, then:

- The system is optimally operated at (uniformly suboptimally operated off) the P-periodic orbit (Theorem 4.5)
- For economic MPC with terminal constraint and cost, the averaged performance $J_\infty^{\mathrm{av}}\left(\boldsymbol{x}^{\mathrm{MPC,cl}}, \boldsymbol{u}^{\mathrm{MPC,cl}}\right)$ equals ℓ_P^* (Remark 5.8) and the P-periodic orbit Π^* is an asymptotically stable set of points for the closed loop system (Theorem 5.6). If, moreover, strict dissipativity of type A holds, then the P-periodic orbit Π^* is an asymptotically stable trajectory for the closed loop system (Theorem 5.6).

The proposed setting straightforwardly extends to the case of multistep MPC [8, Section 7.4]. The major limitations of the current theory, both for the steady state and the periodic case, include

1) while sufficiency of strict dissipativity for stability has been proven in [1] and in the current paper, to the authors' knowledge, no result on its necessity has been obtained yet
2) in general it can be very difficult to prove the existence of a storage function which satisfies the strict dissipativity condition
3) the storage function is assumed to be bounded and continuous in $x_k^{\mathrm{p}*}$ from Π^*
4) in all our examples, functions L_k are identical for all k. So far we were not able to determine whether this is just a coincidence or whether there is a systematic reason for this fact.

Future research will aim at developing the theory further so as to overcome these limitations.

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